1st *Singapore Physics League*

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All diagrams were designed in-house. Some problems' diagrams involved the use of free external graphics acknowledged below:

- *Holding a Pen:* Image of fingers made by Smashicons from www.flaticon.com.
- *Self-Supporting Table:* Image of bucket made by Freepik from www.flaticon.com.
- *Tug of War:* Images of people pulling rope made by Freepik from www.flaticon.com.
- *Tilted Mirrors, Lights over Nuremberg, Mirage, Shortsighted Swimmer:* Image of eye made by Freepik from www.flaticon.com.

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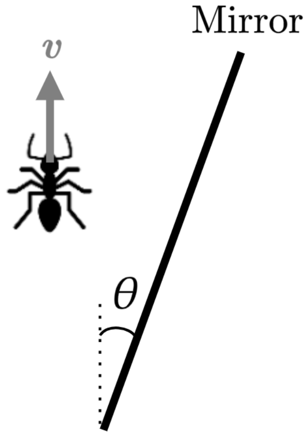
Problem 1: Deeply Reflective Ant (3 points)

An ant travels upwards at constant velocity $v = 4.0$ cm s^{-1}. Beside it, there is a mirror inclined at angle $\theta = 15°$ from the vertical. How fast does the ant perceive its mirror image to be moving?

Leave your answer to 2 significant figures in units of cm s^{-1}.



Solution: The component of the ant's velocity in the direction perpendicular to the surface of the mirror is given by $v_\perp = v \sin\theta$. This means that rate of increase of the perpendicular distance between the ant and the mirror is $v_\perp$.

In any mirror, the perpendicular distance between the object and the mirror is equal to the perpendicular distance between the image and the mirror. This means that the separation between the ant and its mirror image is exactly twice that of the perpendicular distance between the ant and the mirror. As such, the rate of increase of this separation is $2v_\perp = 2v \sin\theta \approx \boxed{2.1 \text{ cm s}^{-1}}$. This is the speed that the ant would perceive its image to be travelling at.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 2: Measuring Contact Angles

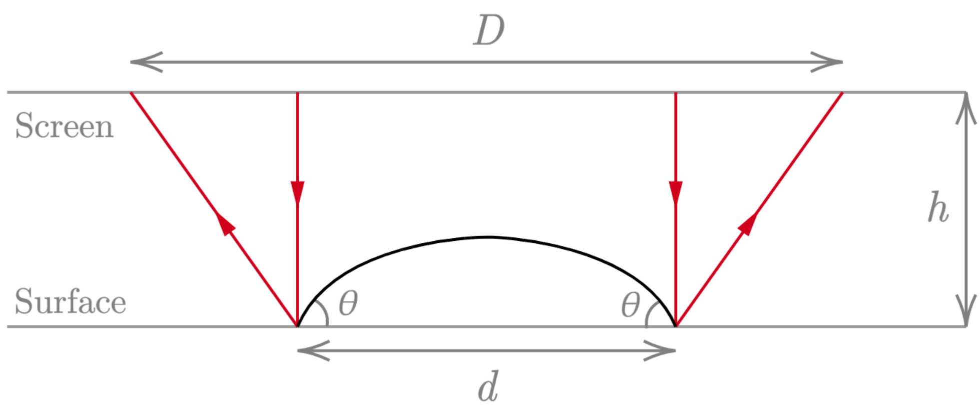
A drop of liquid of diameter $d = 5.0$ mm rests on a horizontal, flat solid surface. In order to find the contact angle θ between the drop and the surface, a collimated laser beam of identical diameter is shone vertically onto the liquid drop, and a translucent screen is set up horizontally at a height $h = 12$ cm above the drop. Assume that $\theta < 90°$. The diagram is not drawn to scale.

(a) Given that an image of diameter $D = 5.10$ cm is formed on the screen, find the contact angle θ between the liquid and solid surface.

Leave your answer to 2 significant figures in units of degrees. (3 points)

(b) The image has blurry edges, and the experimenters could only confidently determine the diameter of the image up to an uncertainty of $\delta D = 0.10$ cm. What is the corresponding uncertainty $\delta\theta$ in the measured contact angle?

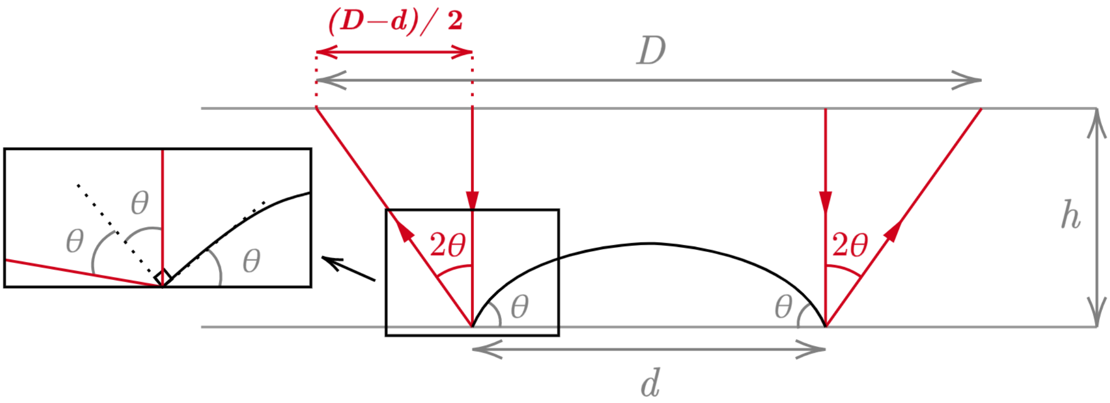
Leave your answer to 2 significant figures in units of degrees. (3 points)



Solution:

(a) The outermost regions of the image are formed by reflection of the laser beam from the edges of the liquid. Since the liquid has a contact angle of θ, the beam will be incident on the liquid surface at this angle and the reflected beam will be diverted by a total angle of 2θ. Thus, we can infer from geometry that:

$$\tan 2\theta = \frac{D-d}{2h} \implies \theta = \frac{1}{2}\tan^{-1}\frac{D-d}{2h} \approx \boxed{5.4°}$$



(b) Letting $\theta(D) = \frac{1}{2}\tan^{-1}\frac{D-d}{2h}$, we can use the relation

$$\delta\theta = \frac{\mathrm{d}\theta}{\mathrm{d}D}\delta D$$

to obtain

$$\delta\theta = \frac{\mathrm{d}\theta}{\mathrm{d}D}\delta D = \frac{h}{(D-d)^2+4h^2}\delta D = 0.0020095\ldots\ \text{rad} \approx \boxed{0.12^\circ}$$

Note that since the formula $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2}$ is only valid when the angles are measured in radians, we need to perform the whole computation in radians before converting to degrees as a final step.

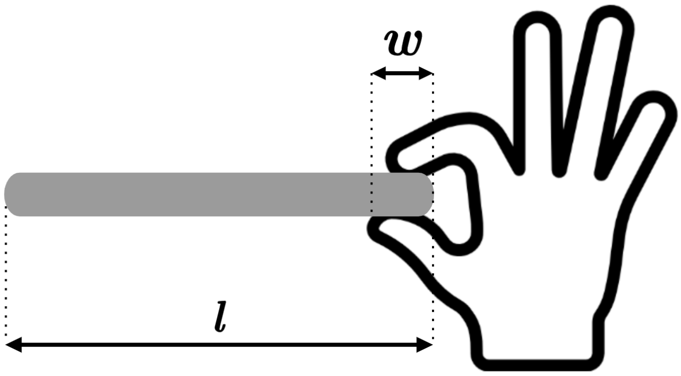
Alternatively, one can simply use a formula of the form $\delta\theta = \theta(D+\delta D) - \theta(D)$ and substitute the appropriate values to obtain the same numerical answer at the required accuracy.

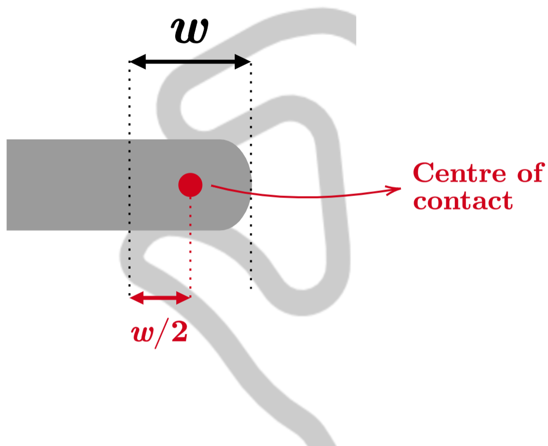
Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Problem 3: Holding a Pen (3 points)

If you try to hold a stationary pen horizontal by clamping its right end with two fingers, your fingers have a tendency to rotate anticlockwise. Consider the pen to be a thin uniform stick of mass $m = 25$ g and length $l = 14$ cm, and suppose that your fingers have contact width $w = 2$ cm with the pen. Calculate the anticlockwise torque exerted by the pen on your fingers about the centre of contact of your fingers.

Leave your answer to 2 significant figures in units of N m.





Solution: Let us consider the pen and the torques acting on it, and take the origin to be the centre of contact of the fingers. For the pen to be in static equilibrium, net torque on the pen around the origin is zero, meaning that the clockwise torque exerted by the fingers on the pen needs to balance the anticlockwise torque due to weight:

$$\begin{aligned}\text{Magnitude of torque exerted by fingers} &= \text{Magnitude of torque due to weight}\\ &= mg\left(\frac{l}{2} - \frac{w}{2}\right)\end{aligned}$$

where $\frac{l}{2} - \frac{w}{2}$ is the distance between the pen's centre and the origin.

Newton's Third Law then tells us that the torque exerted by the pen on the fingers must be equal and opposite to the torque exerted by the fingers on the pen. Hence, the fingers experience a torque of $mg\left(\frac{l}{2} - \frac{w}{2}\right) = \boxed{0.015 \text{ N m}}$ anticlockwise.

Alternative solution: As an alternative approach that is more mathematical, we set the origin to be the pen's centre, and let $\sigma(x)$ be the vertical force (with upwards defined positive) exerted by the fingers on the pen per unit length at distance x from the origin, where $\frac{l}{2} - w < x < \frac{l}{2}$ by definition. [In other words, the force exerted by fingers on a portion of the pen from x to $x + \mathrm{d}x$ is given by $\sigma(x)\,\mathrm{d}x$.]

Static equilibrium requires:

$$\text{Net force on the pen} = 0 \implies \int_{\frac{l}{2}-w}^{\frac{l}{2}} \sigma(x)\,\mathrm{d}x = mg$$

$$\text{Net torque on the pen} = 0 \implies \int_{\frac{l}{2}-w}^{\frac{l}{2}} \sigma(x)x\,\mathrm{d}x = 0$$

To determine the magnitude of torque τ exerted by the pen on the finger around the finger's centre of contact (which is at $\frac{l}{2} - \frac{w}{2}$ away from the origin), it can be written as:

$$\tau = \int_{\frac{l}{2}-w}^{\frac{l}{2}} \sigma(x)\left(x - \frac{l-w}{2}\right)\mathrm{d}x$$

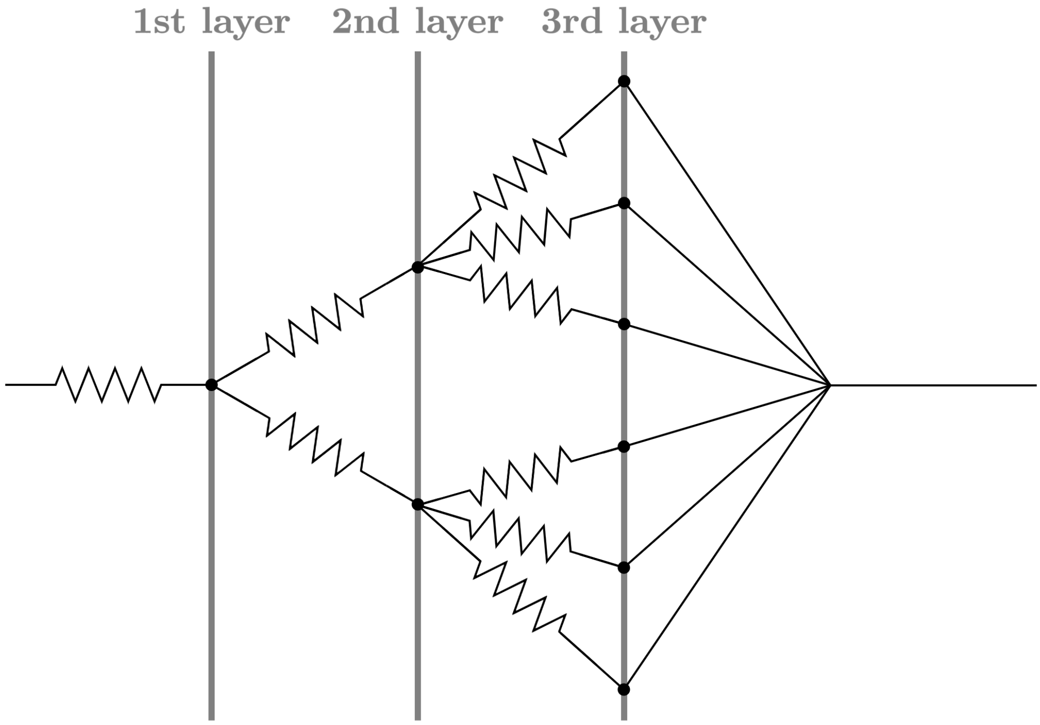
Expanding the integral, and substituting the two previously determined results, the expression for τ reduces to $\tau = mg\left(\frac{l-w}{2}\right)$, which is interestingly independent of the precise form of $\sigma(x)$. As expected, this is in agreement with the result of our previous approach.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

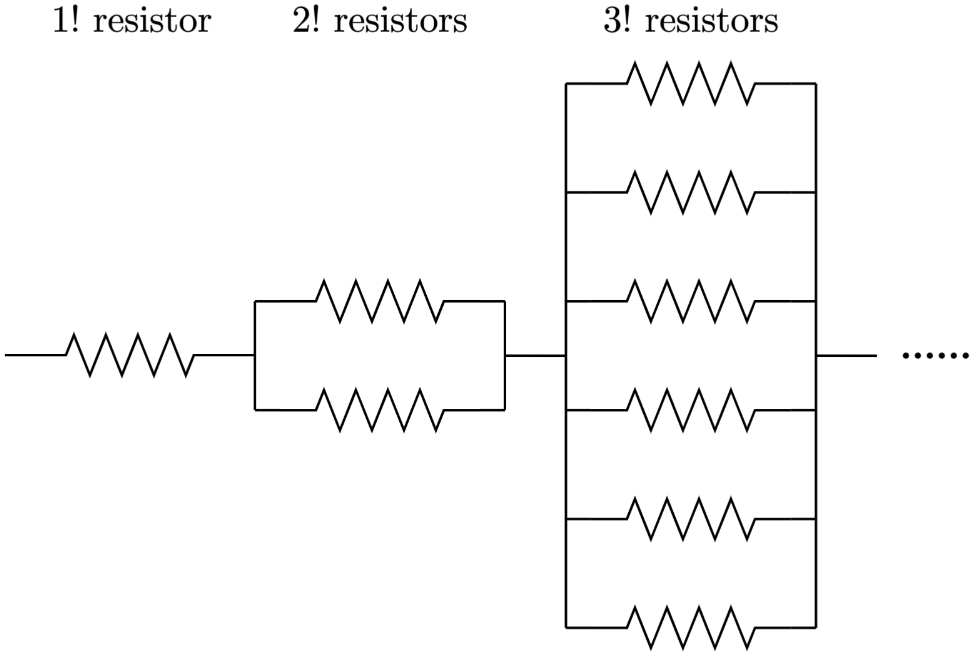
Problem 4: Circuitree (4 points)

Consider a tree of resistors with a number of layers, with nodes on each layer that resistor branches are connected to. Each node at the n^{th} layer branches out to form $(n+1)$ nodes at the $(n+1)^{\text{th}}$ layer. Each branch has resistance $R = 10\ \Omega$. A tree with 3 layers has been illustrated below. What is the effective resistance, R_{eff}, of an infinite-layered variant of the tree (with an infinite number of layers)?

Leave your answer to 4 significant figures in units of Ω.



Solution: Observe that there are $n!$ resistors between the $(n-1)^{\text{th}}$ and n^{th} layers. By symmetry, all the nodes on a layer are equipotential. As such, between two adjacent layers, the resistors can be treated to be connected in parallel, as shown in the circuit below.



The resistance between the $(n-1)^{\text{th}}$ and n^{th} layers, R_n, would follow:

$$\frac{1}{R_n} = n!\left(\frac{1}{R}\right) \implies R_n = \frac{R}{n!}$$

Thus, the total resistance of an infinite-layered tree can be found:

$$R_{\text{eff}} = \sum_{n=1}^{\infty} \frac{R}{n!} = R(e-1) \approx \boxed{17.18\ \Omega}$$

Setter: Galen Lee, galen.lee@sgphysicsleague.org

Problem 5: Cannonballs (4 points)

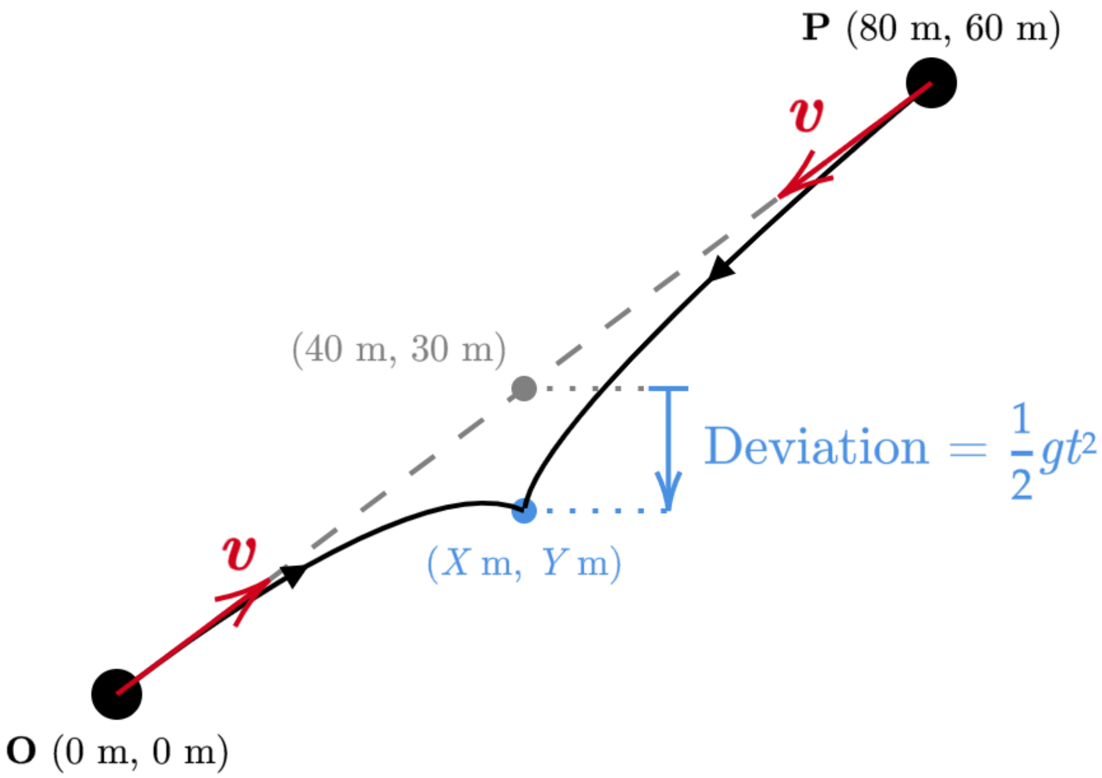
Two cannons, one situated on a cliff at coordinates $P = (80\text{ m},\ 60\text{ m})$ and the other at the origin, are aimed directly at each other. They fire simultaneously, projecting their cannonballs at identical speeds of $v = 50\text{ m s}^{-1}$. Let the coordinates of the position where the cannonballs collide be $(X\text{ m},\ Y\text{ m})$. Find $X + Y$. You can assume that the cannonballs are point objects, and that the trajectories of the cannonballs are not obstructed by the cliff.

Leave your answer to 3 significant figures.

Solution: The simplest approach will be to first consider the situation in the absence of gravity. The first quantity that we are interested in is the time it takes for the two cannonballs to collide, t_c. Since the 2 cannonballs fly directly at each other in this situation, $t_c = \frac{\sqrt{80^2+60^2}}{2v} = 1$ s.

In fact, even in the presence of gravity, the cannonballs take the same time t_c to collide. This is because gravity only changes the vertical component of the cannonball's velocity, while t_c depends only on the horizontal component of the cannonball's velocity, which is the same at any point in time whether gravity is present or not.

In the absence of gravity, the trajectory of each cannonball is a straight line. The presence of gravity causes the cannonballs to deviate from this trajectory, such that the new trajectory is a parabola. The vertical deviation of each cannonball from its straight line trajectory at time t after firing is $-gt^2/2$. Thus at the point of collision, each cannonball will have a vertical deviation of $-gt_c^2/2 = -4.905$ m.



Without gravity, the 2 cannonballs collide at their midpoint (40 m, 30 m). With gravity, the coordinates become (40 m, 30 m − 4.905 m) = (40 m, 25.095 m). Thus, $X + Y = 40 + 25.095 \approx \boxed{65.1}$.

An alternative approach will be to solve for the trajectory of each cannonball and locate their point of intersection, but this approach requires more calculation.

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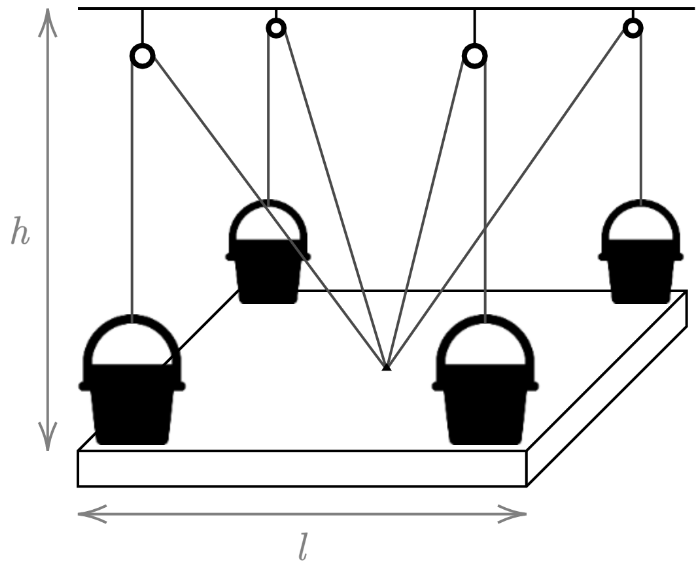
Problem 6: Self-Supporting Table (4 points)

"This table is held up by its own weight." - *r/mildlyinteresting*

A horizontal square table of side length $l = 50.0$ cm is located at vertical distance $h = 100$ cm below the ceiling. The table has mass M when it is empty. Four buckets, each of mass $m = 0.500$ kg, are placed on the corners of the table. Tied to each bucket is a taut light string that loops over a small support at the ceiling, whose other end is attached to the centre of the table. This way, the table stands by itself without requiring any legs.

What is the maximum value of M such that the table can be in equilibrium like that? Assume that the buckets are negligibly small, and that each support is directly above a bucket. Also take the distance between the supports and the ceiling to be negligible.

Leave your answer to 3 significant figures in units of kg.



Solution: Each string has two portions: a vertical portion, and a slant portion. Let θ be the angle of the slant portion from the horizontal. The diagonal distance between the corner and the centre of the square is $\frac{l}{\sqrt{2}}$. From trigonometry, we may derive that $\tan\theta = \frac{\sqrt{2}h}{l}$. Correspondingly, $\sin\theta = \frac{1}{\sqrt{1+\frac{l^2}{2h^2}}}$.

Since the strings are light, each string can be assumed to have uniform tension. By symmetry, each string has the same tension. Let this tension be T.

Consider the vertical forces acting on the entire system of the table and four buckets. The vertical portions of the strings each collectively exert a force of $4T$ upwards, while the slant portions collectively exert a force with an upward component of $4T\sin\theta$.

Balancing vertical forces on this system, we can write:

$$\begin{aligned} 4T(1+\sin\theta) &= (M+4m)g \\ \implies T &= \left(\frac{M}{4}+m\right)\frac{g}{1+\sin\theta} \end{aligned}$$

Now, let us consider the forces on a single bucket. The bucket is pulled up by T, and pushed up by a contact force N from the table, while being pulled down by its weight mg. Considering its net force to be zero at equilibrium, we can determine N:

$$N = mg - T = g\left(\frac{m\sin\theta - \frac{M}{4}}{1+\sin\theta}\right)$$

In order for the bucket to remain in contact with the table, $N \geq 0$. Hence the condition for equilibrium is:

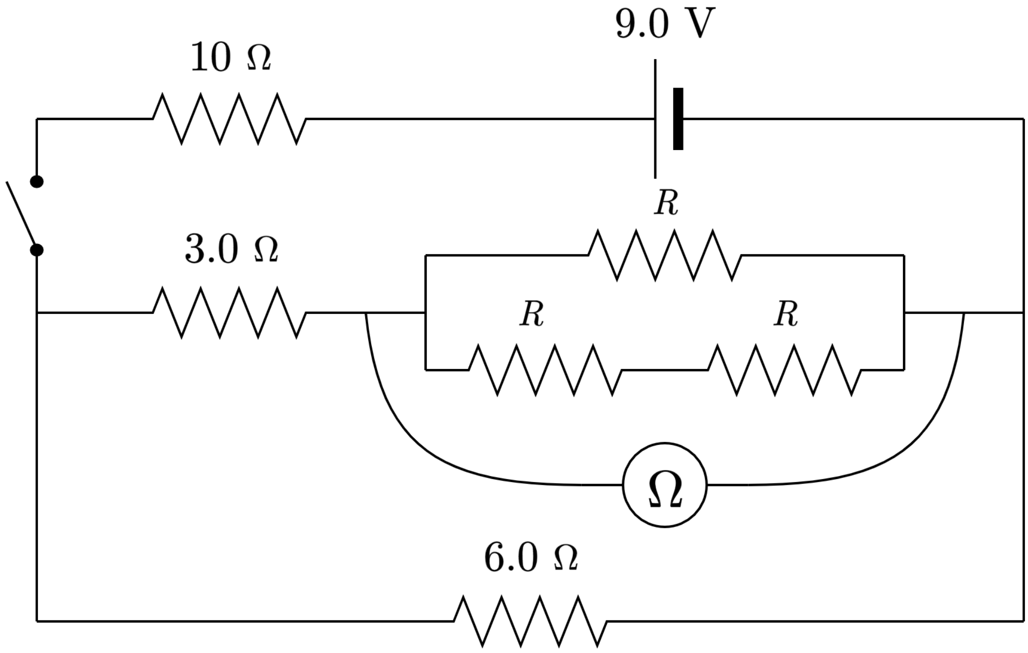
$$M \leq 4m\sin\theta \implies M \leq \frac{4m}{\sqrt{1+\frac{l^2}{2h^2}}} \approx \boxed{1.89\text{ kg}}$$

Setter: Christopher Ong, **chris.ong@sgphysicsleague.org**

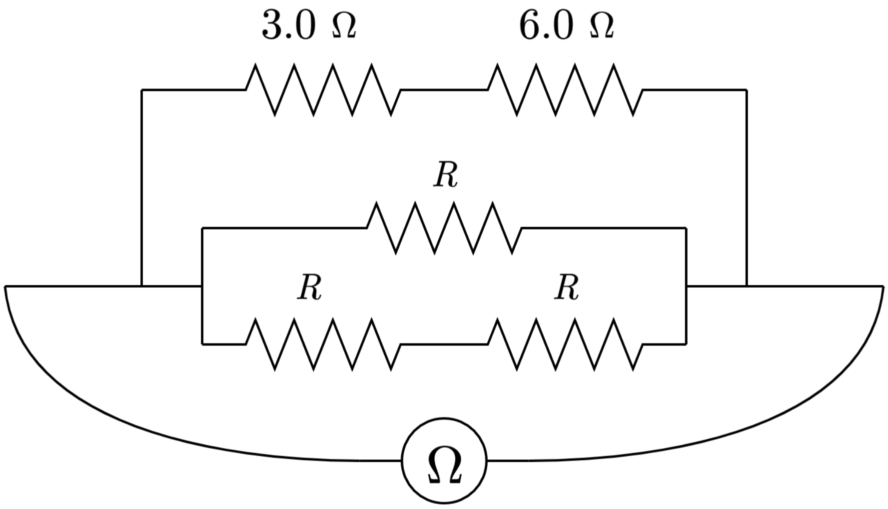
Problem 7: Ohmmeter (3 points)

An ohmmeter is a device that measures the resistance between two points in a circuit. In the circuit below, the ohmmeter produces a reading of $r = 6.0\ \Omega$. Find R.

Leave your answer to 2 significant figures in units of Ω.



Solution: Since the switch is open, the uppermost part of the circuit (the battery with the $10\ \Omega$ resistor) can be ignored. Between the two chosen points, current can flow in two paths. The first path is through the three R resistors. The second path is through the $3.0\ \Omega$ and $6.0\ \Omega$ resistors. The equivalent circuit thus looks like this:



The first path (the arrangement of the three R resistors) has an equivalent resistance of $\frac{1}{\frac{1}{R}+\frac{1}{2R}} = \frac{2R}{3}$, while the second path (the $3.0\ \Omega$ and $6.0\ \Omega$ resistors in series) has an equivalent resistance of $3.0 + 6.0 = 9.0\ \Omega$.

The ohmmeter measures the equivalent resistance r of these two paths in parallel:

$$r = \frac{1}{\frac{1}{9} + \frac{1}{\left(\frac{2R}{3}\right)}} \implies R = \frac{3}{2}\left(\frac{1}{\frac{1}{r} - \frac{1}{9}}\right) = \boxed{27\ \Omega}$$

A common mistake would be to neglect the second path and directly equate $\frac{2R}{3} = r$. In fact, such an error comes up frequently when building circuits if one is not careful.

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Problem 8: Distorted Stick (3 points)

A stationary observer on Earth sees a stick travelling at velocity v in the horizontal direction, with the stick angled at $\theta = 60°$ from the horizontal. However, a second observer who rides along with the stick views it to be oriented at $\phi = 30°$ from the same horizontal. Determine the value of v/c, where c is the speed of light.

Leave your answer to 3 significant figures.

Solution: Let the proper length of the stick be l. In the stick's frame, the stick spans a horizontal distance of $l\cos\phi$, and a vertical distance of $l\sin\phi$. From the Earth's frame, due to length contraction, the stick now only spans a horizontal distance of $\frac{l\cos\phi}{\gamma}$, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Since length contraction only applies in the direction of relative motion, considering that the velocity of the stick has no vertical component, the stick still spans a vertical distance of $l\sin\phi$ in the Earth's frame. We can thus relate these horizontal and vertical lengths to angle θ:

$$\tan\theta = \frac{(l\sin\phi)}{\left(\frac{l\cos\phi}{\gamma}\right)} = \gamma\tan\phi$$

Hence $\gamma = \frac{\tan\theta}{\tan\phi} = 3$. This gives $\frac{v}{c} = \sqrt{\frac{8}{9}} \approx \boxed{0.943}$.

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Problem 9: A Confident Batter (5 points)

There was an oversight for this question. Calculations would yield a negative value of h_1, implying the ball would "phase through the ground", which is physically impossible in the context of the question. Therefore, this question is voided.

Joey is a world-class baseball player, admired and adored by many. However, he has suffered a recent string of poor performances. Thus, he arranges a batting session to prove that he is, indeed, the best baseball player of all time. Joey stands at a horizontal distance of $d = 20.0$ m from the pitcher. The pitcher throws a ball of mass $m = 142$ g at a speed $u_0 = 30$ m s^{-1} horizontally towards Joey, from a height $h_0 = 1.7$ m above the ground. Joey, being a strong, well-trained batter, hits the ball back at an angle $\theta = 32°$ above the horizontal, with a force of $F = 5000$ N for contact duration $t = 0.002$ s.

At what horizontal distance D away from Joey does the ball land after contact with the bat? Ignore effects of air resistance, and any rotational motion of the ball.

Leave your answer to 3 significant figures in units of m.

Solution: Consider the x-axis to be along the horizontal direction, and the y-axis to be along the vertical direction. First, we need to find the velocities, u_x and u_y, as well as the vertical distance of the ball from the ground h_1 at the instant the ball makes contact with the bat. Taking upwards to be the $+y$ direction, and the direction of the ball's initial horizontal motion towards Joey to be the $-x$ direction, we obtain:

$$\begin{aligned} u_x &= -u_0 \\ u_y &= -\frac{gd}{u_0} \\ h_1 &= h_0 - \frac{gd^2}{2u_0^2} \end{aligned}$$

Now, we consider the impulse supplied by the bat during the collision to determine the velocities v_x, v_y of the ball immediately after collision. Accounting for the different components of the force F separately: $F_y = F \sin\theta$ and $F_x = F\cos\theta$. The components of the impulse caused by F would then be $J_y = F_y t$ and $J_x = F_x t$ respectively.

We can now calculate the the velocities of the ball after it has been hit, v_x and v_y:

$$v_x = \frac{J_x + mu_x}{m}$$

$$v_y = \frac{J_y + mu_y}{m}$$

Finally, using kinematics, we can derive D:

$$D = \frac{v_x\left(v_y + \sqrt{v_y^2 + 2gh_1}\right)}{g} \approx \boxed{186\ \mathrm{m}}$$

Setter: Shaun Quek, shaun.quek@sgphysicsleague.org

Problem 10: Straw Trick (4 points)

A vertical cylindrical straw of length $l = 15.0$ cm is partially immersed into a basin of water, causing the inside of the straw to be filled with water to depth $h = 10.0$ cm. Brian caps the top opening of the straw using his finger, and *slowly* raises the still-capped straw. By the time the whole straw has been lifted out of the basin of water, some of the original water column remains held within the straw. Calculate the percentage of water that is retained inside the straw. Assume that surface tension is negligible, and that the process is slow enough to be treated as quasistatic.

Leave your answer to 3 significant figures as a percentage. (For example, if you think the final answer should be 51.0%, *input your answer as* 51.0*)*

Solution: Denote atmospheric pressure as p_0, and cross-sectional area of the straw as A. Let the final depth of water in the straw be H, and the final pressure of the air pocket inside the straw be p.

For the water column to remain held inside the straw, an equilibrium balance of pressures yields the following condition:

$$p + \rho g H = p_0$$

Given that the straw was raised slowly, the temperature of air inside the straw must have remained approximately unchanged. Thus the product pV must have remained constant, which means that:

$$pA(l - H) = p_0 A(l - h) \implies p = p_0 \frac{l - h}{l - H}$$

Substituting this expression for p into the pressure balance equation, and solving for H:

$$H = \frac{p_0 + \rho g l - \sqrt{(p_0 + \rho g l)^2 - 4\rho g h p_0}}{2\rho g} \approx 9.95 \text{ cm}$$

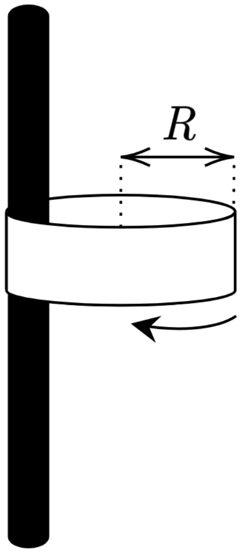
Hence, the percentage of water retained is $\frac{H}{h} \times 100\% = \boxed{99.5\%}$, which is a lot! This is a shocking reminder of how strong atmospheric pressure really is.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 11: Ring around the Rod (3 points)

A vertical rod is placed through a thin ring of radius $R = 5.0$ cm, with coefficient of static friction $\mu_s = 0.30$ between the rod and the ring. When the ring is spun at a sufficiently high frequency f_{min} around the rod, it will remain at the same height for some time before it falls. Determine the value of f_{min}. Assume that the axis of the ring always remains vertical, and that the radius of the rod is negligible.

Leave your answer to 2 significant figures in units of s^{-1}.



Solution: Normal force provides the centripetal force for the ring's rotation around the rod, so $N = mR\omega^2$. To stay at the same height, net vertical force on the ring must be zero, so there must be an upward static friction force $F = mg$. Since $F \leq \mu_s N$, solving for ω gives:

$$\omega \geq \sqrt{\frac{g}{\mu_s R}}$$

This means that the required ω_{min} for the ring to stay at the same height is given by $\omega_{\text{min}} = \sqrt{\frac{g}{\mu_s R}}$, and equivalently:

$$f_{\text{min}} = \frac{1}{2\pi}\sqrt{\frac{g}{\mu_s R}} \approx \boxed{4.1 \text{ s}^{-1}}$$

As an extension to the problem, you may want to consider what happens if the rod had non-negligible radius r.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 12: Hopeless Romance (3 points)

Joshua is stranded alone on an island. In hopes of being found, he uses his radio to transmit signals. His radio operates on voltage $V = 12$ V and draws current $I = 1.0$ A.

As distance makes the heart grow fonder, Isabelle is determined to find him. She attempts to set up a satellite dish, whose geometry is perfectly hemispherical with radius R. The satellite receiver can detect the source of signals that have a minimum strength $P_{\text{min}} = 24\ \mu$W.

Given that Joshua is located at a distance $x = 1600$ km away from Isabelle, what is the minimum radius R of the satellite dish that Isabelle must construct in order to locate him? Assume that Joshua's radio is 100% efficient in emitting signals, and behaves like a point wave source that transmits signals uniformly in all directions. You may neglect the Earth's curvature, and any obstructions in the path of the signals.

Leave your answer to 2 significant figures in units of m.

Solution: The power supplied to Joshua's radio is given by VI. Since the radio is a point source, the intensity L of the radio wave signals at Isabelle's position can be written as:

$$L = \frac{VI}{4\pi x^2}$$

The power picked up by Isabelle's receiver is given by:

$$P_{\text{received}} = L\left(\pi R^2\right) = \frac{VIR^2}{4x^2}$$

This is under the assumption that Isabelle positions her satellite dish at the optimal angle, perpendicular to the direction of the transmitted signals, which would give the smallest required radius. Note that the cross-sectional area used is πR^2 rather than $2\pi R^2$, despite the hemispherical geometry of the satellite dish. This is because, even though it is hemispherical, the area it encloses perpendicular to the waves is still only given by πR^2.

To trace the source of the signals, it is necessary for $P_{\text{received}} \geq P_{\text{min}}$:

$$\frac{VIR^2}{4x^2} \geq P_{\text{min}} \implies R \geq \sqrt{\frac{4x^2 P_{\text{min}}}{VI}} \approx \boxed{4500 \text{ m}}$$

Yup, this is pretty hopeless. Sorry Isabelle.

It is also important to note that our solution had implicitly assumed that $x >> R$ when calculating the power picked up by Isabelle's receiver. As a further exercise, you could try re-attempting the question without making this assumption, and see how your final answer changes.

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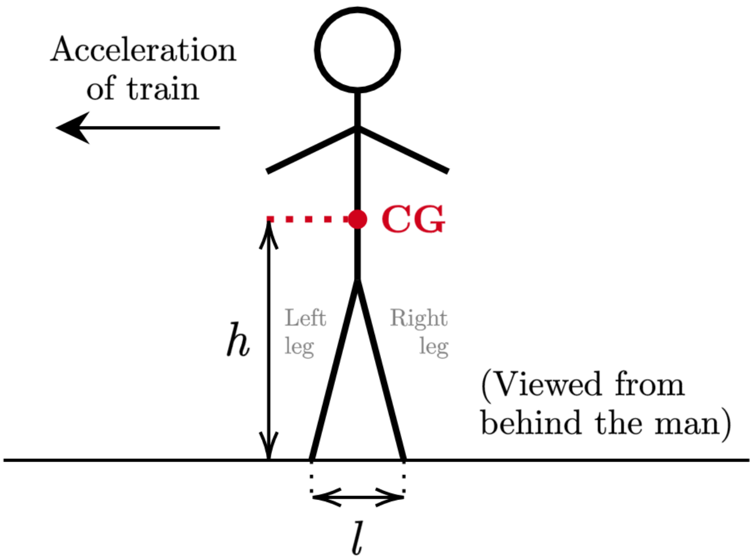
Problem 13: Accelerating MRT (4 points)

A man is in an MRT train that was originally at rest. He stands straight and faces the right side of the train, with his legs at horizontal distance $l = 0.40$ m apart from each other. The centre of gravity of the man is at vertical distance $h = 0.90$ m above the floor. The train starts travelling with uniform acceleration $a = 1.5$ m s^{-2} forward on a level track. Assuming that the man continues standing straight, what percentage of the man's total weight is transferred between his legs?

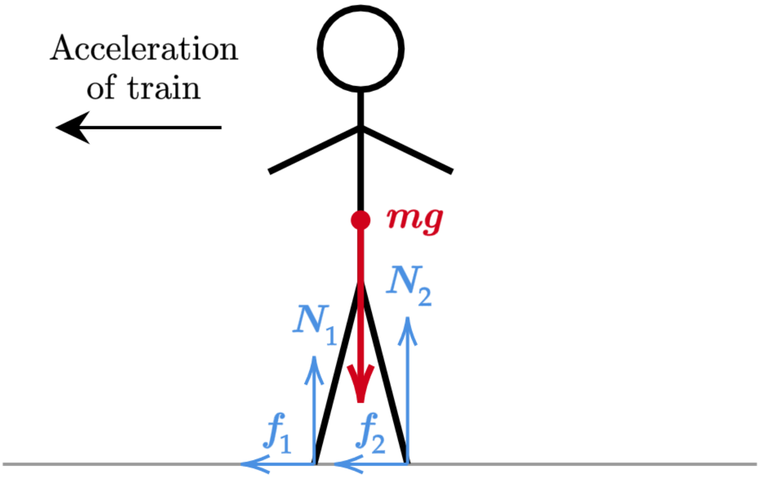
Leave your answer as positive if you think that weight is transferred from his left leg to his right leg.

Leave your answer as negative if you think instead that weight is transferred from his right leg to his left leg.

Leave your answer to 3 significant figures as a percentage. (For example, if you think the final answer should be 51.0%, *input your answer as* 51.0*)*



Solution: Consider the forces acting on the man, including two horizontal frictional forces f_1 and f_2, and two vertical normal forces N_1 and N_2 on the man's left and right legs respectively.



The frictional forces provide the horizontal acceleration for the man to travel along with the train, so:

$$f_1 + f_2 = ma$$

The man clearly does not have any vertical acceleration, so the two normal forces combined must balance the man's weight:

$$N_1 + N_2 = mg$$

The man must also be in rotational equilibrium, so the net torque around the man's centre of gravity is zero:

$$N_1 \frac{l}{2} + f_1 h + f_2 h = N_2 \frac{l}{2}$$

We have four unknowns f_1, f_2, N_1, N_2, and three equations. This means that we can't solve for all of the unknowns – in particular, variables f_1 and f_2 cannot be decoupled. However, we have enough information to solve for N_1 and N_2:

$$N_1 = \frac{mg}{2} - \frac{mah}{l}$$

$$N_2 = \frac{mg}{2} + \frac{mah}{l}$$

Since $N_2 > N_1$, "weight" was transferred from his left leg to his right leg, with magnitude $\frac{mah}{l}$. Qualitatively speaking, this is because with the two frictional forces pointing forward, N_1, f_1 and f_2 exert clockwise moments on the man. Since N_2 is the only anti-clockwise moment, its magnitude has to increase to counteract the combined clockwise moments of the other forces.

Expressing this transfer of weight as a percentage of his total weight:

$$\text{Weight transferred} = \frac{ah}{gl} \times 100\% \approx \boxed{+34.4\%}$$

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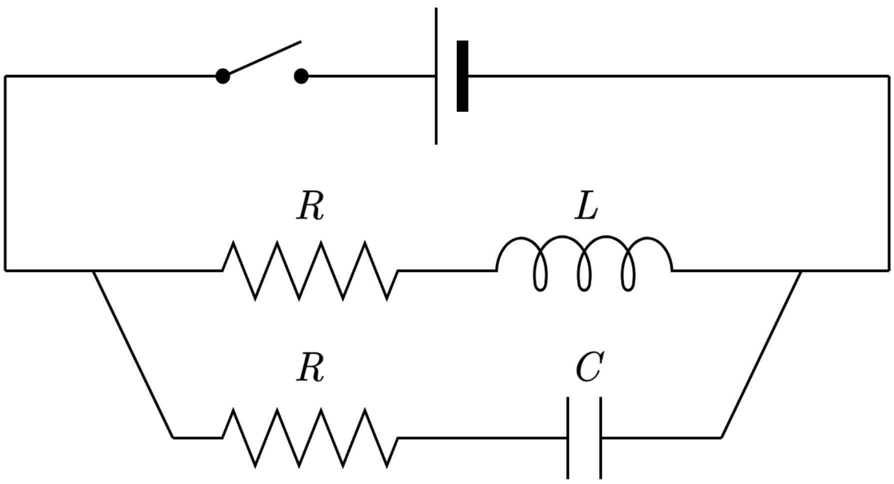
Problem 14: Current Stabiliser (4 points)

In a highly inductive circuit with inductance $L = 15.0$ H and resistance $R = 8.00\ \Omega$ connected to a battery, it takes some time after closing the switch for the current supplied by the battery to reach its maximum value. This may be avoided by connecting a much less inductive branch (which can be assumed to have zero inductance) with the same resistance $R = 8.00\ \Omega$ and an initially uncharged capacitor with capacitance C in parallel with the original circuit, as drawn below. This way, upon closing the switch, the current drawn from the battery may instantly reach its final value and stay there indefinitely. Determine the value of C required to make this possible.

Leave your answer to 3 significant figures in units of F.



Solution: We can treat the top and bottom branches of the circuit independently, as an RL circuit and RC circuit respectively that each connect to an emf E. Let the current flowing through the top branch be I_L, and the current flowing through the bottom branch be I_C. The standard results from RL and RC circuits give:

$$I_L = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$I_C = \frac{E}{R}e^{-\frac{t}{RC}}$$

The total current drawn from the battery I, by Kirchhoff's current law, is simply a sum of both currents:

$$I = I_L + I_C = \frac{E}{R}\left(1 + e^{-\frac{t}{RC}} - e^{-\frac{R}{L}t}\right)$$

For I to be independent of t, the two exponential terms must cancel for all t. This requires their exponents to be equal:

$$\frac{1}{RC} = \frac{R}{L} \implies C = \frac{L}{R^2} \approx \boxed{0.234\ \text{F}}$$

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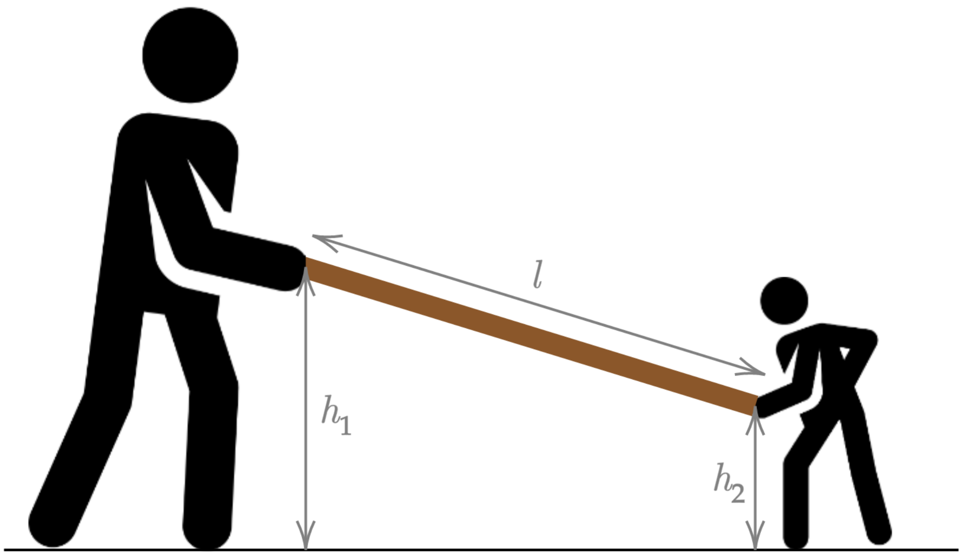
Problem 15: Tug of War (4 points)

Alice and Bob engage in a game of Tug of War. Alice has mass $m_1 = 20$ kg and pulls the rope at height $h_1 = 1$ m above the ground. Bob pulls the other end of the rope at a height of $h_2 = 0.2$ m above the ground. The rope has length $l = 2$ m and negligible mass. The coefficient of static friction between the feet of either person and the ground is $\mu = 0.5$.

Find the minimum mass of Bob, m_2, such that he can beat Alice.

Leave your answer to 3 significant figures in units of kg.

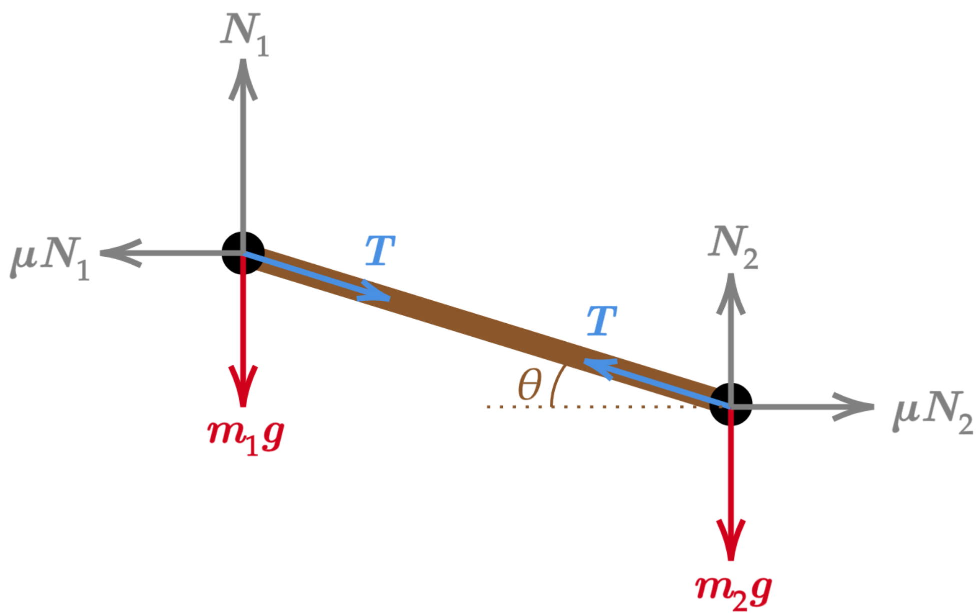


Solution: The game ends when either Alice or Bob starts to slip. Since the rope is light, we can assume that the tension at both ends of the rope have equal magnitude. The condition for Bob to beat Alice is that Bob is sufficiently heavy such that Alice slips before he does.

Let θ be the angle of the rope from the horizontal. We can determine θ in terms of h_1, h_2 and l using trigonometry:

$$\theta = \sin^{-1}\left(\frac{h_1 - h_2}{l}\right)$$

Let us find the tension in the rope when Alice is just about to slip. This occurs when the horizontal force exerted by the rope is equal to the maximum static friction between Alice and the ground.



Considering the free body diagram of Alice, we obtain the following equations by balancing forces in the horizontal and vertical directions respectively:

$$\mu N_1 = T\cos\theta \tag{1}$$

$$N_1 = m_1 g + T\sin\theta \tag{2}$$

By substituting Eq. (2) into Eq. (1), we can obtain an expression for T:

$$T = \frac{\mu m_1 g}{\cos\theta - \mu\sin\theta} \tag{3}$$

For Bob to beat Alice, the non-slip condition $\mu N_2 > T\cos\theta$ must be met for Bob. We refer to the free body diagram of Bob. By balancing forces on Bob in the vertical direction, we may express $N_2 = m_2 g - T\sin\theta$. Then, by substituting this expression for N_2 into the above inequality, we get the following:

$$\mu\left(m_2 g - T\sin\theta\right) > T\cos\theta$$

We can then isolate m_2 and plug in the expression for T that we found earlier in Eq. (3), to obtain:

$$m_2 > \frac{T\cos\theta + \mu T\sin\theta}{\mu g}$$
$$\implies m_2 > m_1\frac{\cos\theta + \mu\sin\theta}{\cos\theta - \mu\sin\theta} \approx \boxed{31.2\text{ kg}}$$

Thus, Bob must weigh a minimum of 31.2 kg. This should be expected, since Bob is shorter than Alice.

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Problem 16: Rolling Pennies

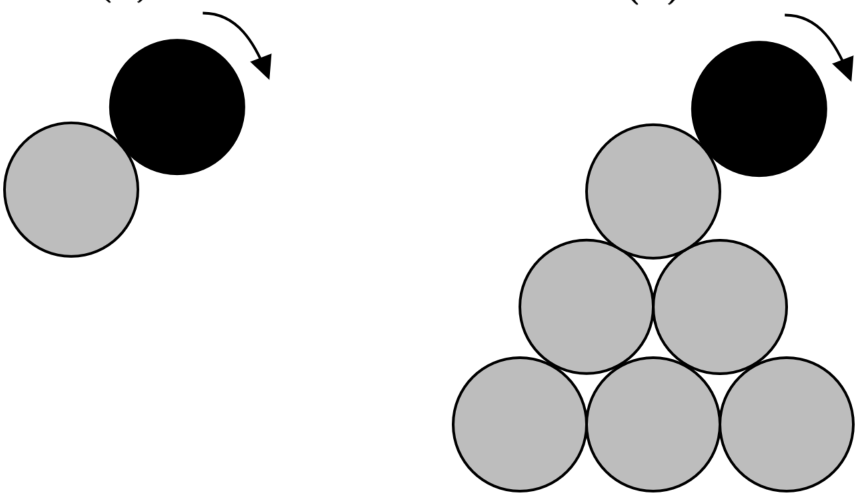
(a) One penny rolls without slipping around another identical penny which is held stationary. How many revolutions does the penny complete relative to you, when it returns to its starting position?

Leave your answer to 2 significant figures. (2 points)

(b) Six pennies are arranged in the triangle formation shown below and held stationary. A seventh penny rolls around the six pennies without slipping. How many revolutions will it complete, relative to you, when it returns to its starting position?

Leave your answer to 2 significant figures. (3 points)

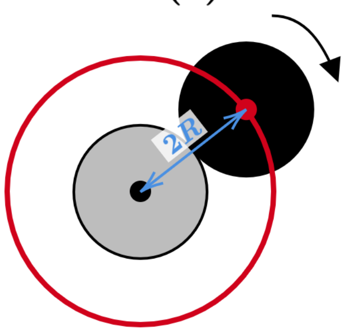
(a) **(b)**



Solution: The problem can be solved by considering the distance covered by the centre of the rolling penny. Denote the penny's radius as R. If the penny does not slip, each revolution causes its centre to travel distance $2\pi R$.

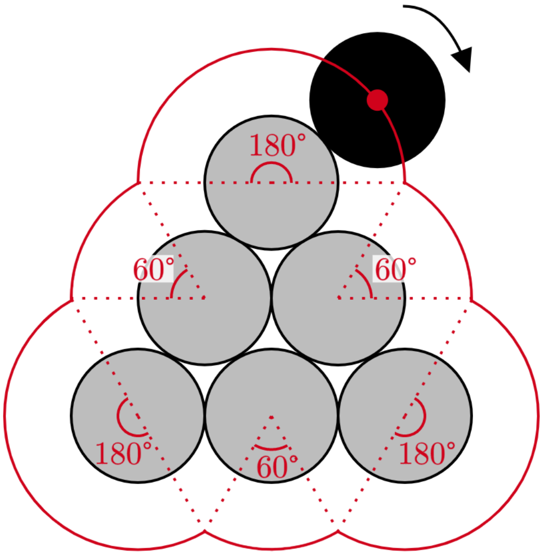
(a) The trajectory of the centre of the rolling penny evidently traces a circle of radius $2R$, as shown below. Thus, the distance covered by the penny's centre is $4\pi R$. The number of revolutions the penny covers is then given by $\frac{4\pi R}{2\pi R} = \boxed{2}$.

(a)



(b) The trajectory of the penny's centre is not as straightforward. It is composed of 3 arcs that each subtend 180°, and 3 arcs that each subtend 60° as drawn below. All of these arcs have radii $2R$.

(b)

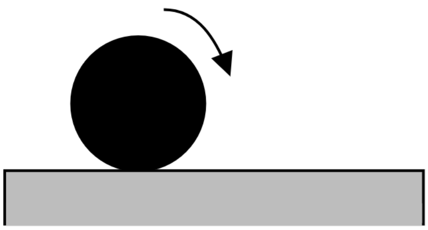


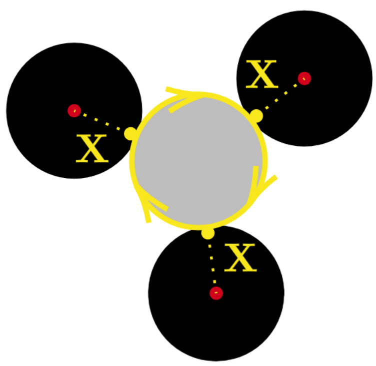
The total length of these arcs can be calculated to be $3\left[\pi(2R)\right]+3\left[\frac{\pi}{3}(2R)\right] = 8\pi R$. Consequently, the number of revolutions the penny rotates is $\frac{8\pi R}{2\pi R} = \boxed{4}$.

Alternative solution: It might be unintuitive that the answer for the number of revolutions traversed is dependent on whether the surface is flat or curved. For instance, in (a), if we unwind the stationary penny and roll the other penny along its straightened circumference without slipping, it would cover 1 revolution rather than 2. Here, we consider a more intuitive explanation for this effect.

(a) Suppose that coin A is the stationary penny, while coin B is the rolling penny. The rotation of the coin could be decoupled into two sources:

(i) **(ii)**





First, the coin is rolling without slipping. If you unwind the circumference of coin A into a straight line and roll coin B along it, then a chosen point on the circumference of coin B will complete a full revolution with respect to you.

Second, the coin is traversing a curved surface. Suppose you slide coin B along coin A without any rolling. That is, you mark out a point X on coin B and make sure that as you slide coin B, that point is always in contact with coin A. Notice that even though the coin is not rolling, coin B does appear to complete a full revolution with respect to you. To see this, imagine a line connecting point X to the centre of coin B, as coin B slides, this line makes one revolution.

Thus, the total number of revolutions can be obtained by summing the revolutions due to the 2 effects.

(b) Let us break down the rotation of the penny in the same way we approached the previous part.

First, let us straighten out the arcs of the circles that are contacted by the rolling penny, and combine them. This is the length that the penny has rolled across. Now, we consider the number of revolutions if we roll a penny across this straightened surface without slipping. The total distance traversed by the coin is hence $3(\pi R) + 3\left(\frac{\pi R}{3}\right) = 4\pi R$. Thus the number of revolutions is $\frac{4\pi R}{2\pi R} = 2$.

Second, we find the number of revolutions when we slide the coin without rolling around the arrangement of pennies. One may erroneously deduce that the number of revolutions due to this effect is 1, since the coin travels one round along a closed loop. However, this neglects that the point that the coin is rotating with respect to is changing (in other words, the "fixed point" on the moving coin changes from coin to coin). Hence, we will have to sum the revolutions due to each of the 6 arcs that the coin traverses. The number of revolutions is $\frac{3\times\pi+3\times\frac{\pi}{3}}{2\pi} = \frac{4\pi R}{2\pi R} = 2$.

The total number of revolutions is $2 + 2 = \boxed{4}$.

Of course, being an open-book competition, the fastest way to solve this would be to take a few coins and work it out by experimentation.

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Problem 17: Releasing a Spring

An ideal light spring with spring constant $k = 25$ N m^{-1} is placed on the ground with its axis vertical. The spring's top end is attached to a massless plate, while its bottom end is fixed onto the ground. A mass $m = 80$ g is placed on the plate and pressed down, compressing the spring by distance x compared to its unstretched state. The mass is then released from rest.

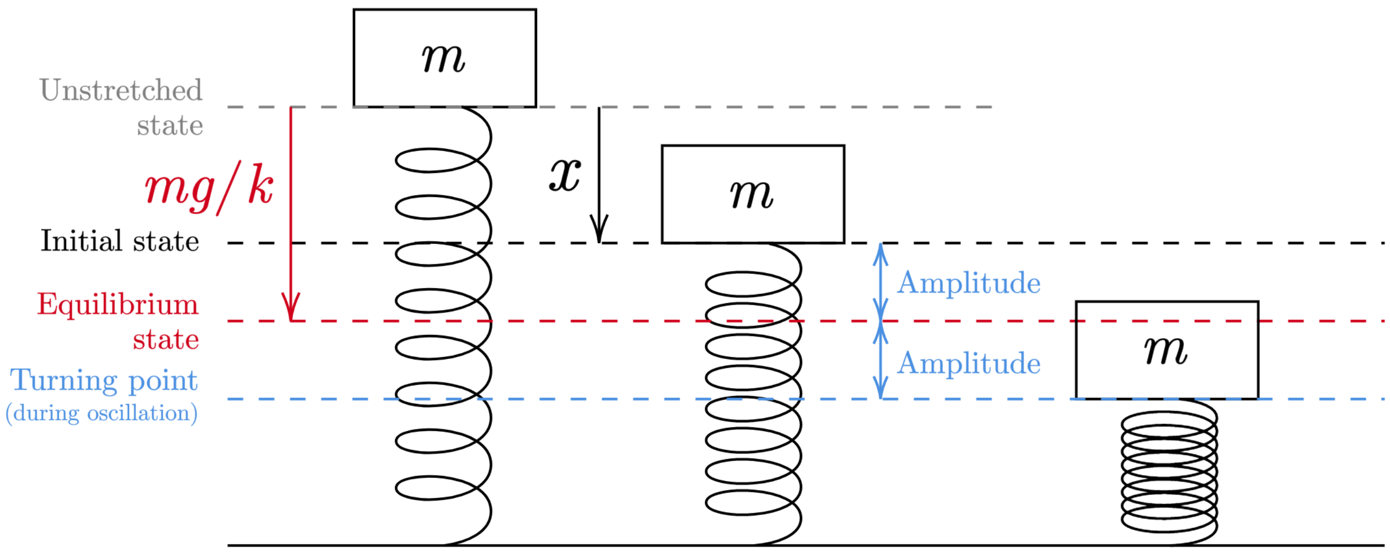
(a) If the relaxed (unstretched) length of the spring were $l = 10$ cm, find the *maximum* value of x such that the subsequent motion of the mass can be exactly described as simple harmonic motion.

Leave your answer to 2 significant figures in units of cm. (3 points)

(b) If the relaxed (unstretched) length of the spring were $l = 5$ cm, find the *minimum* value of x such that the subsequent motion of the mass can be exactly described as simple harmonic motion.

Leave your answer to 2 significant figures in units of cm. (3 points)

Solution: In the traditional case of a spring-mass system in the vertical plane, the system executes simple harmonic motion about its equilibrium position. The mass is at equilibrium when the upward spring force exactly cancels its downward weight, which occurs when the spring is compressed by $\frac{mg}{k}$. When the mass is released, the resulting amplitude of its motion is given by the distance between the mass' initial position and its equilibrium position, as shown in the diagram below.



However, in the context of the problem, there are a few requirements to be fulfilled for this mass-and-plate system to execute simple harmonic motion.

(a) One of these requirements is that the spring force must be upward at all times.

To prove this, suppose on the contrary that the spring force was downward. Then the plate would have an acceleration greater than g. However, since the mass is only placed on the plate and is not attached to it, the plate is unable to pull the mass down, so the acceleration of the mass can only take on a maximum value g. As a result, the mass loses contact with the plate and undergoes free-fall motion, rather than simple harmonic motion.

As such, for the spring force to be always upward, the spring can never be in a state of extension. This means that the plate can rise no higher than in its position when the spring was unstretched. Thus, the maximum allowable amplitude of oscillations of the mass is $\frac{mg}{k}$. The corresponding maximum x is where the mass starts out at $\frac{mg}{k}$ below its equilibrium position, which corresponds to a maximum $x = \frac{2mg}{k} \approx \boxed{6.3\text{ cm}}$.

(b) Another requirement is that the mass does not contact the ground at any point in its motion. If it does, the external normal force exerted by the ground on the mass will result in the motion no longer being simple harmonic.

Since the equilibrium position is at vertical distance $l - \frac{mg}{k}$ from the ground, the maximum amplitude of oscillations such that the mass does not hit the ground is also $l - \frac{mg}{k}$. The amplitude of oscillations, expressed in terms of x, as $\frac{mg}{k} - x$. Hence the condition is:

$$\frac{mg}{k} - x < l - \frac{mg}{k} \implies x > \frac{2mg}{k} - l \approx \boxed{1.3\text{ cm}}$$

Alternative solution: A more mathematically involved method would explicitly find the position of the mass as a function of time $y(t)$, from which the conditions for simple harmonic motion can be derived.

In this model, we assume that the mass indeed undergoes simple harmonic motion. Let our coordinate system set the floor as $y = 0$ and take the upward direction as positive. The equilibrium position of the mass is at $y = l - \frac{mg}{k}$, while the initial position of the mass is at $y = l - x$. Thus the initial displacement of the mass relative to the equilibrium position is $(l - x) - (l - \frac{mg}{k}) = \frac{mg}{k} - x$; this is also the amplitude of motion. The position of the mass over time can therefore be written as:

$$y(t) = \left(\frac{mg}{k} - x\right)\cos\omega t + \left(l - \frac{mg}{k}\right)$$

Its acceleration can be found by differentiating $y(t)$ twice:

$$y''(t) = \left(\frac{kx}{m} - g\right)\cos\omega t$$

(a) The system in question is special since the mass is not directly attached to the plate, but is merely placed on top of the plate. This means that the spring force can only push the mass up, but cannot pull it down. As such, the downward acceleration of the mass can only be at most g, otherwise it will lose contact with the plate, and the motion is no longer simple harmonic. Thus we have a constraint for the minimum value of $y''(t) > -g$.

The minimum value of $y''(t)$ is $g - \frac{kx}{m}$ (when $\cos \omega t = -1$). Imposing the constraint, we deduce that $x < \frac{2mg}{k} \approx \boxed{6.3 \text{ cm}}$. This is the maximum x for which the motion is simple harmonic.

(b) There is an additional constraint that the mass does not contact the floor. If it does, the external normal force exerted by the floor on the mass will result in the motion no longer being simple harmonic. This means that the minimum value of $y(t) > 0$. Since the minimum value of $y(t) = l + x - \frac{2mg}{k}$, we can thus conclude that $x > \frac{2mg}{k} - l \approx \boxed{1.3 \text{ cm}}$.

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Problem 18: Gravitational Blueshift (4 points)

Photons have an effective gravitational mass that is always equal to their momentum divided by their speed. Suppose a photon with some initial frequency f is emitted from a satellite orbiting $x = 20000$ km above the surface of the Earth. When it reaches the surface of the Earth, its new frequency is f'. By treating the photon as a massive particle with mass equal to its effective gravitational mass, find the natural logarithm of fractional change in frequency of the photon, $\ln \frac{f'-f}{f}$. Take the radius of the Earth to be $R = 6370$ km, and the mass of the Earth to be $M = 5.97 \times 10^{24}$ kg. You may assume the photon travels at constant speed c.

Leave your answer to 3 significant figures.

Solution: The effective gravitational mass of a photon with frequency ν is $\frac{p}{c} = \frac{h}{\lambda c} = \frac{h\nu}{c^2}$.

Let the initial and final effective gravitational mass of the photon be m and m' respectively. The gravitational potential energy of the photon at the satellite is $-\frac{GMm}{R+x}$, while the gravitational potential energy of the photon at the Earth's surface is $-\frac{GMm'}{R}$.

Let the frequency of the photon at the satellite and at the surface of the Earth be f and f' respectively. Then, by conservation of energy, we have:

$$hf - \frac{GMm}{R+x} = hf' - \frac{GMm'}{R}.$$

Substituting in the expressions for m and m' and simplifying, we obtain:

$$\begin{aligned} f\left(1 - \frac{GM}{c^2}\frac{1}{R+x}\right) &= f'\left(1 - \frac{GM}{c^2}\frac{1}{R}\right) \\ \implies f' &= \left(\frac{1 - \frac{GM}{c^2}\frac{1}{R+x}}{1 - \frac{GM}{c^2}\frac{1}{R}}\right) f \end{aligned}$$

We rewrite the final result as $f' = \alpha f$, where $\alpha = \frac{1-\frac{GM}{c^2}\frac{1}{R+x}}{1-\frac{GM}{c^2}\frac{1}{R}}$. The fractional change in frequency is thus given by $\frac{f'-f}{f} = \frac{(\alpha-1)f}{f} = \alpha - 1$. Its natural logarithm $\ln \frac{f'-f}{f} = \ln(\alpha - 1) \approx \boxed{-21.4}$.

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Problem 19: Eddy Current Braking (3 points)

A long, thin, hollow, conducting cylinder of radius a and length $l \gg a$ is rotated about its axis at angular velocity ω in a uniform magnetic field B normal to its axis. This induces the formation of eddy currents on the cylinder and results in a retarding torque. If the cylinder has sheet resistance s and permeability μ, then the torque τ can be written in the form:

$$\tau = \frac{4\pi\omega s a^n B^2 l}{4s^2 + \omega^2\mu^2 a^2}$$

Find the constant n.

Leave your answer to 3 significant figures.

Solution: Let T denote dimensions of time, L denote dimensions of length, M denote dimensions of mass and I denote dimensions of current. We first write down the dimensions of all relevant quantities:

$$[\tau] = ML^2T^{-2}, [\omega] = T^{-1}, [a] = [l] = L, [B] = MT^{-2}I^{-1}, [\mu] = MLT^{-2}I^{-2}$$

Sheet resistance has the same dimensions as resistance, although its physical meaning is slightly different. Even without knowing this, we can still deduce its dimensions from the expression $4s^2 + \omega^2\mu^2a^2$ in the denominator, since the fact that the two terms can be added together implies that both terms are of the same dimension. Thus, $[s^2] = [\omega^2\mu^2a^2] \implies [s] = [\omega\mu a] = T^{-1} \times MLT^{-2}I^{-2} \times L = ML^2T^{-3}I^{-2}$.

Thus:

$$[\tau] = \left[\frac{4\pi\omega s a^n B^2 l}{4s^2 + \omega^2\mu^2 a^2}\right]$$

$$\implies ML^2T^{-2} = \frac{T^{-1} \times L^n \times ML^2T^{-3}I^{-2} \times (MT^{-2}I^{-1})^2 \times L}{(ML^2T^{-3}I^{-2})^2} = ML^{n-1}T^{-2}$$

Therefore, $n - 1 = 2 \implies n = \boxed{3.00}$.

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Problem 20: Heating a Blackbody (4 points)

A heat pump of ideal efficiency consumes power $W = 90.7$ kW from an external supply and draws from a reservoir held at constant temperature $T_0 = 200$ K. It is used to heat a blackbody that simultaneously loses heat through radiation. Given that the total surface area of the blackbody over which radiation occurs is $A = 125\ \mathrm{m}^2$, find the equilibrium temperature T of the blackbody.

Leave your answer to 3 significant figures in units of K.

Solution: Let Q be the rate at which the pump delivers heat energy to the blackbody. The efficiency $\eta = W/Q$ of the heat pump is limited by the first and second laws of thermodynamics (or Carnot's theorem) to $\eta_{\max} = 1 - \frac{T_0}{T}$. Since the pump operates at this ideal efficiency, it will pump heat energy into the body at a rate:

$$Q = \frac{W}{\eta} = \frac{W}{1 - \frac{T_0}{T}}$$

At the same time, the blackbody radiates heat energy at a rate $\sigma A T^4$, in accordance to the Stefan–Boltzmann law. At equilibrium, these two rates are equal:

$$\frac{W}{1 - \frac{T_0}{T}} = \sigma A T^4$$

If we set $x = \frac{T}{T_0}$ and rearrange this equation, we arrive at:

$$x^4 - x^3 = \frac{W}{\sigma A T_0^4}$$

Since $\frac{W}{\sigma A T_0^4} = 7.998... \approx 8$ to an accuracy of better than $0.002/8 = 2.5 \times 10^{-4}$, we merely need to solve the equation $x^4 - x^3 - 8 = 0$. This polynomial can be factorized as $(x-2)(x^3 + x^2 + 2x + 4) = 0$, and thus we see that the solution is simply $x = 2$, or $T = \boxed{400\text{ K}}$, and we can be confident that this answer is accurate to 3 significant figures.

Alternatively, a direct numerical solution yields $x = 1.9999118...$, which still gives the same answer at the desired accuracy.

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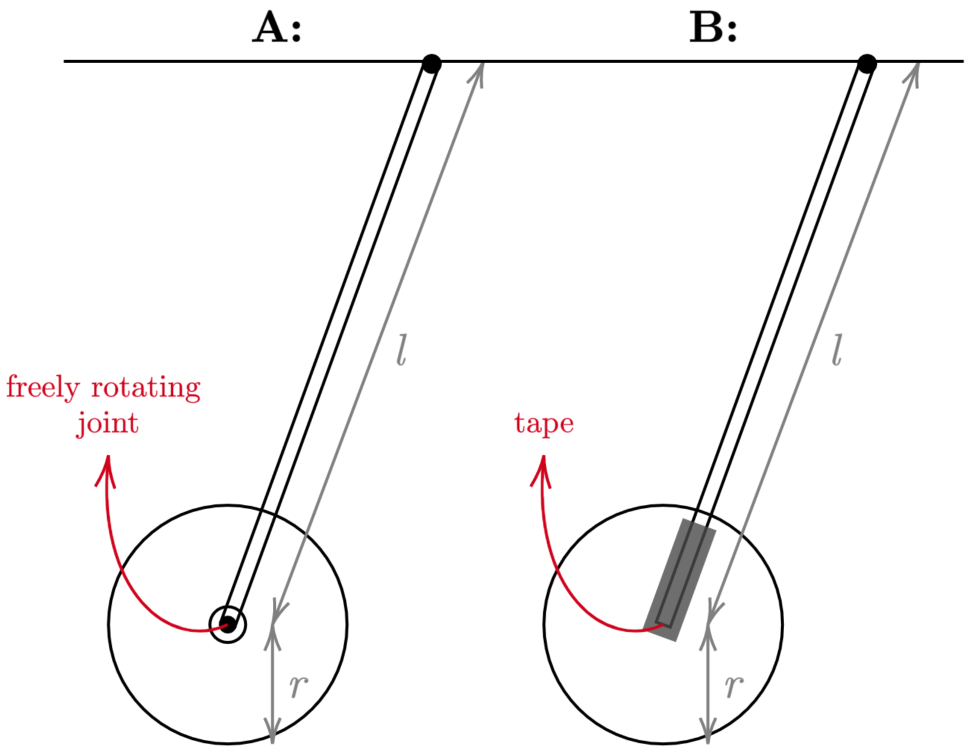
Problem 21: Disc Pendulums (4 points)

Two pendulums, A and B are pivoted from the ceiling. Each pendulum is made from a flat disc of mass $m = 1.0$ kg and radius $R = 0.2$ m which is joined at its centre to a massless rod of length $l = 0.5$ m. The oscillations occur in the plane of the disc.

In pendulum A, the rod only contacts the disc at a single point at the disc's centre, via a joint that can rotate without friction. On the other hand, in pendulum B, the rod is taped securely onto the disc with many points of contact.

Let the periods of pendulum A and B be T_A and T_B respectively. For oscillations of small angles, find the ratio T_A/T_B. Assume that there are no dissipative forces in the system.

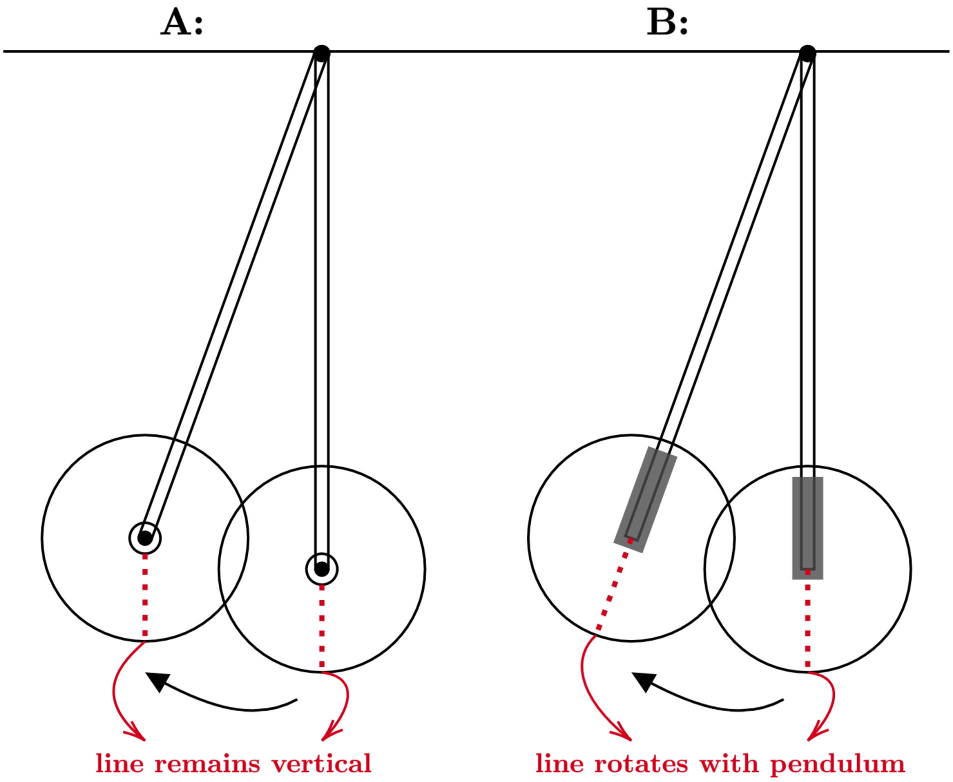
Leave your answer to 3 significant figures.



Solution: In order to find the period of the pendulums, we first need to find their moments of inertia about their pivots. While the two pendulums may appear similar, their moments of inertia differ.

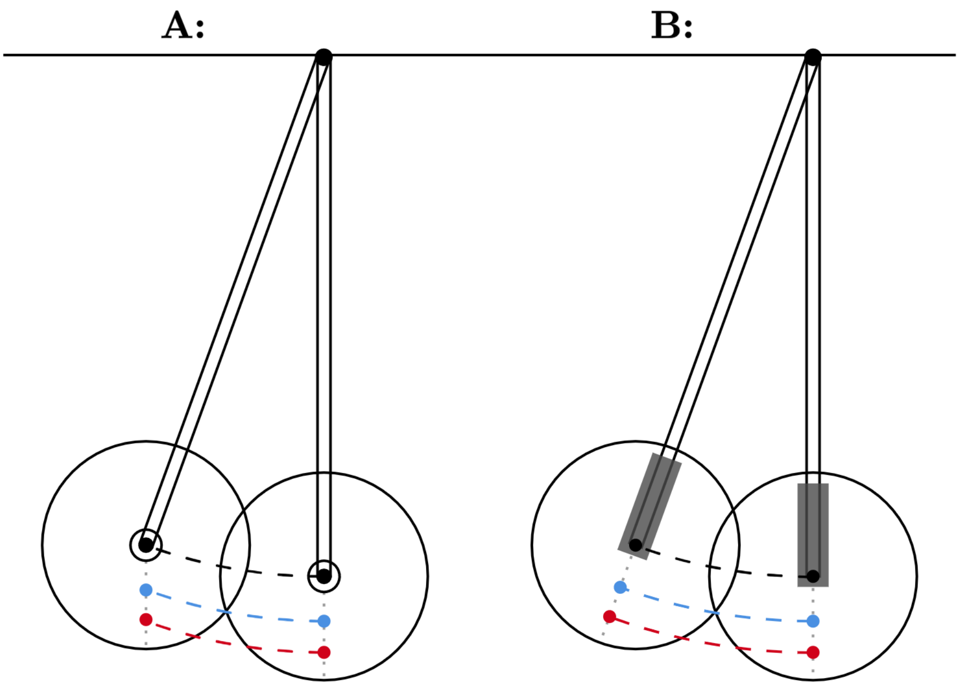
This can be explained through the following visualisation: mark out a vertical line on each disc, and observe what happens to the line as the pendulums oscillate. In pendulum A, since the rod can only exert a force at the centre of the disc, the disc does not experience any torque about its axis, so it cannot rotate. We would thus expect the line on disc A to remain in the vertical position at all times.

On the other hand, the line drawn on disc B will rotate such that it is always parallel to the rod, since the tape makes the entire pendulum rigid. As such, disc B rotates about its own axis, causing the line to rotate.



With that in mind, let's calculate the moments of inertia for each pendulum about their respective pivot points at the ceiling.

The moment of inertia of pendulum A, I_A, will surprisingly, remain unchanged if we substitute disc A for a point mass of the same mass at its centre. To explain this, if we consider the path traced out by each infinitesimal mass element of disc A as it oscillates, they are identical to the path traced out by the point mass. In contrast, each infinitesimal mass element in disc B traces a different arc, as illustrated in the following diagram.



Hence, I_A is given by that of a point mass:

$$I_A = ml^2$$

whereas I_B can be found via the parallel axis theorem:

$$\begin{aligned} I_B &= I_{\text{disc about CM}} + ml^2 \\ &= \frac{1}{2}mR^2 + ml^2 \end{aligned}$$

The period of a physical pendulum is given by the expression $T = 2\pi\sqrt{\frac{I}{mgl}}$. Thus, we can compute the ratio of periods:

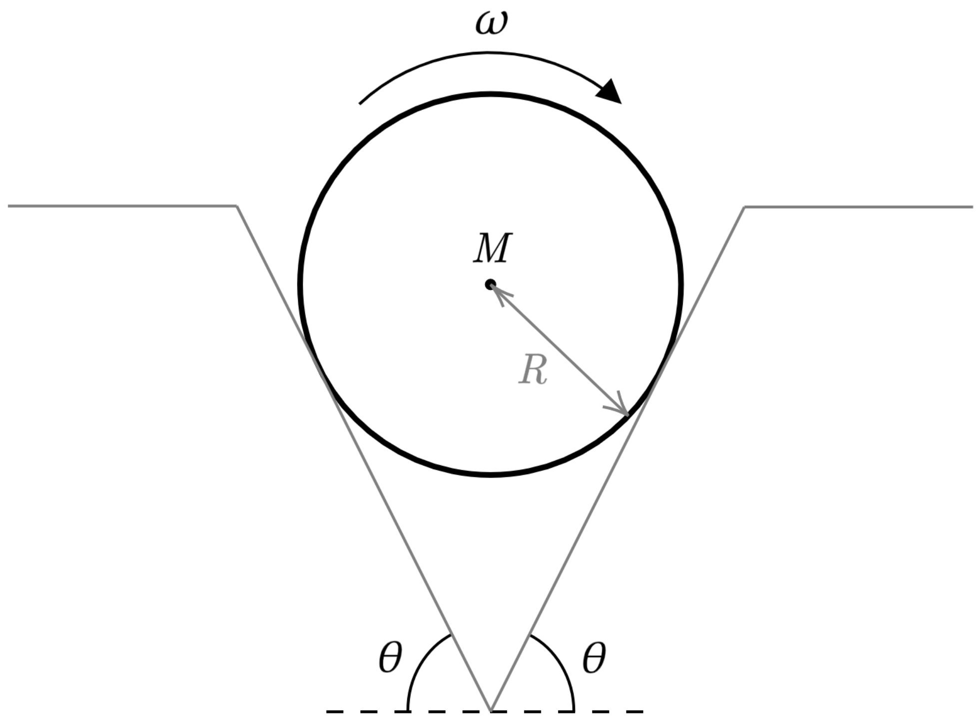
$$\begin{aligned} \frac{T_A}{T_B} &= \sqrt{\frac{I_A}{I_B}} \\ &= \sqrt{\frac{l^2}{\frac{1}{2}R^2 + l^2}} \\ &\approx \boxed{0.962} \end{aligned}$$

Setter: Luo Zeyuan, zeyuan.luo@sgphysicsleague.org

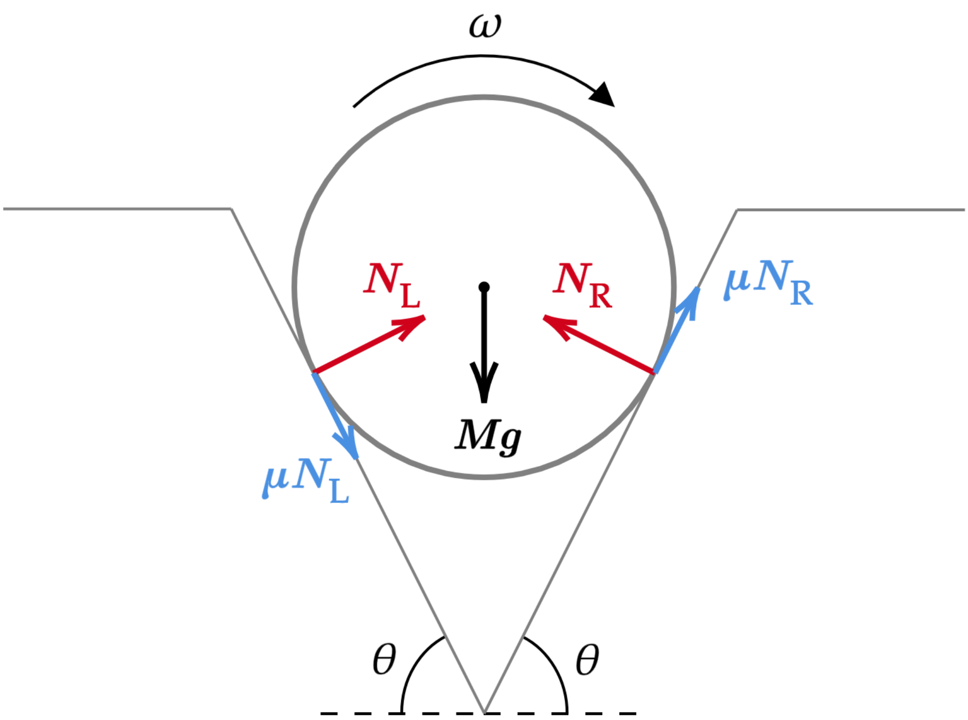
Problem 22: Cylinder in Groove (4 points)

A cylinder is placed in a V-shaped groove, with its axis parallel to the groove. Each side of the groove makes an angle of $\theta = 30°$ with the horizontal. The cylinder has mass $M = 500$ g and radius $R = 25$ cm, and is driven to rotate at a constant angular velocity $\omega = 30$ rad s^{-1}. The coefficient of friction between the cylinder and each surface is $\mu = 0.35$. Calculate the external torque τ required to rotate the cylinder.

Leave your answer to 2 significant figures in units of N m.



Solution: The free body diagram of the set-up is shown below, with N_L and N_R being the normal forces acting on the left and right side respectively.



Applying Newton's Second Law in the vertical and horizontal directions respectively, we obtain:

$$Mg + \mu N_L \sin\theta = \mu N_R \sin\theta + N_L \cos\theta + N_R \cos\theta$$

$$N_L \sin\theta + \mu N_L \cos\theta + \mu N_R \cos\theta = N_R \sin\theta$$

Letting $A = N_L + N_R$, $B = N_R - N_L$, we may simplify calculations drastically:

$$Mg - \frac{1}{2}\mu B = \frac{\sqrt{3}}{2}A$$

$$\frac{1}{2}B = \frac{\sqrt{3}}{2}\mu A$$

Solving the simultaneous equations, we obtain $A = \frac{2Mg}{\sqrt{3}(\mu^2+1)}$.

Since the external torque supplied must balance the total torque due to friction, we have:

$$\tau = \mu N_L R + \mu N_R R = \mu A R = \frac{2\mu MgR}{\sqrt{3}(\mu^2+1)} \approx \boxed{0.44 \text{ N m}}$$

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Problem 23: Strange Release

A light inextensible string of length $l = 5$ cm has its top end fixed at point P and its bottom end tied to a mass. Treat air resistance to be negligible.

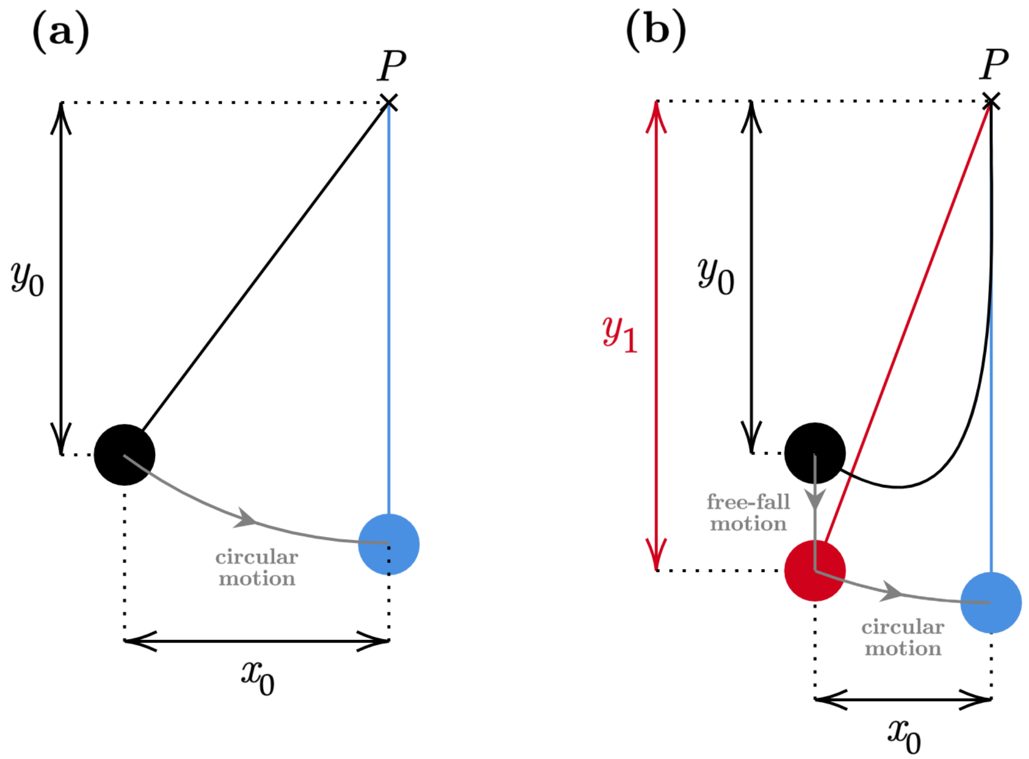
(a) The mass is held at horizontal distance $x_0 = 3$ cm from P, and vertical distance $y_0 = 4$ cm below P. It is then released from rest. What is the velocity of the mass at the later instant when the string is vertical?

Leave your answer to 2 significant figures in units of cm s^{-1}. (2 points)

(b) Solve again for $x_0 = 2$ cm and $y_0 = 4$ cm.

Leave your answer to 2 significant figures in units of cm s^{-1}. (4 points)

Solution: The distinction between (a) and (b) is that in (a), the string started out taut, since $\sqrt{x_0^2 + y_0^2} = 5 \text{ cm} = l$; whereas in (b), the string was slack, as $\sqrt{x_0^2 + y_0^2} \approx 4.5 \text{ cm} < 5 \text{ cm}$.



(a) Since the string is always taut in this case, the mass undergoes circular motion around point P. As such, the string tension is always perpendicular to the mass' velocity, so there is no energy dissipation due to the string. Hence, energy is conserved:

$$mg(l - y_0) = \frac{1}{2}mv^2$$

$$\implies v = \sqrt{2g(l - y_0)} \approx \boxed{44 \text{ cm s}^{-1}}$$

(b) The string starts out slack, meaning that it does not exert any force on the mass initially. As such, the mass is initially in free-fall, falling straight downwards. This goes on until the mass reaches a critical vertical distance y_1 below point P, where the string becomes taut again. This critical distance is given geometrically by:

$$y_1 = \sqrt{l^2 - x_0^2}$$

Consider the motion of the mass before it reaches y_1. Since the mass is in free-fall, we can determine its velocity v_0 just before it reaches y_1:

$$v_0 = \sqrt{2g(y_1 - y_0)}$$

Upon reaching y_1, the string becomes taut. At this point, the mass is still travelling straight downwards at v_0. This has some radially outward component (i.e. away from P).

As the string is inextensible, its length must be fixed at l. In order to keep its length fixed, the string exerts a brief impulse on the mass directed radially inwards (i.e. towards P) which instantly reduces the mass' radial velocity to zero. In other words, only the tangential velocity of the mass is retained. Let this tangential velocity be v_1. Geometrically, v_1 can be expressed:

$$v_1 = v_0 \left(\frac{x_0}{l}\right)$$

Note that this process is dissipative, with the tension doing negative work on the mass. The kinetic energy of the mass is now given by:

$$\frac{1}{2}mv_1^2 = mg(y_1 - y_0)\frac{x_0^2}{l^2}$$

From here on out, energy is conserved since the tension is always perpendicular to the mass' velocity. Let v_2 be the velocity of the mass when the string is vertical. Conserving energy, we obtain:

$$\begin{aligned}
\frac{1}{2}mv_2^2 &= \frac{1}{2}mv_1^2 + mg(l - y_1) \\
v_2^2 &= 2g\left[l - \frac{x_0^2}{l^2}y_0 - \left(1 - \frac{x_0^2}{l^2}\right)\sqrt{l^2 - x_0^2}\right] \\
\implies v_2 &\approx \boxed{32 \text{ cm s}^{-1}}
\end{aligned}$$

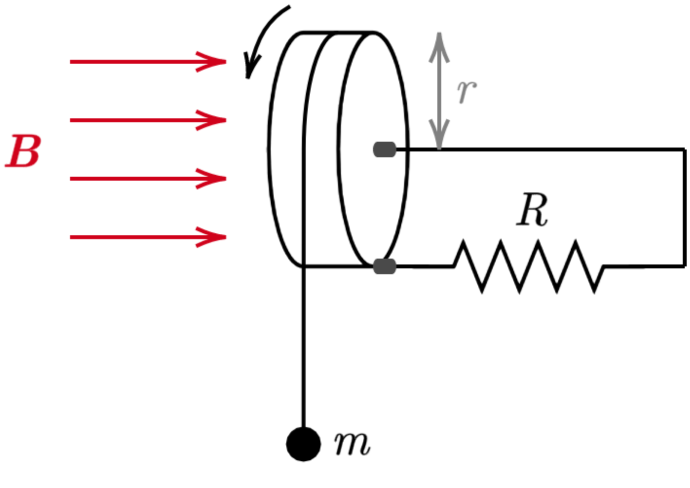
Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 24: Faraday's Wheel (4 points)

A conducting wheel of radius $r = 0.50$ m is mounted on a horizontal axle, and connected in series with a resistor of resistance $R = 1.2\ \Omega$ as shown in Figure 1. The resistor is connected via stationary frictionless brush contacts at its ends. One end touches the conducting central axis, while the other end touches the edge of the wheel. A uniform magnetic field of $B = 1.8$ T is directed along the rotational axis of the wheel.

A thin insulating string is wound around the circumference of the wheel, from which a weight of mass $m = 5.0$ kg is hung. Upon release, the mass begins to fall and eventually reaches a constant velocity. Find P, the power dissipated by the resistor in this steady state. You can assume that the string does not slip in this process.

Leave your answer to 2 significant figures in units of W.



Solution: The situation can be analysed qualitatively from an energy perspective. As the mass falls, it pulls on the string, which in turn causes the wheel to spin. According to Faraday's Law, this generates an induced emf across the wheel, leading to a current through the circuit. As the wheel spins faster, the induced emf and induced current flowing through the resistor increases. Thus, the power dissipated by the resistor will increase. At the steady state, the power dissipated by the resistor will equal the rate of change of the gravitational potential energy of the mass, so all the GPE is dissipated as heat, and the KE of the mass, and thus its velocity, no longer changes.

Now, we quantify our solution at the steady state. Let the angular velocity of the disc at the steady state be ω. Consider a line on the conducting disc drawn from its centre to its edge. During the time period Δt, the area of the sector swept out by this line is $\frac{\omega r^2}{2}\Delta t$. Thus the total magnetic flux swept out by this line is $\frac{B\omega r^2}{2}\Delta t$.

According to Faraday's Law, the magnitude of induced emf generated across the conducting wheel is directly proportional to the rate of change of magnetic flux swept by the line. Thus $|\varepsilon| = \frac{B\omega r^2}{2}$, where ε is the induced emf across the wheel. The power dissipated by the resistor is $\frac{|\varepsilon|^2}{R}$.

At the steady state, the power delivered to the resistor is equal to the rate of GPE loss by the mass:

$$\frac{|\varepsilon|^2}{R} = mgv$$

Expanding the above expression and applying non slip condition $v = \omega r$, we get:

$$\frac{B^2\omega^2 r^4}{4R} = mgr\omega \implies \omega = \frac{4mgR}{B^2 r^3}$$

With our expression for ω, we may determine P:

$$P = mg\omega r = \frac{4m^2g^2R}{B^2r^2} \approx \boxed{14000\text{ W}}$$

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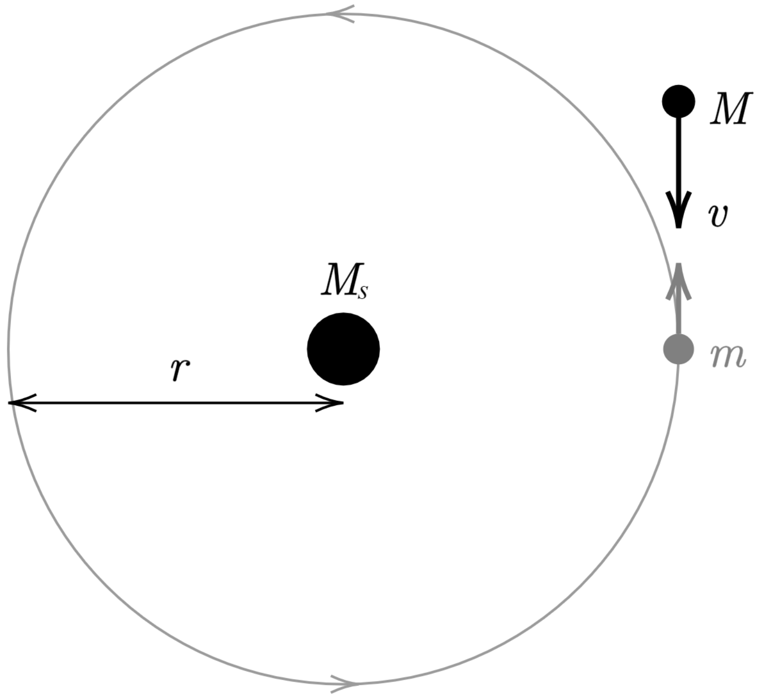
Problem 25: Colliding Asteroids (4 points)

An asteroid of mass $m = 3.35 \times 10^5$ kg is initially in a circular orbit of radius $r = 1.60 \times 10^{12}$ m about the sun. Another asteroid of mass M travelling in the opposite direction strikes it with velocity $v = 1.50 \times 10^5$ m s^{-1}. Both asteroids subsequently stick together to form a new body. Given that the sun's mass is $M_s = 1.99 \times 10^{30}$ kg, what is the minimum mass M required for the resulting body to enter an unbound orbit?

Leave your answer as 0 if you think the resulting body will always be in a bound orbit for all M.

Leave your answer to 2 significant figures in units of kg.



Solution: Let the initial speed at which m travels be u. Since m is in a circular orbit, the gravitational force must provide its centripetal force:

$$\frac{GM_s m}{r^2} = \frac{mu^2}{r} \implies u = \sqrt{\frac{GM_s}{r}}$$

The total momentum in the direction of M's motion is $p = Mv - mu$. Since M and m stick together after the collision, the combined body has mass $M + m$. Its post-collision velocity v' is therefore:

$$v' = \frac{Mv - mu}{M + m}$$

For this combined body to escape a bound orbit, its kinetic energy must exceed the difference in gravitational potential energy between its current position and that of a very distant position from the sun:

$$\frac{1}{2}(m+M)v'^2 \geq \frac{GM_s(M+m)}{r}$$

Substituting the expression for v' and rearranging slightly gives:

$$v'^2 = \left(\frac{Mv - mu}{M+m}\right)^2 \geq \frac{2GM_s}{r} = 2u^2$$

We can take square roots of both sides and divide the numerator and denominator of the left hand side by m to obtain:

$$\frac{\left(\frac{M}{m}\right)v - u}{1 + \left(\frac{M}{m}\right)} \geq \sqrt{2}u \implies \frac{M}{m} \geq \frac{1+\sqrt{2}}{\frac{v}{u} - \sqrt{2}}$$

Substituting in the expression for u and all given values gives us the final answer:

$$M \geq \left(\frac{1+\sqrt{2}}{v\sqrt{\frac{r}{GM_s}} - \sqrt{2}}\right)m = \boxed{5.4 \times 10^4 \text{ kg}}$$

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Problem 26: Thermodynamic Board

A mass $m = 0.100$ kg sits on a very long board of mass $M = 0.500$ kg, which rests on horizontal frictionless ground. The mass and the board start out at the same temperature $T_i = 20.0°\text{C}$, and share the same specific heat capacity $c = 400$ J kg^{-1} K^{-1}. Kinetic friction exists between the mass and the board, with coefficient $\mu = 0.800$. Now, the mass is imparted a small horizontal velocity u. Neglect any energy transfer to the surroundings (including the ground), and assume that the mass does not rotate.

(a) The final change in temperature of the mass can be written as αu^2, where α is a numerical constant. Calculate the value of α.

Leave your answer to 2 significant figures in SI units. (3 points)

If you were unable to solve (a), you may use $\alpha = 1$ for (b). However, the maximum attainable score for (b) will be reduced by 1 point.

(b) The final change in entropy of the universe can be approximated as βu^2, where β is a numerical constant. Calculate the value of β.

Leave your answer to 2 significant figures in SI units. (3 points)

Solution: Throughout the solution, the value of friction coefficient μ is irrelevant.

(a) Consider the combined system of the mass and the board. There are no external forces on the system, so momentum of the system is conserved. Due to friction between the mass and the board, they will move at a shared final velocity v. Conservation of momentum allows us to find this v:

$$mu = (M+m)v \implies v = \frac{m}{M+m}u$$

Also consider the energy transfers in the system. Friction causes a conversion of some of the mechanical energy of the mass and the board into thermal energy. Let us determine the change in kinetic energies of the mass ΔKE_m and the board ΔKE_M:

$$\begin{aligned}\Delta\text{KE}_m &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = -\frac{Mm}{(M+m)^2}u^2\left(\frac{M}{2}+m\right)\\ \Delta\text{KE}_M &= \frac{1}{2}Mv^2 \qquad\qquad\;\; = \frac{Mm}{(M+m)^2}u^2\left(\frac{m}{2}\right)\end{aligned}$$

The total change in mechanical energy ΔE can be found by summing up these two changes in kinetic energies, which gives:

$$\Delta E = \Delta\text{KE}_m + \Delta\text{KE}_M = -\frac{1}{2}\frac{Mm}{M+m}u^2$$

This loss in mechanical energy must have gone into the rise in thermal energy, thus $\Delta E = (M+m)c\Delta T$, where ΔT is the change in temperature. Solving for ΔT:

$$\Delta T = \frac{Mm}{2c(M+m)^2}u^2$$

Thus, $\alpha = \frac{Mm}{2c(M+m)^2} \approx \boxed{0.00017}$.

(b) Let us now consider the change in entropy of the mass ΔS_m and the board ΔS_M.

$$\Delta S_m = \int_{T_i}^{T_i+\Delta T} \frac{mc\,\mathrm{d}T}{T} = mc\ln\left(1+\frac{\Delta T}{T_i}\right)$$
$$\Delta S_M = \int_{T_i}^{T_i+\Delta T} \frac{Mc\,\mathrm{d}T}{T} = Mc\ln\left(1+\frac{\Delta T}{T_i}\right)$$

This totals up to a change in entropy ΔS of the system:

$$\Delta S = \Delta S_m + \Delta S_M = c(M+m)\ln\left(1+\frac{\Delta T}{T_i}\right)$$

If the final temperature is only slightly larger than the initial temperature, as one would expect for a large mass and board, and a small initial speed, then it is valid the apply the approximation $\ln(1+x) \approx x$, which gives:

$$\Delta S \approx \frac{c(M+m)\alpha}{T_i}u^2 = \frac{Mm}{2T_i(M+m)}u^2$$

Since no heat is transferred to the surroundings, there is no change in entropy of the surroundings. Thus the change in entropy of the universe is equal to the change in entropy ΔS of the mass and the board. Hence, $\beta = \frac{Mm}{2T_i(M+m)} \approx \boxed{0.00014}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 27: Enlarged Merlion (5 points)

The Merlion has height $h = 8.6$ m. A water pump, which operates at power P with ideal efficiency, delivers water to the Merlion at a constant rate, via a pipe that transports water at uniform speed from a stationary reservoir at ground level to the top of the Merlion. There, water is ejected horizontally, forming the iconic fountain. The horizontal range of the Merlion's fountain is $l = 13$ m.

Authorities plan to double the fountain's horizontal range. To perform this, the operating power of the pump is to be increased to P'. Find the ratio P'/P.

Leave your answer to 3 significant figures.

Solution: Let the speed of water ejected from the Merlion be u. Firstly, we consider the energy supplied by the water pump. In a time $\mathrm{d}t$, the mass of water ejected from the top of the water pump is $\rho A u \, \mathrm{d}t$, where ρ is the water's density and A is the cross-sectional area of the internal pipe within the Merlion. Note that the area of the pipe must be uniform along its length by mass conservation, given that the water's speed is uniform.

By mass conservation, a mass $\mathrm{d}m = \rho A u \, \mathrm{d}t$ of water must enter the Merlion at its bottom, so the kinetic energy it gains is $\frac{1}{2}u^2 \, \mathrm{d}m = \frac{1}{2}\rho A u^3 \, \mathrm{d}t$. In addition, the water column also gains gravitational potential energy $gh \, \mathrm{d}m = \rho A u g h \, \mathrm{d}t$. The total energy $\mathrm{d}E$ gained by water within time $\mathrm{d}t$ is thus:

$$\mathrm{d}E = \rho A u \left(gh + \frac{1}{2}u^2 \right) \mathrm{d}t$$

Hence, the power supplied by the pump P is:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \rho A u \left(gh + \frac{1}{2}u^2 \right)$$

We can also express u in terms of h and l. This is a problem of kinematics. Given that the water is ejected from the Merlion's mouth horizontally, its initial velocity in the vertical direction is zero, so by considering a water particle's motion in the vertical direction, the time taken t for it to reach ground level satisfies $h = \frac{1}{2}gt^2$. Correspondingly, analysing its horizontal motion gives $l = ut = u\sqrt{\frac{2h}{g}} \implies u = l\sqrt{\frac{g}{2h}}$. Thus P can be re-expressed in terms of l and h:

$$P = \rho A l g \sqrt{\frac{gh}{2}} \left(1 + \frac{l^2}{4h^2} \right)$$

We can now determine the final power required P' if l is doubled. Replacing l with $2l$

and simplifying, we obtain:

$$P' = 2\rho A l g \sqrt{\frac{gh}{2}} \left(1 + \frac{l^2}{h^2}\right)$$

Hence the required power increased by a factor of:

$$\frac{P'}{P} = \frac{8(h^2 + l^2)}{4h^2 + l^2} \approx \boxed{4.18}$$

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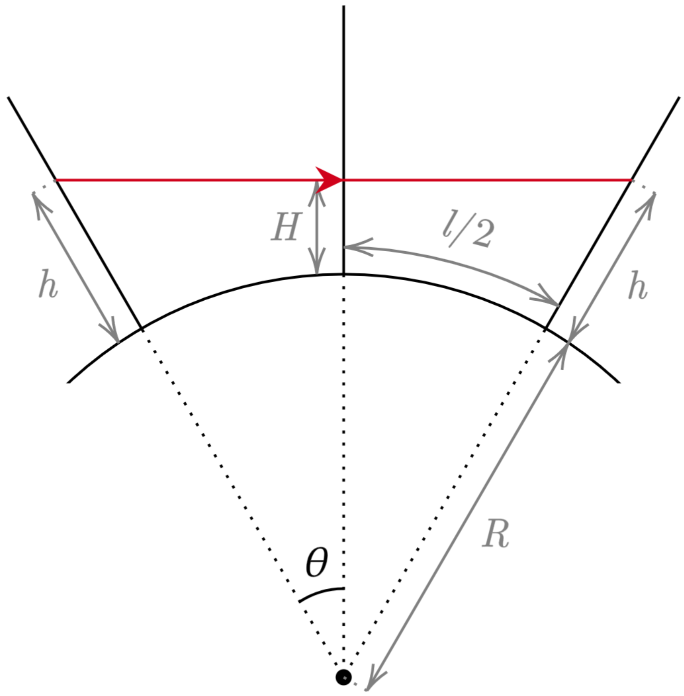
Problem 28: Flat Earth Experimentalists (3 points)

In the 2018 documentary *Behind the Curve*, Flat Earthers devised an experiment to test the Earth's curvature. Two measuring posts are placed at geographical distance $l = 2.50$ km apart. The posts are adjusted to be perfectly vertical using a spirit level. A laser is calibrated such that the emitted beam intersects each post at height $h = 3.00$ m above the ground. An identical vertical post is then placed at the exact midpoint of the two existing posts. Keeping the laser's calibration unchanged, the intersection between the laser beam and this newly added post is measured to be at height H above the ground. What is the percentage deviation of measured H from the prediction of Flat Earth theory? You may treat the Earth to be spherical with radius $R = 6371$ km.

Leave your answer to 2 significant figures as a percentage. (For example, if you think the final answer should be 51%*, input your answer as* 51*)*

Your answer should be positive.

Solution: The lengths of each measuring post are perpendicular to the surface of the Earth, since they were each made vertical using a spirit level. The diagram below shows a drawing of the actual setup under the assumption of a spherical Earth of radius R. The laser beam intersects the first and last measuring posts at the same distance h from the ground.



Let θ represent the angle between the central post and a side post at the Earth's centre. Since $\frac{l}{2}$ is the length of the arc of radius R subtended by angle θ, we can write $\theta = \frac{l}{2R}$. By symmetry about the central post, the laser beam makes a right angle with the axis of the central post. Consider the right-angle triangle whose three vertices are located at the centre of the Earth, the point at which the laser beam intersects a side post, and the point at which the laser beam intersects the central post. We can write:

$$(h+R)\cos\theta = R+H$$

$$\therefore H = (R+h)\cos\frac{l}{2R} - R$$

Flat Earth theory will predict that $H = h$ since the posts stand on the same "horizontal ground". Hence, the required percentage deviation is given by:

$$\frac{h-H}{h}\times 100\% = \left(\frac{R}{h}+1\right)\left(1-\cos\frac{l}{2R}\right)\times 100\% \approx \boxed{4.1\%}$$

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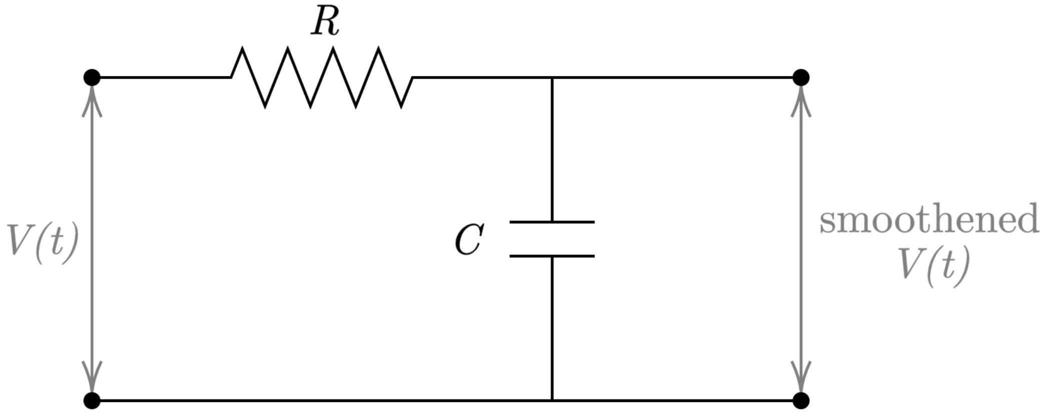
Problem 29: Noisy Voltage (5 points)

When the operation of a regular direct voltage source is affected by significant periodic fluctuations, its output voltage is best described as a sinusoidal variation centred around a positive value, taking the form $V(t) = A\cos(Bt) + C$ for positive constants A, B, C. As a result of these fluctuations, the ratio $V_{\max}/V_{\min} = 1.5$, where $V_{\max}$ and $V_{\min}$ are the maximum and minimum voltages of the source respectively. The fluctuations come at frequency $f = 200$ Hz, which may be taken to be the frequency of $V(t)$.

In efforts to reduce this noise, the voltage of the source is now fed through a low-pass filter (shown below). This comprises a resistor $R = 15.0\ \Omega$ and a capacitor $C = 120\ \mu$F. The new output voltage, taken as the voltage across the capacitor, is a smoothened version of $V(t)$ with a lowered $V_{\max}/V_{\min}$ ratio. Find this ratio.

Leave your answer to 3 significant figures.



Solution: We may treat the voltage $V(t)$ of the noisy DC source to be the superposition of two ideal sinusoidal AC components, one with frequency 0 (representing the original DC voltage), and one with frequency f (representing the periodic fluctuations). As such, we focus our attention on finding the output of the low-pass filter $V_{\text{out}}(t)$ for a general AC input $V_{\text{in}}(t) = V_0 \cos\omega t$.

In complex form, we may write $\tilde{V}_{\text{in}} = V_0 e^{i\omega t}$. Correspondingly, the output voltage may be written as $\tilde{V}_{\text{out}} = \frac{Z_C}{Z_C + Z_R}\tilde{V}_{\text{in}}$, where $Z_C = \frac{1}{i\omega C}$ is the impedance of the capacitor and $Z_R = R$ is the impedance of the resistor. Simplifying for $\tilde{V}_{\text{out}}$, we obtain:

$$\tilde{V}_{\text{out}} = \frac{V_0}{\sqrt{\omega^2 R^2 C^2 + 1}} e^{i(\omega t + \phi)}$$

where ϕ is a phase imparted that can be deduced and expressed in terms of ω, R and C (but will not be shown here as it is irrelevant in solving the problem). Hence, the real output voltage $V_{\text{out}}(t)$ can be written as:

$$V_{\text{out}}(t) = \text{Re}\left(\tilde{V}_{\text{out}}\right) = \frac{V_0}{\sqrt{\omega^2 R^2 C^2 + 1}} \cos(\omega t + \phi)$$

Now, let's apply it to this specific noisy voltage source. We can express $V(t) = V_0(\alpha + \cos\omega t)$ where α is a dimensionless constant to be determined. In this form, $V_{\text{max}} = V_0(\alpha + 1)$, and $V_{\text{min}} = V_0(\alpha - 1)$, so the ratio $\frac{V_{\text{max}}}{V_{\text{min}}} = \frac{\alpha+1}{\alpha-1}$. Given that the value of this ratio $= 1.5$, we can deduce that $\alpha = 5$.

Treating $V_{\text{in}}(t)$ as a superposition of αV_0 (AC component with $\omega = 0$ representing the DC voltage) and $V_0 \cos\omega t$ (AC component with $\omega = 2\pi f$), we can use our previously derived result on each of these two components, and superpose them together to obtain the corresponding V_{out}:

$$\begin{aligned} V_{\text{out}}(t) &= \alpha V_0 + \frac{V_0}{\sqrt{\omega^2 R^2 C^2 + 1}} \cos(\omega t + \phi) \\ \therefore V_{\text{max}} &= V_0 \left(\alpha + \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}\right) \\ \therefore V_{\text{min}} &= V_0 \left(\alpha - \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}\right) \end{aligned}$$

As such, the new ratio $\frac{V_{\text{max}}}{V_{\text{min}}}$ can be written as:

$$\frac{V_{\text{max}}}{V_{\text{min}}} = \frac{\alpha + \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}}{\alpha - \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}} \approx \boxed{1.18}$$

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Problem 30: Nailed It (4 points)

A uniform rod lies flat on a horizontal table. One of its ends is nailed to the table. This nailed end serves as a pivot about which the rod is free to rotate. On its other end, a horizontal force $|\vec{F}| = 60$ N is applied perpendicular to the length of the rod. What is the force exerted by the nail on the rod at this instant? Assume that there is no friction between the rod and the nail or the table.

Leave your answer as positive if you think the force by the nail on the rod acts in the direction of $\vec{F}$.

Leave your answer as negative if you think the force by the nail on the rod opposes the direction of $\vec{F}$.

Leave your answer to 2 significant figures in units of N.

Solution: Let the rod have mass m and length l. The moment of inertia of the rod about its end is $\frac{1}{3}ml^2$. Let α be the angular acceleration associated with the rotation of the rod. Writing rotational Newton's Second Law with the nailed end of the rod taken as the origin:

$$Fl = \frac{1}{3}ml^2\alpha \implies \alpha = \frac{3F}{ml}$$

Here, the torque is only due to the external force $\vec{F}$. The torque exerted by the nail on the rod may be neglected due to our choice of origin.

As such, the acceleration a of the rod's centre of mass can be written:

$$a = \frac{l}{2}\alpha = \frac{3F}{2m}$$

Taking the direction of $\vec{F}$ to be positive, let the force by the nail on the rod be R. Then, writing translational Newton's Second Law on the rod (where the acceleration of the centre of mass of the rod is used),

$$F + R = ma = \frac{3F}{2} \implies R = \frac{F}{2}$$

Thus, the nail exerts a force on the rod that is along $\vec{F}$, with magnitude $\frac{F}{2} = \boxed{+30 \text{ N}}$. It may seem surprising and unintuitive that this force is in the direction of $\vec{F}$. To better understand this, imagine that the nail was not present. In that case, the end of the rod (opposite to the end that the force was applied on) will travel in the opposite direction of $\vec{F}$ at the instant it was applied; you may prove this by calculation. Hence the presence of the nail serves to keep this end stationary. Therefore, the nail exerts a force along $\vec{F}$.

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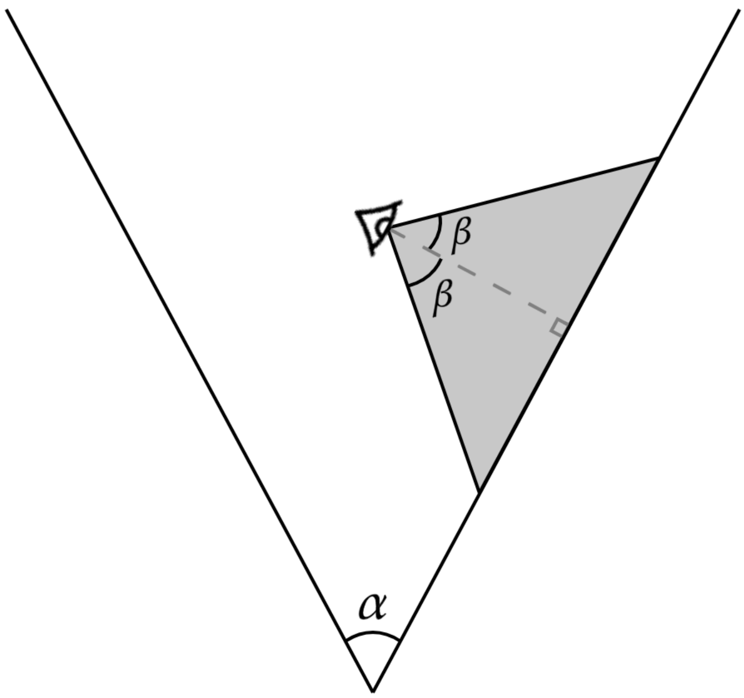
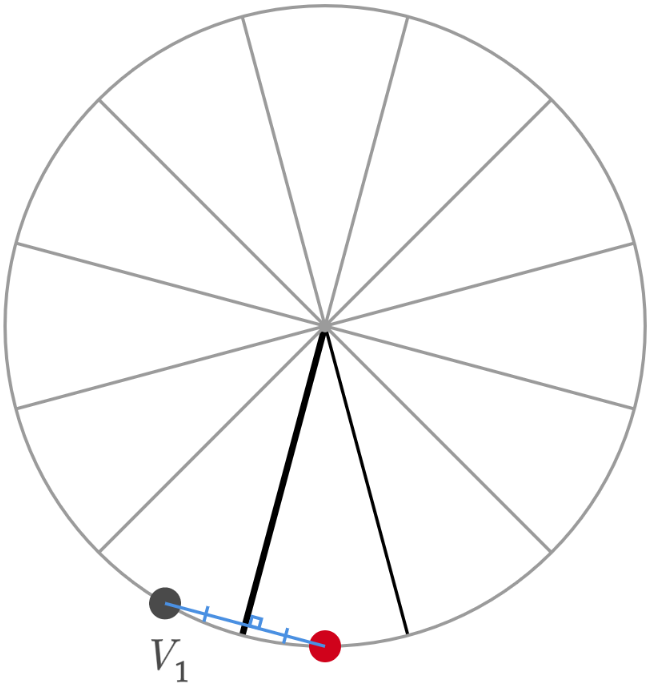
Problem 31: Tilted Mirrors (4 points)

Two large plane mirrors are oriented at an angle $\alpha = 12°$ with respect to each other. A man stands halfway between the two mirrors and orients himself such that his line of sight is perpendicular to one of the mirror surfaces. The field of view of the man is $\beta = 39°$ towards each side. How many images of himself does he see?

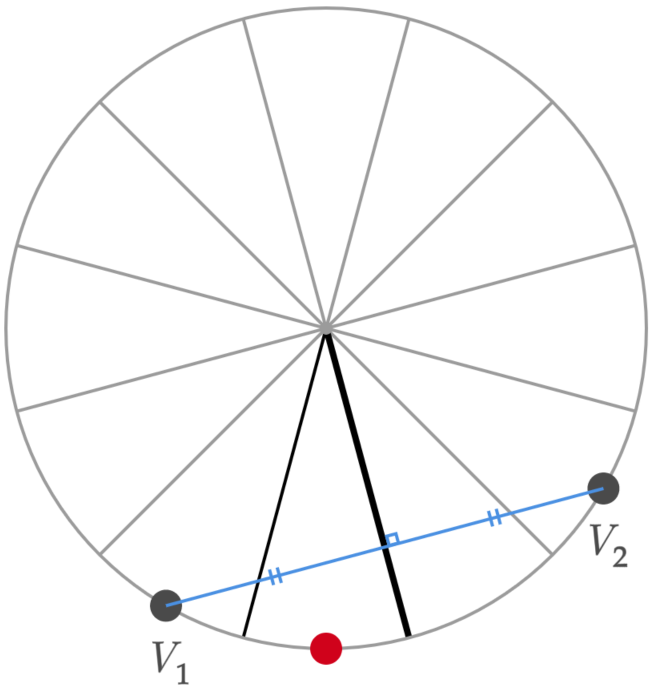
Leave your answer as an integer.



Solution: Let the first image on the person's left (from the reflection in the left mirror) be V_1. Let's construct a circle O centred on the intersection point of the two mirrors and passing through the person and the image. We will prove that all other images must lie on this circle. Here, we ignore the fact that the person's field of view is limited, and we are concerned with all physically possible positions of the person's images.



Consider the reflection of V_1 in the right mirror.[1] Let's call this V_2. According to ray optics, V_1 and V_2 will be the same distance from the mirror but on opposite sides. Furthermore, the line connecting V_1 and V_2 will be perpendicular to the mirror, as shown in the figure below.

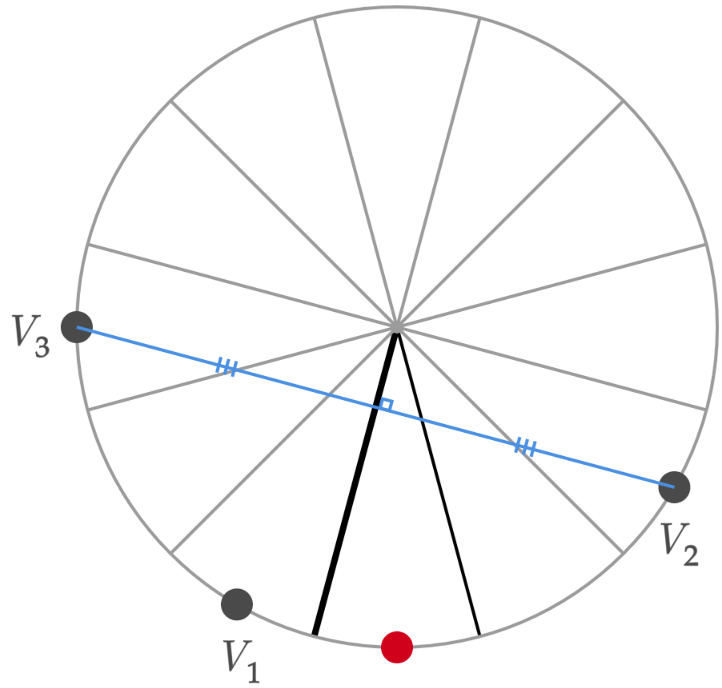


Now, look at the geometry of the setup. Let the intersection of line V_1V_2 and the right mirror be X. Then $|V_1X| = |V_2X|$. V_1 lies on circle O, and the right mirror

[1]Technically, we are considering here the image of a virtual image. If this disturbs you, simply imagine replacing this virtual image with a real object. This is because the geometry of the light rays from V_1 and the real object are exactly the same in the region between the two mirrors.

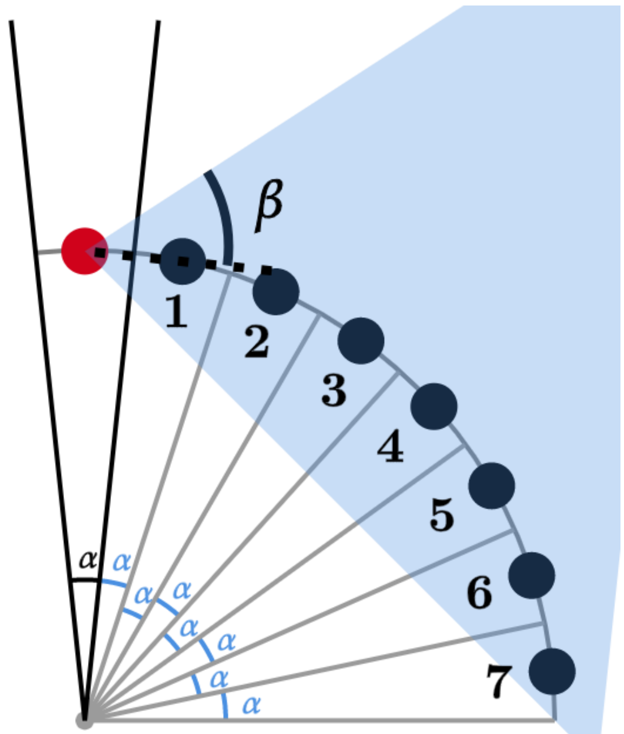
passes through the centre of circle O. This means that V_1V_2 is a chord of circle O, and therefore V_2 must lie on circle O.

The same idea can be used to find V_3, V_4 and so on. The example for V_3 is shown in the figure below.



Notice that the above procedure leaves some "gaps" along the circle. These gaps can be filled up by considering the first image on the man's right and repeating the above procedure.

Let us now consider how many of these images will land in the man's field of view.



We simply count the number of images that fall within the field of view of the man, which covers a quarter of the circle. The number of images contained within is $\left\lfloor \frac{90^\circ}{12^\circ} \right\rfloor = 7$. Thus, the man can see $\boxed{7}$ images of himself. Note that the image closest to him (of his front) is laterally inverted, the subsequent image (of his back) is not, and so on.

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Problem 32: Dazzling Supernovae (3 points)

In observational astronomy, the magnitude of an object is a dimensionless measure of the intensity of light from it. Magnitude is measured on a logarithmic scale. For every 100-fold increase in light intensity, the magnitude will decrease by 5 units. Therefore, if light from object A is 100 times more intense than light from another object B whose magnitude is 3.2, the magnitude of A will be $3.2 - 5 = -1.8$.

The apparent magnitude refers to the magnitude of an object as viewed from Earth while the absolute magnitude refers to the magnitude of an object as viewed from a fixed distance of 32.6 light years.

A type-1A supernova has an absolute magnitude of $M = -19.3$. If such a supernova occurred 1000 light years from Earth, what would its apparent magnitude, m, be? Assume that nothing obstructs the line of sight between the supernova and Earth.

Leave your answer to 3 significant figures.

Solution: If light from an object is 100^n times more intense, by definition, its magnitude will change by $-5n$, i.e. its magnitude will decrease by $5n$. If we let $x = 100^n$, we have $n = \log_{100} x$, and thus an object with light that is x times more intense will have its magnitude change by $-5\log_{100} x$.

Since nothing obstructs the line of sight from Earth to the supernova, the intensity of light from the supernova will obey the inverse-square law, i.e. it will decrease proportionally to the inverse square of the distance.

Suppose that the intensity of the supernova from a distance of r light years is $I(r)$. The inverse square law tells us that $I(r) \propto \frac{1}{r^2}$, i.e. $I(r) = \frac{k}{r^2}$, where k is a constant. Therefore, $I(32.6) = \frac{k}{32.6^2}, I(1000) = \frac{k}{1000^2}$, which implies that:

$$k = I(32.6) \cdot (32.6)^2 = I(1000) \cdot (1000)^2$$

Rearranging this, we have:

$$\frac{I(1000)}{I(32.6)} = \frac{32.6^2}{1000^2} = 1.06276 \times 10^{-3}$$

Thus, light from the supernova is 1.06276×10^{-3} times as intense from Earth as it is from 32.6 light years away. Its magnitude will thus change by $\Delta m = -5\log_{100}(1.06276 \times 10^{-3}) \approx +7.43$. Since the magnitude in the latter case is $M = -19.3$, the magnitude in the former case is:

$$m = M + \Delta m = -19.3 + 7.4 = \boxed{-11.9}$$

This is just slightly dimmer than a full moon, whose apparent magnitude is -12.7.

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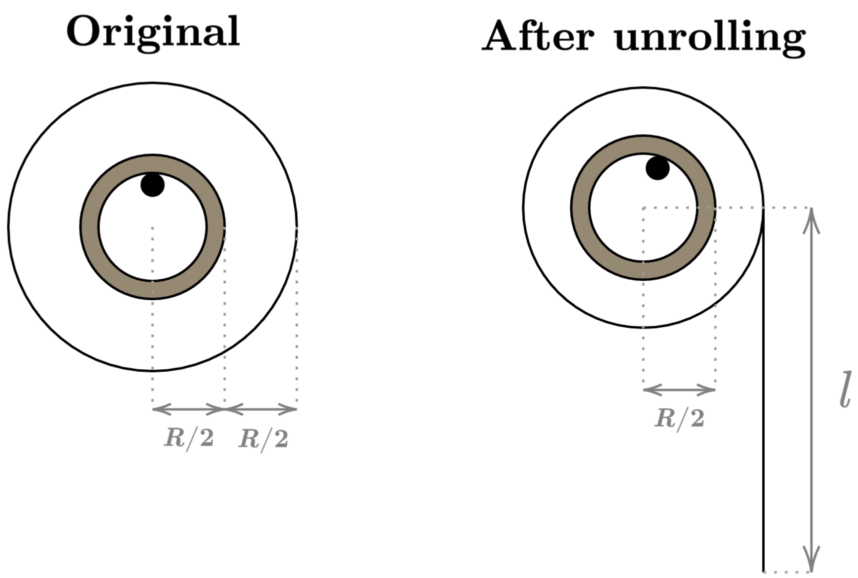
Problem 33: A Toilet Nightmare (5 points)

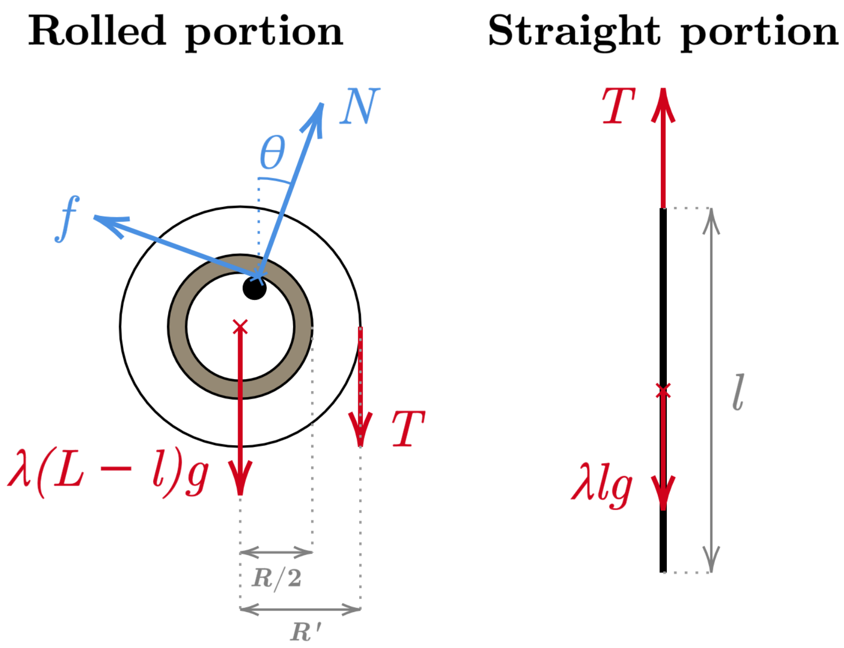
A toilet roll with initial outer radius R is made by winding length L of paper with uniform mass per unit length λ, around a cardboard tube of radius $r = R/2$ and negligible thickness and mass. The cardboard roll in the centre experiences friction with the toilet roll holder, a thin rod upon which the inner surface of the cardboard tube rests. The static friction is characterised by coefficient $\mu_s = 0.300$.

The roll is unrolled by grabbing its end and pulling it downwards slowly. When a minimum length, l, of toilet paper has been unrolled, it is released and the remaining toilet paper unrolls spontaneously. Find l/L. Assume that there is sufficient friction between the paper and the roll such that the paper never slips relative to the roll. You may also assume that the roll is elevated such that the paper never touches the floor.

Leave your answer to 3 significant figures.



Solution: When part of the toilet paper is unrolled, the centre of mass of the toilet paper now lies to the right of the centre of the roll. At equilibrium, the rod will therefore be positioned to the right of the roll's centre. Let the angle of displacement, θ, of the centre of the roll from its original position be defined as shown in the diagram below.



The free-body diagram on the left considers the external forces on the system of the rolled paper and the cardboard roll. Note that friction between the roll and the paper, as an internal force, is not depicted. The free-body diagram on the right considers only the unrolled portion of paper.

When length l of the paper has been unrolled, the remaining roll will have radius $R' < R$. At the onset of spontaneous unwinding, the setup is in a state of equilibrium. At equilibrium, net vertical force on the unrolled portion must be zero. As such, the gravitational force induces a tension $T = \lambda l g$ on the top of the unrolled length of toilet paper. This tension then exerts a torque TR' on the system depicted on the left.

At the limit of equilibrium, frictional force $f = \mu_s N$, where N is the normal contact force exerted by the holder on the roll. By balancing forces in the horizontal direction, we obtain:

$$f\cos\theta = N\sin\theta \implies \mu_s N\cos\theta = N\sin\theta$$

It therefore follows that:

$$\tan\theta = \mu_s \implies \theta = \tan^{-1}\mu_s$$

By balancing torques about the point of contact between the roll and the holder, we derive that:

$$\lambda\left(L-l\right)gr\sin\theta = T\left(R' - r\sin\theta\right)$$
$$\frac{R}{2}\sin\theta\left[\lambda\left(L-l\right)g + T\right] = TR'$$

Substituting $T = \lambda l g$, the above expression reduces to:

$$\frac{R}{2}\sin\theta\left(\lambda L g\right) = R'\lambda l g \implies \frac{R'}{R} = \frac{L\sin\left(\tan^{-1}\mu_s\right)}{2l}$$

To transform this ratio of radii to a ratio of lengths, we must consider the geometry of the roll. Taking the thickness of the toilet paper to be α, the entire roll of toilet paper has a cross-sectional area of $\alpha L = \pi R^2 - \pi r^2$, of which the toilet paper of length l contributes a cross-sectional area of $\alpha l = \pi R^2 - \pi R'^2$. This gives us the following relation:

$$\frac{l}{L} = \frac{R^2 - R'^2}{R^2 - r^2} = \frac{R^2 - R'^2}{R^2 - \frac{R^2}{4}} = \frac{4}{3}\left(1 - \frac{R'^2}{R^2}\right)$$

By substituting our result for R'/R, we form an equation in l/L:

$$\frac{l}{L} = \frac{4}{3}\left[1 - \left(\frac{L\sin\left(\tan^{-1}\mu_s\right)}{2l}\right)^2\right]$$

The equation can be rewritten as:

$$3\frac{l^3}{L^3} - 4\frac{l^2}{L^2} + \sin^2\left(\tan^{-1}\mu_s\right) = 0$$

This yields the solutions $l/L = -0.1368, 0.1527, 1.317$

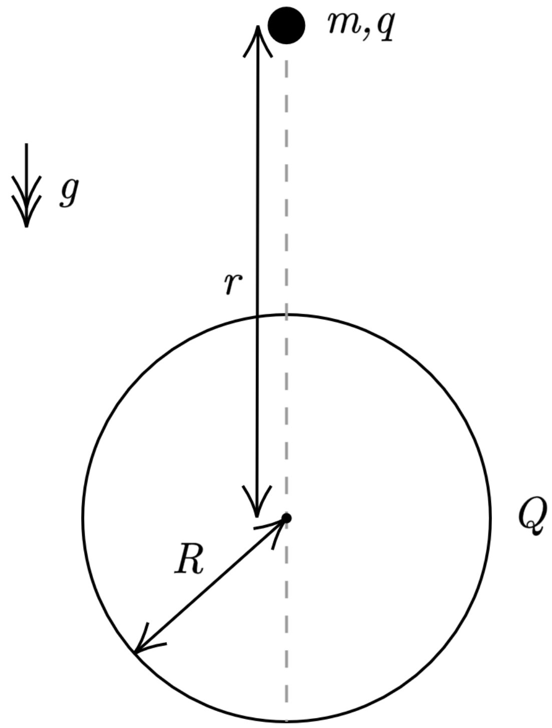
Since $0 < l/L < 1$, $l/L \approx \boxed{0.153}$

Setter: Galen Lee, galen.lee@sgphysicsleague.org

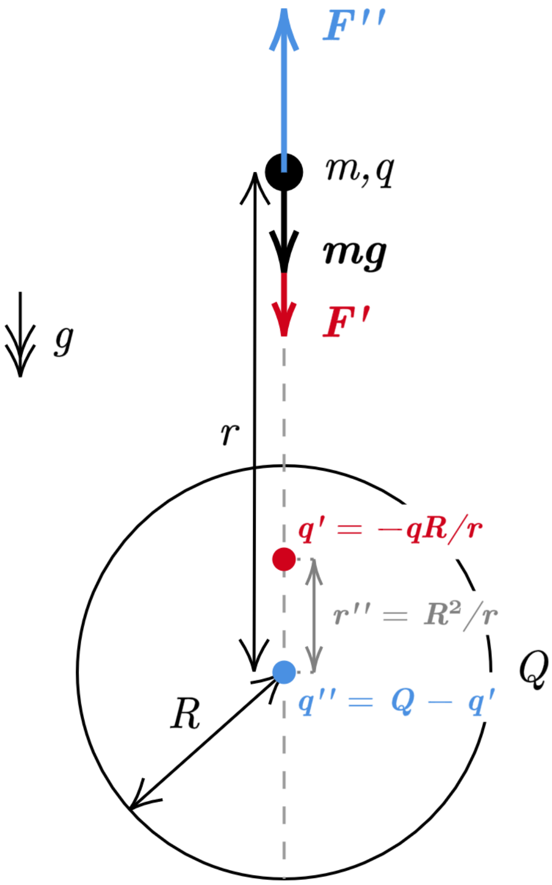
Problem 34: A Charged Conducting Sphere (5 points)

A conducting sphere of radius $R = 0.500$ m has an unknown charge Q initially distributed uniformly over its surface. To find this unknown charge, a particle with known mass $m = 1.00$ g and charge $q = 0.520$ μC is placed above the sphere and constrained to move along a vertical line passing through the centre of the sphere. It was found that the particle settled into a stable equilibrium at a distance $r = 2R$ above the sphere's centre. Find the charge Q on the conducting sphere.

Leave your answer to 3 significant figures in units of μC.



Solution: To model the effects of the induced charges that form on the conducting sphere, we place an image charge $q' = -qR/r$ along the line segment between the particle and the centre of the sphere, at a distance $r' = R^2/r$ from the sphere centre. The force on q from this image charge will be equal to the force from the induced negative charge that forms on the sphere. However, since the total charge on the sphere is conserved (i.e. the sphere is not grounded), excess positive charge will still be left on the sphere and exert a repulsive force on q. We account for this by placing another charge $q'' = Q - q'$, which ensures that the sum of the two image charges equals the total charge on the original sphere, Q.



Thus, three forces act on q: its weight $-mg$, as well as forces from the image charges q' and q'', which are F' and F'' respectively. Defining the forces to be positive if they act upwards:

$$F' = \frac{q}{4\pi\varepsilon_0}\frac{q'}{(r-r')^2}$$
$$F'' = \frac{q}{4\pi\varepsilon_0}\frac{q''}{r^2}$$

Since q is in stable equilibrium, these forces cancel out, which gives us $F' + F'' = mg$:

$$\frac{q}{4\pi\varepsilon_0}\frac{q''}{r^2} + \frac{q}{4\pi\varepsilon_0}\frac{q'}{(r-r')^2} = mg$$

Substituting q', q'', r' as defined previously and solving for Q gives:

$$Q = q\left[\frac{4\pi\varepsilon_0 r^2 mg}{q^2} - \frac{R}{r} + \frac{r^3 R}{(r^2 - R^2)^2}\right] = \boxed{2.30\ \mu\text{C}}$$

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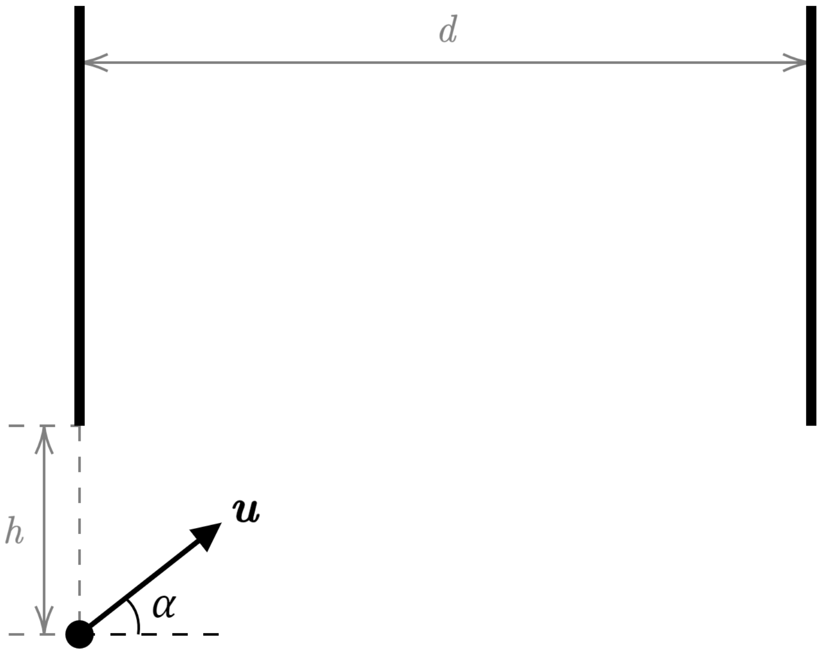
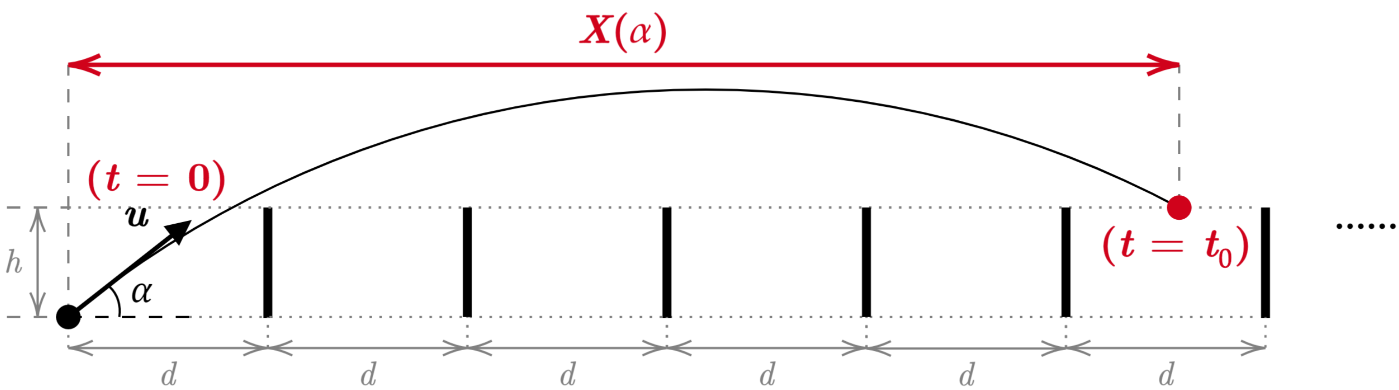
Problem 35: Boing Boing (5 points)

Two vertical walls of infinite height have their bases elevated $h = 3.40$ m above the ground, and are separated by a distance $d = 10.0$ m. A mass on the ground directly under the left wall is fired towards the right wall at angle α above the horizontal with an initial speed $u = 30$ m s^{-1}. For the angles $\alpha_1 < \alpha < \alpha_2$, the mass undergoes a maximal number of collisions with the walls before falling to the ground. Assume all collisions are perfectly elastic and ignore resistive forces. Find the ratio α_1/α_2.

Leave your answer to 2 significant figures.



Solution: Due to the elastic assumptions, both the horizontal speed and the vertical velocity of the ball are preserved between collisions. As such, this problem is equivalent to that of a projectile clearing consecutive barriers of height h and separated by a distance d, as illustrated below. The clearing of a barrier in this context is analogous to a collision in the original setup.



The number of barriers cleared, N, would be obtained by $N = \lfloor \frac{X(\alpha)}{d} \rfloor$, where $X(\alpha)$ is the horizontal distance travelled by the mass between the time of launch (at $t = 0$), and the time when its vertical displacement drops below h in the descending phase of its motion (at $t = t_0$). The equivalent of $X(\alpha)$ in the original context would be the horizontal distance travelled before the ball falls below the base of the two walls, after which no further collision with the walls can occur. Two equations can then be obtained, with t_0 denoting the time taken for the ball to traverse horizontal distance $X(\alpha)$:

$$\begin{aligned} h &= ut_0 \sin\alpha - \frac{1}{2} g t_0^2 \\ X(\alpha) &= ut_0 \cos\alpha \end{aligned}$$

Solving for t_0:

$$t_0 = \frac{u\sin\alpha + \sqrt{u^2 \sin^2\alpha - 2gh}}{g}$$

We reject the smaller solution for t_0, as that is the time at which the mass has traversed a vertical displacement h, but is travelling upward.

Substituting t_0 into the expression for $X(\alpha)$, we obtain:

$$X(\alpha) = \frac{u\cos\alpha}{g}(u\sin\alpha + \sqrt{u^2\sin^2\alpha - 2gh})$$

By plotting $N = \lfloor \frac{X(\alpha)}{d} \rfloor$ against α for $0 \le \alpha \le \frac{\pi}{2}$, we find that maximum $N = 8 \implies$ all values of α for which $X(\alpha) > 8d$ correspond to the maximum number of collisions. We obtain numerically that $\alpha_1 \approx 0.597$ and $\alpha_2 \approx 1.016$. Hence, $\frac{\alpha_1}{\alpha_2} \approx \boxed{0.59}$.

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Problem 36: Stacked Blocks

A block of mass $m = 3.0$ kg is stacked above another block of mass $M = 5.0$ kg, which lies on horizontal ground. The ground is frictionless, but there exists friction between the two blocks, with coefficient $\mu_s = 1.0$ for static friction and coefficient $\mu_k = 0.7$ for kinetic friction. With both blocks initially at rest, the lower block is pushed with a constant horizontal force F. Assume that the lower block is sufficiently long and wide, such that the upper block never falls off, and the blocks do not rotate.

(a) Determine the value of F required so that the lower block has an acceleration $a = 5.0\ \mathrm{m\ s^{-2}}$.

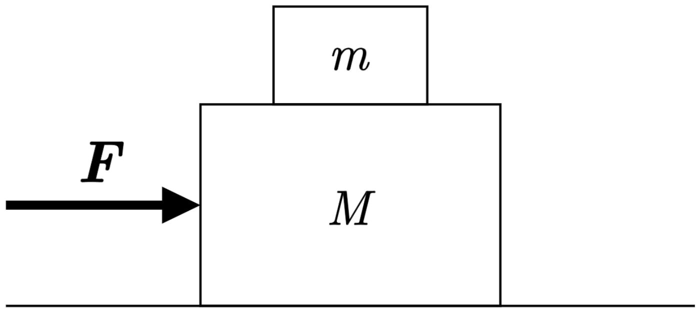
Leave your answer to 3 significant figures in units of N. (2 points)

(b) Determine the value of F required so that the lower block has an acceleration $a = 15.0\ \mathrm{m\ s^{-2}}$.

Leave your answer to 3 significant figures in units of N. (2 points)

(c) There is a range of accelerations $a_0 < a < a_1$ that is impossible for the lower block to attain regardless of the value of F. Find $a_1 - a_0$.

Leave your answer to 3 significant figures in units of $\mathrm{m\ s^{-2}}$. (4 points)



Solution: Let the magnitude of friction between the two blocks be f. If the acceleration a is sufficiently low and f is sufficiently high such that the two blocks do not slip with respect to each other, i.e. they have the same acceleration, the two equations of motion for the top and bottom block respectively would look like this:

$$f = ma$$
$$F - f = Ma$$

which gives $F = (m + M)a$ and $f = ma$. We note, however, that there is a maximum $f \leq \mu_s mg$ (since friction is static in the non-slipping case). So this result is only valid if $a \leq \mu_s g$.

If instead $a > \mu_s g$, friction is not large enough to prevent slipping between the two blocks, so the two blocks do not take on the same acceleration. In this case, friction becomes kinetic, so that $f = \mu_k mg$. Writing Newton's Second Law on the lower block gives:

$$F - \mu_k mg = Ma$$

Hence within this range of a, required $F = Ma + \mu_k mg$.

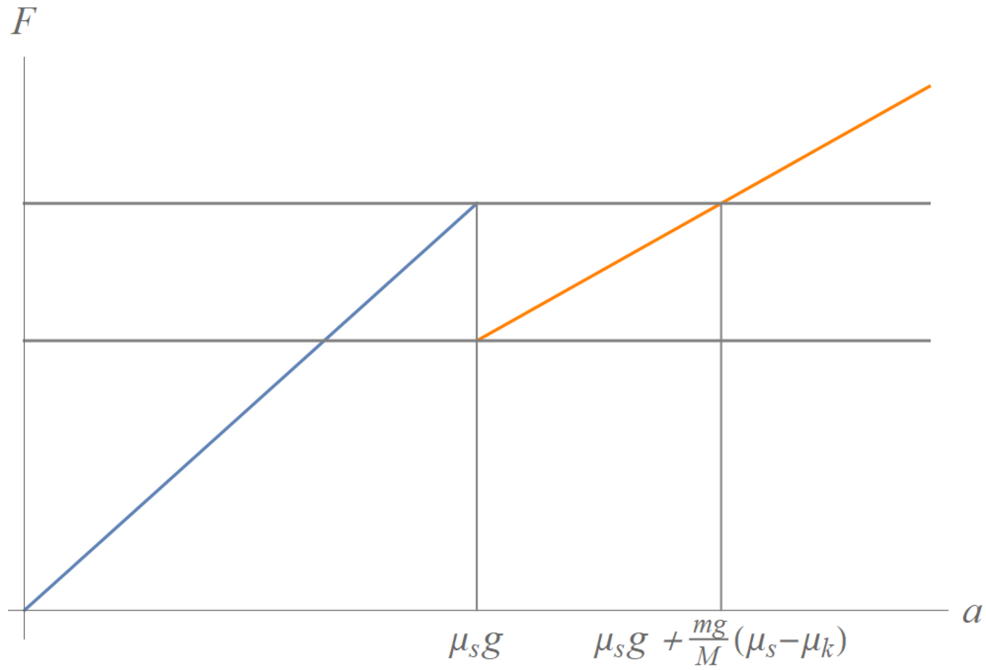
(a) In this part, $a = 5.0 \text{ m s}^{-2} \leq \mu_s g = 9.81 \text{ m s}^{-2}$, so this value of a falls in the non-slipping regime. Thus $F = (M + m)a = \boxed{40.0 \text{ N}}$.

(b) As for this part, $a = 15.0 \text{ m s}^{-2} > \mu_s g = 9.81 \text{ m s}^{-2}$, so the two blocks slip with respect to each other. The required $F = Ma + \mu_k mg \approx \boxed{95.6 \text{ N}}$.

(c) Altogether, $F(a)$ can be written as a piecewise function as follows:

$$F(a) = \begin{cases} (M+m)a, & a \leq \mu_s g \\ Ma + \mu_k mg, & a > \mu_s g \end{cases}$$

The graph plot of $F(a)$ against a is shown below. The blue line represents the non-slip regime, and the orange line represents the slip regime.



Consider the behaviour of the system for different ranges of constant horizontal force F. Let us start from $F = 0$ and explore what happens when a larger value of F is chosen. At first, the acceleration a is unambiguously defined by the x-value of the intersection point between the blue line and the horizontal line whose y-value is at F.

Once F enters the region bounded by the two gray horizontal lines as shown, it appears as though there are two possible values of a that it could take on. However, noting still that $a < \mu_s g$, and the fact that both blocks started out at rest, it is impossible for the resulting motion to be in the slip regime, so the value of a for these values of F is still given by the blue line.

Finally, once F leaves this region and is raised above the upper horizontal line, the block starts slipping with a given by the orange line. At this point, however, $a = \mu_s g + \frac{mg}{M}(\mu_s - \mu_k)$. Evidently, throughout the variation of F, a range of values of a was essentially skipped:

$$\mu_s g < a < \mu_s g + \frac{mg}{M}(\mu_s - \mu_k)$$

It is thus impossible for the block to be set in motion at a value of a within this range. As such:

$$a_1 - a_0 = \frac{mg}{M}(\mu_s - \mu_k) \approx \boxed{1.77\ \mathrm{m\ s^{-2}}}$$

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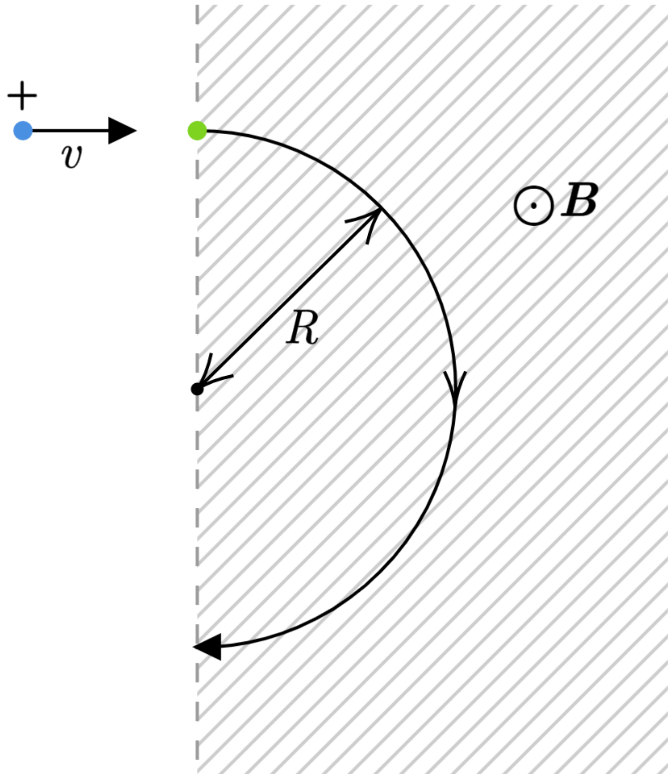
Problem 37: Collisions in a Magnetic Field (5 points)

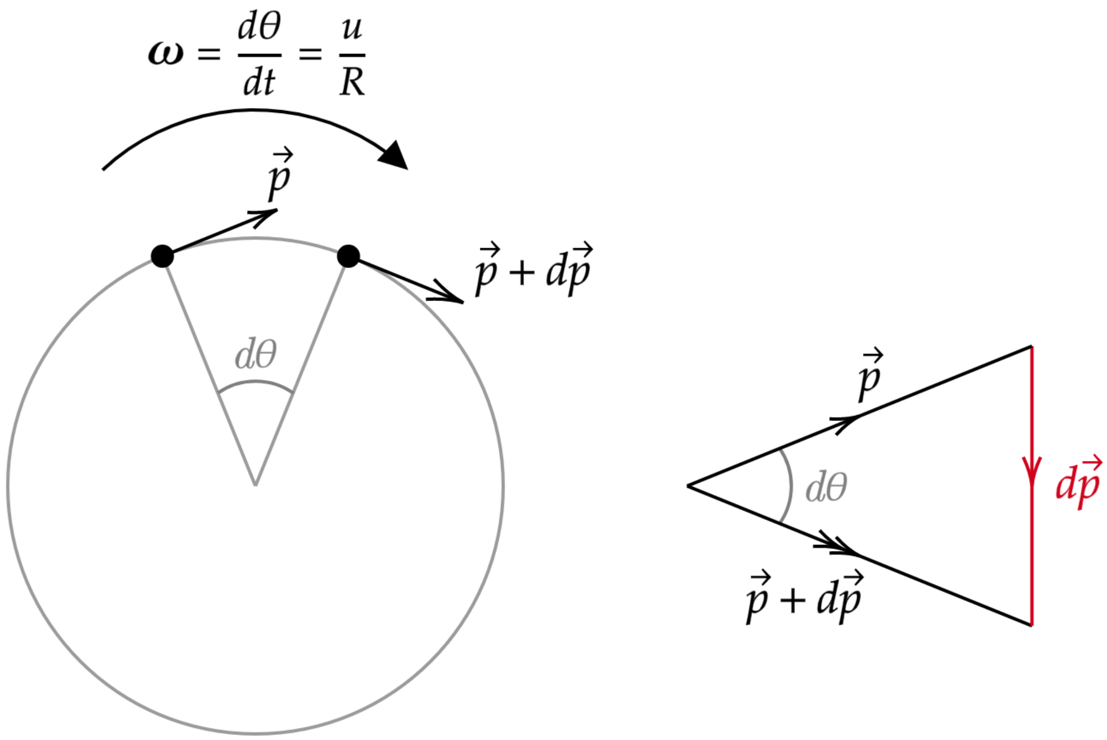
A proton moving at relativistic speed v in the lab frame collides with and sticks to a stationary neutron at the edge of a magnetic field of strength $B = 0.100$ T. The magnetic field is directed perpendicular to the plane of the proton's motion. If the resulting particle moves in a circular arc of radius $R = 1.00$ m through the magnetic field, find the value of v/c, where c is the speed of light. Take the neutron to have the same rest mass as the proton.

Leave your answer to 2 significant figures.



Solution: We will be analyzing the whole problem within the lab frame. Inside the region with the magnetic field, the combined particle, which has charge $+e$ and a velocity we will denote as u, experiences a magnetic force of magnitude $F = euB$ in the lab frame. While the usual Newtonian expression for centripetal force $F_c = mu^2/R$ no longer works in our relativistic problem, we can proceed as follows:

Let the relativistic momentum of the combined particle be $\vec{p}$. As the particle moves in a uniform circular trajectory, $\vec{p}$ changes only in its direction, not its magnitude. This is shown in the diagram below.



When a short amount of time $\mathrm{d}t$ passes, the particle rotates by a small angle $\mathrm{d}\theta = \omega\,\mathrm{d}t = \frac{u}{R}\,\mathrm{d}t$, as shown. From the vector triangle drawn above, we can determine $|\mathrm{d}\vec{p}|$ in terms of $\mathrm{d}\theta$:

$$|\mathrm{d}\vec{p}| = 2p\sin\left(\frac{\mathrm{d}\theta}{2}\right) \approx p\,d\theta$$

Hence, dividing by $\mathrm{d}t$:

$$\left|\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right| = \frac{pu}{R}$$

Notice that all of this has been deduced merely from geometrical considerations arising from the uniform circular motion of the particle. We have yet to invoke any physical laws, either relativistic or otherwise. We now apply the relativistic Newton's second law, $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$, to obtain:

$$\left|\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right| = \frac{pu}{R} = |\vec{F}| = euB$$

Hence, the relativistic momentum of the combined particle immediately after the collision has magnitude:

$$p_c = eBR$$

Note that this is actually the same formula as its non-relativistic counterpart, except that p here is now the relativistic instead of the classical momentum.

It now suffices to deduce the original velocity of the proton by conserving relativistic momentum. The initial relativistic momentum of the proton is $p_p = \frac{mv}{\sqrt{1-v^2/c^2}}$, while that of the stationary neutron is $p_n = 0$. Since no external forces influenced the collision, we have $p_p + p_n = p_c$, thus:

$$\frac{mv}{\sqrt{1-(\frac{v}{c})^2}} = eBR$$

Solving this for v/c gives:

$$\frac{v}{c} = \frac{1}{\sqrt{1+\left(\frac{mc}{eBR}\right)^2}} \approx \boxed{0.032}$$

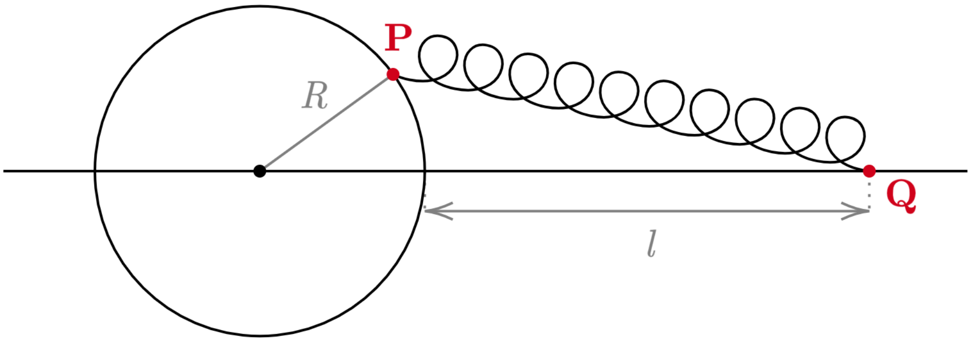
Notice that this approach does not require us to find u, which would involve the conservation of relativistic energy and a lot more tedious algebra.

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

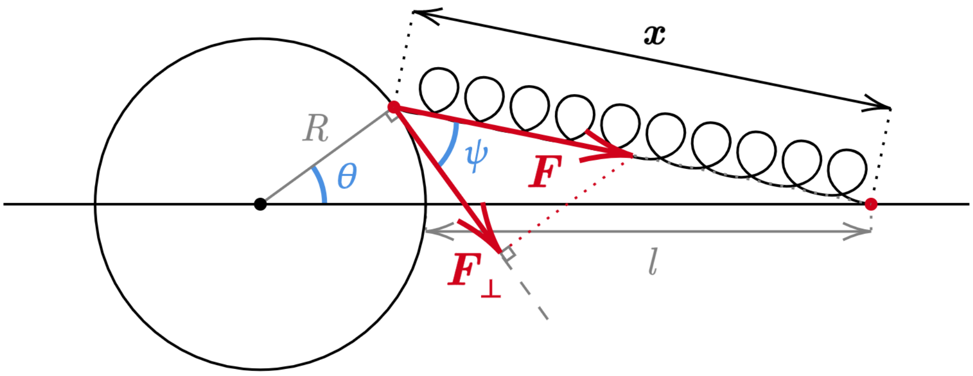
Problem 38: Springs and Discs (4 points)

A uniform solid disc with mass $m = 1.0$ kg and radius $R = 0.30$ m is free to rotate around a fixed frictionless axle through its centre. An ideal light spring with zero rest length and spring constant $k = 5.0 \times 10^3$ N m^{-1} has one end attached to the disc's edge, at the point P, while the other end is attached to a fixed point a distance $l = 1.00$ m from the disc's edge, at the point Q. What is the frequency f of small oscillations of the disc about its equilibrium position?

Leave your answer to 3 significant figures in units of s^{-1}.



Solution: Suppose the disc is displaced by an angle θ from equilibrium. Let the distance between points P and Q be x. The spring exerts a force $F = kx$ on the disc, the tangential component of which we will denote as $F_\perp$. The restoring torque on the disc is thus $\tau = -F_\perp R$.



In the diagram above, ψ is defined as the angle between the spring force and the tangent to P. First, from geometry, $F_\perp = F\cos\psi$. Since $\sin\left(\frac{\pi}{2} + \psi\right) = \cos\psi$, and the sine rule gives $\sin(\frac{\pi}{2} + \psi) = (R + l)\frac{\sin\theta}{x}$, we can re-express $F_\perp$:

$$F_\perp = F(R + l)\frac{\sin\theta}{x}$$

Furthermore, since $F = kx$, we can deduce that $F_\perp = k(R + l)\sin\theta$. By rotational Newton's Second Law, we equate the restoring torque τ with $I\ddot{\theta}$, where $I = \frac{1}{2}mR^2$ is the moment of inertia of the disc about its centre, and hence obtain an equation of motion for θ:

$$\ddot{\theta} = -\frac{\tau}{I} = -\frac{F_\perp R}{\frac{1}{2}mR^2} = -\frac{2k(R+l)}{mR}\sin\theta \approx -\frac{2k(R+l)}{mR}\theta$$

where the last approximation $\sin\theta \approx \theta$ is made for small θ. Comparing this with the equation of a simple harmonic oscillator with angular frequency ω, $\ddot{\theta} = -\omega^2\theta$, we conclude that:

$$\omega = \sqrt{\frac{2k(R+l)}{mR}}$$

Therefore, the frequency is:

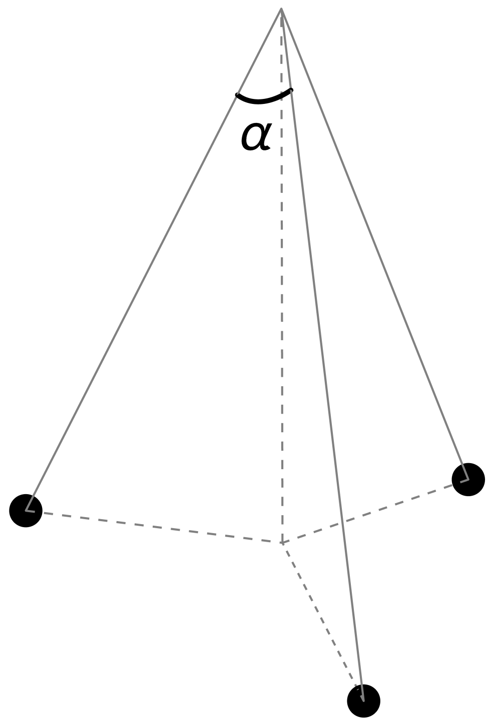
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2k(R+l)}{mR}} \approx \boxed{33.1\ \mathrm{s}^{-1}}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

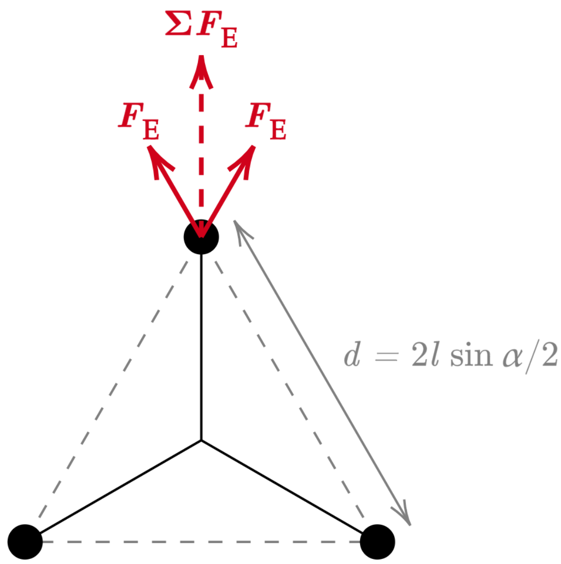
Problem 39: 3 Charged Balls (5 points)

Three identical point charges, each with a charge of $Q = 10^{-6}$ C and mass $m = 1.0$ kg are suspended from the same point using massless, insulating threads of fixed length $l = 1.0$ m. Find the angle α between two strings when the system is at its stable equilibrium configuration.

Leave your answer to 2 significant figures in units of degrees.



Solution: At stable equilibrium, the positions of the 3 balls form an equilateral triangle. Consider the setup from the top view.



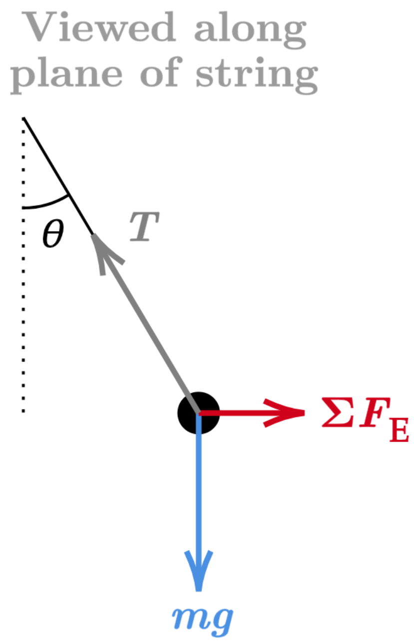
The distance between two balls is $d = 2l \sin \frac{\alpha}{2}$. Invoking Coulomb's law, the electrostatic force between two balls F_E can be written as:

$$F_E = \frac{Q^2}{4\pi\varepsilon_0 d^2} = \frac{Q^2}{16\pi\varepsilon_0 l^2 \sin^2\left(\frac{\alpha}{2}\right)}$$

The net electrostatic force ΣF_E on one ball can be found by considering the vector sum of the F_E from each of the other two balls:

$$\Sigma F_E = 2F_E \cos 30^\circ = \frac{\sqrt{3}Q^2}{16\pi\varepsilon_0 l^2 \sin^2 \frac{\alpha}{2}}$$

Let us consider the free-body diagram of a ball, viewed along the plane of a string.



At equilibrium, net force on the ball is zero. Let θ be the angle between the string and the vertical. Balancing forces in the vertical and horizontal directions respectively:

$$T\cos\theta = mg$$
$$T\sin\theta = \Sigma F_E$$

$$\implies \tan\theta = \frac{\Sigma F_E}{mg} = \frac{\sqrt{3}Q^2}{16\pi\varepsilon_0 mgl^2 \sin^2\frac{\alpha}{2}} \tag{1}$$

By purely considering the geometry of the setup, we can find a constraint for θ in terms of α. We start by finding the distance from one vertex of the triangle to its centre (measured along the horizontal plane). Let's call this distance x. By the cosine rule:

$$4l^2\sin^2\frac{\alpha}{2} = 2x^2 - 2x^2\cos 120^\circ \implies x = \frac{2l\sin\frac{\alpha}{2}}{\sqrt{3}}$$

Using basic trigonometry:

$$\sin\theta = \frac{x}{l} \implies \theta = \sin^{-1}\left(\frac{2\sin\frac{\alpha}{2}}{\sqrt{3}}\right)$$

Using the identity $\tan(\sin^{-1} y) = \frac{y}{\sqrt{1-y^2}}$, we can find an expression for $\tan\theta$:

$$\tan\theta = \frac{2\sin\frac{\alpha}{2}}{\sqrt{3-4\sin^2\frac{\alpha}{2}}} \tag{2}$$

To find α, we can equate the two different expressions we have for $\tan\theta$ in Eq. (1) and (2) to form an equation in $\sin\frac{\alpha}{2}$:

$$\frac{\sqrt{3}Q^2}{16\pi\varepsilon_0 mgl^2} = \frac{2\sin^3\frac{\alpha}{2}}{\sqrt{3-4\sin^2\frac{\alpha}{2}}}$$

We can solve for $\sin\frac{\alpha}{2}$ numerically, and obtain $\sin\frac{\alpha}{2} \approx 0.06996 \implies \alpha \approx \boxed{8.0^\circ}$.

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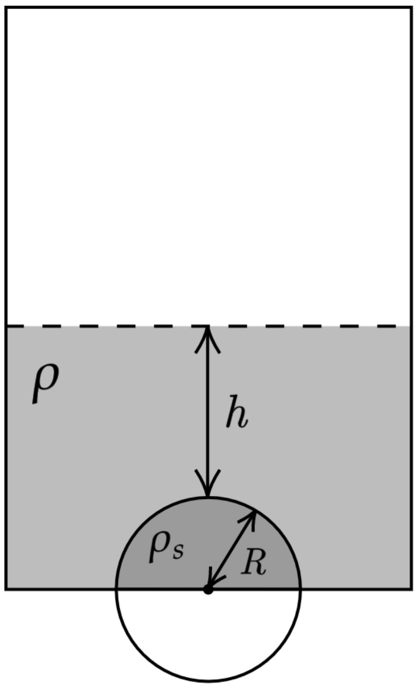
Problem 40: Force on Sphere (5 points)

A circular hole of radius $R = 15.0$ cm is cut out at the bottom of a large tank. A copper sphere with density $\rho_s = 8960$ kg m^{-3} and radius infinitesimally larger than R is lodged into the hole. The tank is then filled with water until the top of the sphere is a distance $h = 20$ cm below the water surface. Hence, the upper half of the sphere is now surrounded by water, and the lower half surrounded by air. Calculate the total upward force F exerted on the sphere by the hole at equilibrium. You may take atmospheric pressure to be constant all around the set-up.

Leave your answer to 3 significant figures in units of N.



Solution: Since atmospheric pressure is constant all around the set-up, the upward force due to air on the bottom half of the sphere cancels the force due to air above the water surface pushing down on the upper half of the sphere. Therefore, it suffices to consider only the forces due to water, the hole, and gravity acting on the sphere itself.

The hydrostatic pressure acting on an infinitesimal area element $\mathrm{d}A$ of the sphere at a polar angle θ (from the vertical) is $\rho g[h + R(1 - \cos\theta)]$.

The downward component of force acting on $\mathrm{d}A$ is $\rho g[h + R(1 - \cos\theta)]\cos\theta\,\mathrm{d}A$.

In spherical coordinates, $\mathrm{d}A = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, where ϕ is the azimuthal angle.

The downward force due to hydrostatic pressure can be obtained by integrating the above expression across the upper hemisphere ($0 < \theta < \pi/2$):

$$\begin{aligned}\text{Downward force due to water} &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho g[h + R(1-\cos\theta)]\cos\theta R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= 2\pi R^2 \rho g \int_0^{\frac{\pi}{2}} [h + R(1-\cos\theta)] \sin\theta\cos\theta \, \mathrm{d}\theta \\ &= \pi R^2 \rho g \left(h + \frac{R}{3}\right)\end{aligned}$$

To obtain the final answer, we note that at equilibrium, the force from the hole must balance the weight of the sphere and the downward force due to water. Plugging in the given constants, we have the required upward force:

$$F = \pi R^2 \rho g \left(h + \frac{R}{3}\right) + \frac{4}{3}\pi R^3 \rho_s g \approx \boxed{1420 \text{ N}}$$

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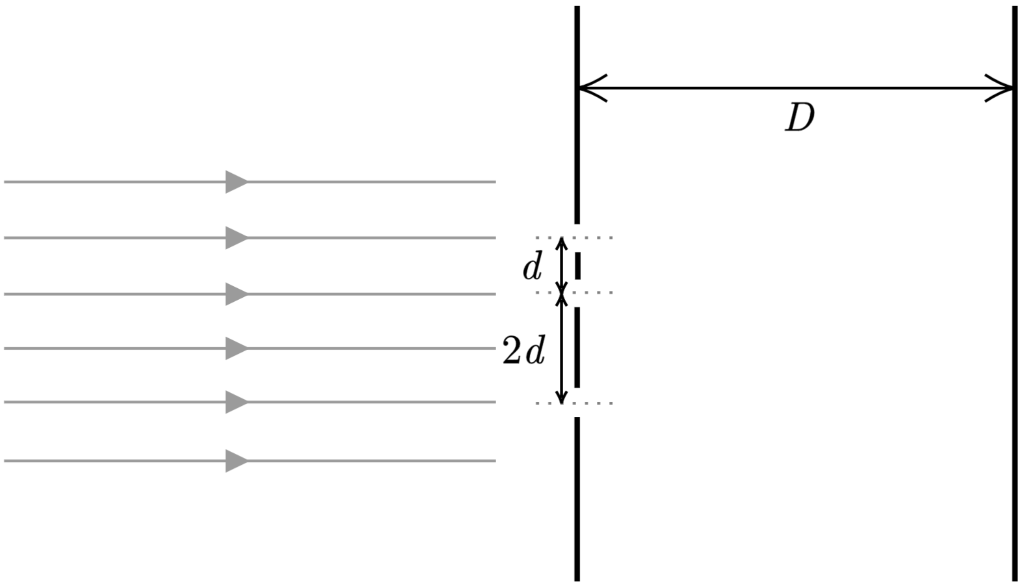
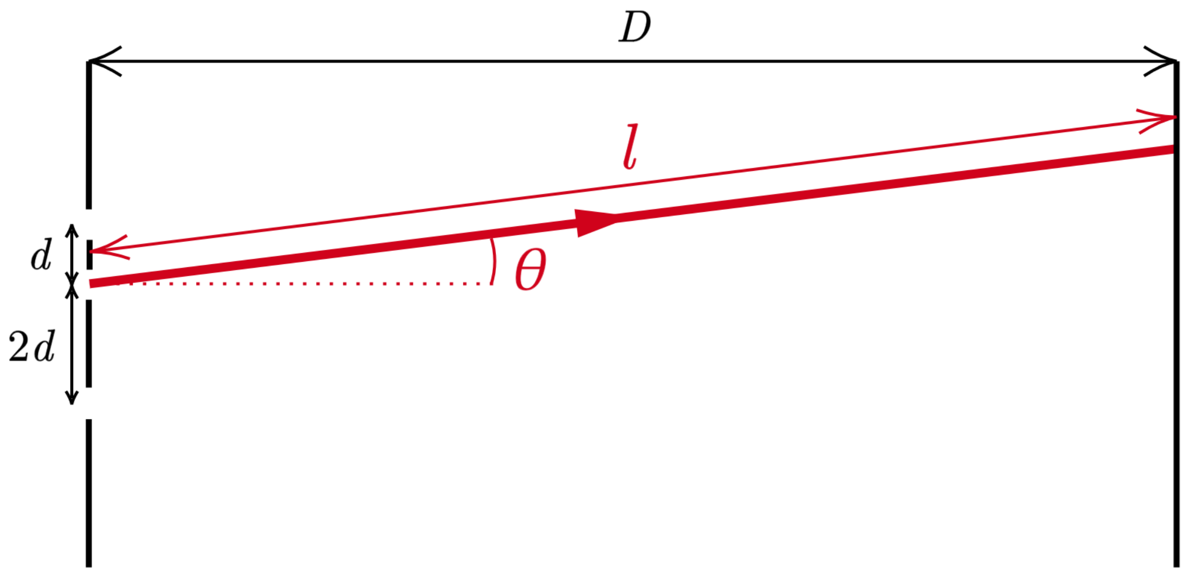
Problem 41: Triple Slit Interference (5 points)

A triple slit interference experiment is set up. The slit separation is unequal, with one slit being a distance $d = 18\ \mu\text{m}$ away from the central slit, while the other slit is a distance $2d$ away from the central slit. The screen is placed a distance $D = 1.5$ m from the slits, and monochromatic light of wavelength $\lambda = 600$ nm is shone normally onto the slits. Compute the distance x between the first intensity minima on either side of the central maxima. Take the central maxima as the maxima directly in front of the central slit. You may take the slit width to be negligible.

Leave your answer to 2 significant figures in units of mm.



Solution: Define the quantities l and θ for light through the central slit as shown in the diagram below.



Let $\omega = \frac{2\pi}{T}$ be the angular frequency of the light, where T is the period of the light. Let $A(\theta)$ be the amplitude of the interference pattern on the screen. Then, since $D \gg d$, we may take the three light rays to be almost parallel. The distances travelled by light rays from the top and bottom slit are therefore $l - d\sin\theta$ and $l + 2d\sin\theta$ respectively, giving us:

$$A(\theta) = \alpha[\cos(\omega t - kl) + \cos(\omega t - k(l - d\sin\theta)) + \cos(\omega t - k(l + 2d\sin\theta))]$$

where α is the amplitude of the light ray from each individual slit, while k is the wavenumber of the light.

Since intensity is proportional to amplitude squared, the time-averaged intensity I may be found as follows:

$$\begin{aligned} I &\propto \frac{1}{T}\int_0^T [A(\theta)]^2 \, \mathrm{d}t \\ &= \frac{\omega}{2\pi}\alpha^2 \int_0^{\frac{2\pi}{\omega}} [\cos(\omega t - kl) + \cos(\omega t - k(l - d\sin\theta)) + \cos(\omega t - k(l + 2d\sin\theta))]^2 \, \mathrm{d}t \end{aligned}$$

This simplifies to:

$$I \propto \left[\frac{3}{2} + \cos(kd\sin\theta) + \cos(2kd\sin\theta) + \cos(3kd\sin\theta)\right]$$

Note that the intensity pattern is symmetrical about the central maxima at $\theta = 0$.

To find the locations of the intensity minima and maxima, we differentiate the above expression with respect to θ and set the result to zero. This gives us:

$$\cos\theta[\sin(kd\sin\theta) + 2\sin(2kd\sin\theta) + 3\sin(3kd\sin\theta)] = 0$$

Since clearly $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos\theta \neq 0$ and therefore:

$$\sin(kd\sin\theta) + 2\sin(2kd\sin\theta) + 3\sin(3kd\sin\theta) = 0$$

We can find the smallest positive value for $\theta \approx 6.8593 \times 10^{-3}$ numerically.

Since θ is small, we may use the approximation $\tan\theta \approx \theta$. Hence, the required distance is given by:

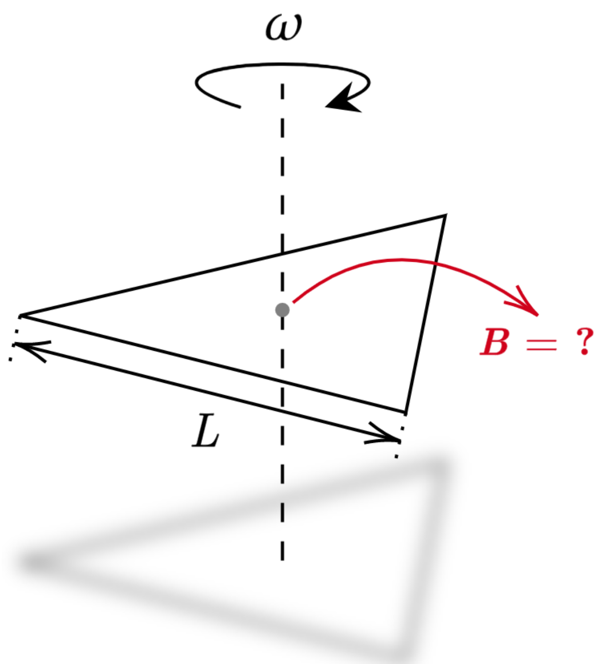
$$x = 2D\tan\theta \approx 2D\theta \approx \boxed{21 \text{ mm}}$$

Setter: Brian Siew, brian.siew@sgphysicsleague.org

Problem 42: Rotating Triangle (5 points)

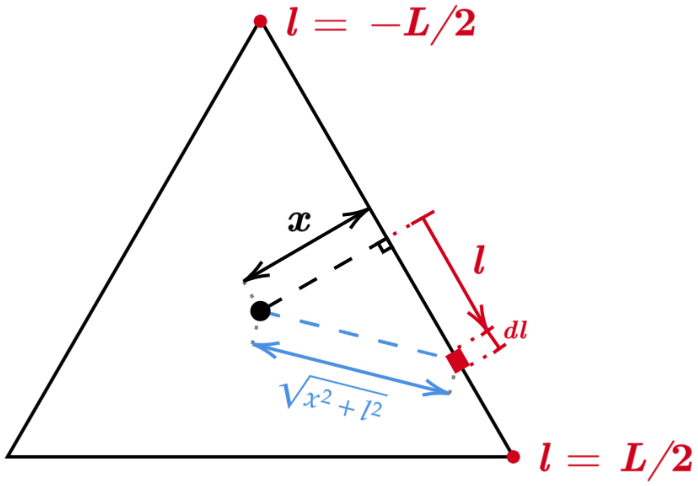
A perfectly insulating thin rigid wire, in the shape of an equilateral triangle, has side length $L = 12$ cm and a linear charge density of $\lambda = 10$ C m^{-1}. It is rotated about an axis perpendicular to its plane and passing through its centre at an angular velocity of $\omega = 320$ rad s^{-1}. Find the magnetic field B at the centre of the triangle. You may assume that ω is sufficiently large, such that the currents formed from the motion of the triangle are steady.

Leave your answer to 2 significant figures in units of T.



Solution: Recall that the magnetic field strength at the centre of a ring carrying a current I is given by $B = \frac{\mu_0 I}{2r}$. Now suppose we have a point charge q undergoing circular motion with angular velocity ω. If ω is sufficiently large, we may still approximate it as a steady current of magnitude $I = \frac{q}{\frac{2\pi}{\omega}} = \frac{q\omega}{2\pi}$. The magnetic field at the centre of the circle would then be $\frac{\mu_0}{2r}\left(\frac{q\omega}{2\pi}\right) = \frac{\mu_0 q\omega}{4\pi r}$.

Now, consider a single side of the triangle in the question. Let the perpendicular distance of this side from the centre of the triangle be x. We chop up the side into infinitesimal pieces with length $\mathrm{d}l$. Each of these pieces has a charge $\mathrm{d}q = \lambda\,\mathrm{d}l$ and travels in circular motion with radius $\sqrt{l^2 + x^2}$, where l is the distance of the piece from the centre of the side.



Hence, each piece produces a magnetic field dB given by:

$$\mathrm{d}B = \frac{\mu_0 \lambda \omega \, \mathrm{d}l}{4\pi\sqrt{l^2+x^2}}$$

The magnetic field generated by the rotation of the whole side can then be computed via integrating from $l = -L/2$ to $l = L/2$. As a triangle has three sides, the final answer can be obtained by multiplying by a factor of 3:

$$B = 3\int \mathrm{d}B = 3\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\mu_0 \lambda \omega \, \mathrm{d}l}{4\pi\sqrt{l^2+x^2}} = \frac{3\mu_0 \lambda \omega}{2\pi} \ln(2+\sqrt{3})$$

where x has been re-expressed by invoking the geometric relation $x = L\tan 30^\circ/2$. The expression for B is surprisingly independent of L.

Substituting in given values yields $B \approx \boxed{2.5 \times 10^{-3}\ \mathrm{T}}$.

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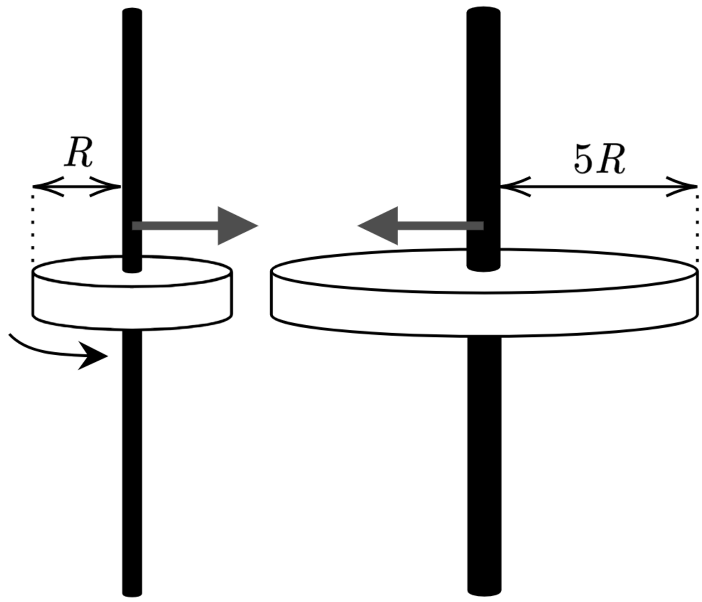
Problem 43: Contacting Discs (5 points)

Two uniform discs made from the same material and of equal thickness have radii R and $5R$ respectively. Each disc has a fixed axle through its centre that lets it rotate frictionlessly about its central axis. The disc with radius R is imparted some initial rotational velocity, while the other disc is initially stationary. The two discs are brought in contact and allowed to settle to a steady state. What percentage of the system's mechanical energy is dissipated in the process?

Leave your answer to 2 significant figures as a percentage. (For example, if you think the final answer should be 51.0%*, input your answer as* 51.0*)*



Solution: The axles are fixed in place, which means that external forces are exerted on each disc to keep it stationary. These external forces result in external torques being exerted on the system of both discs, and as such, angular momentum of the system is not conserved.

Denote the disc with radius R as disc A, and the other disc with radius $5R$ as disc B. Assume, without loss of generality, that the direction of initial rotation of disc A is clockwise, with angular velocity ω_0. When the two discs come into contact, there will be friction acting tangentially on the edges of discs A and B due to the relative motion between the surfaces of both discs. For disc A, the friction will create an anticlockwise torque. By Newton's Third Law, the friction acting on B will take on the same magnitude but opposite direction to that of A, so the friction on B results in an anticlockwise torque around its centre.

This friction causes the magnitude of ω_A (in the clockwise direction) to decrease over time, and the magnitude of ω_B (in the anticlockwise direction) to increase. Let the frictional force be f. f may be a function of time, since it is unclear from the given information whether f depends on the relative velocity. Also, let the mass per unit area of both discs be σ. This quantity is equal for both discs since they are made from the same material. We treat the clockwise direction to be positive. Applying rotational Newton's Second Law on both discs, we can find their angular accelerations α_A and α_B:

$$\frac{1}{2}\left(\sigma\pi R^2\right)R^2\alpha_A = -fR \qquad \implies \alpha_A = -\frac{2f}{\sigma\pi R^3}$$
$$\frac{1}{2}\left[\sigma\pi\left(5R\right)^2\right]\left(5R\right)^2\alpha_B = -f\left(5R\right) \implies \alpha_B = -\frac{2f}{125\sigma\pi R^3}$$

With that, we can determine how ω_A and ω_B change with time t, where the discs first make contact at $t = 0$:

$$\omega_A\left(t\right) = \omega_0 - \frac{2}{\sigma\pi R^3}\int_0^t f\,\mathrm{d}t$$
$$\omega_B\left(t\right) = -\frac{2}{125\sigma\pi R^3}\int_0^t f\,\mathrm{d}t$$

This proceeds until the velocity of the contact points between both discs equalize, i.e. when there is no more slipping between the surfaces of both discs. In other words:

$$R\omega_A = -5R\omega_B$$

After which, a steady-state is reached where ω_A and ω_B no longer change. Let this occur at time T. Applying this non-slipping condition:

$$R\omega_0 - \frac{2}{\sigma\pi R^2}\int_0^T f dt = \frac{2}{25\sigma\pi R^2}\int_0^T f\ dt$$
$$\implies \int_0^T f\ dt = \frac{25\sigma\pi R^3\omega_0}{52}$$

With this, we can find the steady-state angular velocities $\omega_A\left(T\right)$ and $\omega_B\left(T\right)$:

$$\omega_A\left(T\right) = \frac{\omega_0}{26}, \qquad \omega_B\left(T\right) = -\frac{\omega_0}{130}$$

Now, we can compare the initial and final rotational kinetic energies. Initially, since disc B started out stationary, we only need to consider disc A to compute the total rotational kinetic energy E_i:

$$E_i = \frac{1}{2}\left[\frac{1}{2}\left(\sigma\pi R^2\right)R^2\right]\omega_0^2 = \frac{1}{4}\sigma\pi R^4\omega_0^2$$

At the end, we sum the contributions from both discs using the previously derived steady-state angular velocities:

$$E_f = \frac{1}{2}\left[\frac{1}{2}\left(\sigma\pi R^2\right)R^2\right]\left(\frac{\omega_0}{26}\right)^2 + \frac{1}{2}\left[\frac{1}{2}\left(\sigma\pi\left(5R\right)^2\right)\left(5R\right)^2\right]\left(-\frac{\omega_0}{130}\right)^2 = \frac{1}{104}\sigma\pi R^4\omega_0^2$$

The percentage of energy dissipated due to friction is thus given by $\frac{E_i - E_f}{E_i} \times 100\% \approx \boxed{96\%}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 44: Joule Heating (5 points)

A long cylindrical copper wire has radius $R = 0.5$ mm, electrical resistivity $\rho = 1.68 \times 10^{-8}\ \Omega\ \mathrm{m}^{-1}$, and thermal conductivity $\kappa = 385\ \mathrm{W\ m^{-1}\ K^{-1}}$. The wire carries uniform current density $J = 2.55 \times 10^6\ \mathrm{A\ m^{-2}}$ directed along the axis of the wire. The surface of the wire is maintained at temperature $T_0 = 300$ K. Given that T is the temperature at a distance $R/2$ from the central axis of the wire at steady state, find $T - T_0$.

Leave your answer to 3 significant figures in units of mK.

Solution: Recall the heat equation, where H is the rate of energy production per unit volume, and C is the heat capacity per unit volume:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{C}\nabla^2 T + \frac{H}{C}$$

Since we are only concerned with the steady state temperature, $\partial T/\partial t = 0$, and the equation reduces to:

$$\kappa \nabla^2 T + H = 0$$

Now, we set up a cylindrical coordinate system, with z being the axis of the cylinder, s being the perpendicular distance from the axis, and ϕ being the azimuthal angle. The equation then becomes:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{H}{\kappa} = 0$$

Note that the set-up has cylindrical symmetry, thus T has no dependence on either z or ϕ. The above partial differential equation can therefore be simplified to an ordinary differential equation:

$$\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}T}{\mathrm{d}s}\right) + \frac{H}{\kappa} = 0$$

When solved, we obtain the following expression for $T(s)$, where A, B are constants to be determined by boundary conditions:

$$T(s) = -\frac{H}{4\kappa}s^2 + A\ln s + B$$

Evidently, $A = 0$, otherwise the temperature would blow up at the centre of the cylinder. Furthermore, $T(R) = T_0$ implies $B = T_0 + \frac{H}{4\kappa}R^2$. Hence:

$$T(s) = T_0 + \frac{H}{4\kappa}(R^2 - s^2)$$

Required temperature:

$$T = T\left(\frac{R}{2}\right) = T_0 + \frac{H}{4\kappa}\left(R^2 - \left(\frac{R}{2}\right)^2\right) = T_0 + \frac{H}{4\kappa}\left(\frac{3R^2}{4}\right) = T_0 + \frac{3HR^2}{16\kappa}$$

To find H, we apply the Joule heating law to a segment of wire of length l. The volume of this segment is $\pi R^2 l$. The resistance of this wire is $\frac{\rho l}{\pi R^2}$, implying that the power dissipated when current flows in this segment is $\frac{\rho l}{\pi R^2}(J\pi R^2)^2$. Thus:

$$H = \frac{\frac{\rho l}{\pi R^2}(J\pi R^2)^2}{\pi R^2 l} = J^2\rho$$

Plugging in the values, the final answer is:

$$T - T_0 = \frac{3HR^2}{16\kappa} = \boxed{0.0133 \text{ mK}}$$

Setter: Brian Siew, brian.siew@sgphysicsleague.org

Problem 45: Damped Oscillator (5 points)

A mass $m = 0.50$ kg is connected to an ideal light spring with spring constant $k = 60$ N m^{-1} on a rough horizontal surface. The mass is pulled, giving the spring an initial extension $A = 1.2$ m from its rest length, and released from rest. Throughout its subsequent motion, there is a resistive frictional force with constant magnitude $f = 0.40$ N opposing the oscillatory motion of the mass. Determine the time taken, t, for the mass to come to a permanent rest. Neglect static friction.

Leave your answer to 3 significant figures in units of s.

Solution: Let x be the extension of the spring from its rest length. The equations of motion for the mass are as follows:

$$m\ddot{x} = -kx - f \text{ for } \dot{x} > 0$$
$$m\ddot{x} = -kx + f \text{ for } \dot{x} < 0$$

The sign of f is always opposite that of $\dot{x}$ since f acts in the direction to oppose the mass' velocity. Evidently, the presence of friction will only serve to change the equilibrium position of the mass' oscillations, and will not have any direct effect on its period. Due to the constant switching of the mass' equation of motion between the two differential equations, solving it would be difficult. Instead, we opt for a simpler approach.

Let s_n be the n^{th} maximum point in the graph of $|x(t)|$ against t, starting with $s_0 = A$. Physically speaking, s_n represents the amplitude of the mass' oscillations (i.e. maximum extension or compression of the spring) after n half-periods. As one might expect, s_n would be a decreasing sequence due to the dissipative effects of friction.

Consider the motion of the mass as $|x(t)|$ goes from s_n to s_{n+1}. During this half-period, the mass travels a distance of $s_n + s_{n+1}$. Hence, the amount of negative work done on the mass by the frictional force is $-f\,(s_n + s_{n+1})$. Over the course of this half-period, there is a loss of elastic potential energy, from $\frac{1}{2}ks_n^2$ to $\frac{1}{2}ks_{n+1}^2$. Kinetic energy is zero when $|x(t)| = s_n$ and $|x(t)| = s_{n+1}$ since these are points of maximum extension or compression. Putting this all together, by considering the energy changes during this half-period:

$$\frac{1}{2}ks_n^2 - \frac{1}{2}ks_{n+1}^2 = f\,(s_n + s_{n+1}) \implies s_{n+1} = s_n - \frac{2f}{k}$$

Given that $s_0 = A$, we can obtain a simple general expression for s_n:

$$s_n = A - \frac{2f}{k}n$$

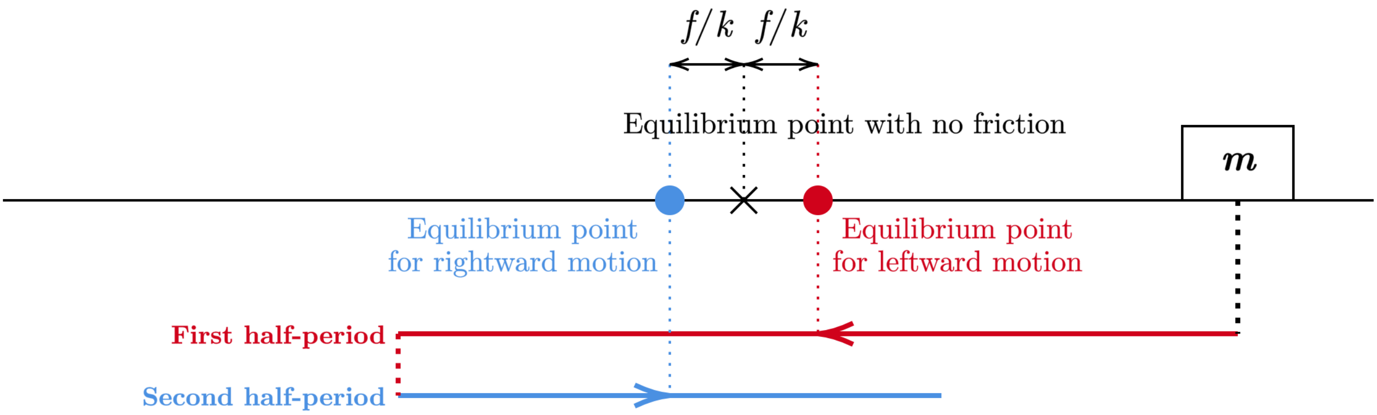
When the mass comes to rest, the amplitude must have gone to zero, so $s_n = 0$, which occurs at integer $n = \frac{kA}{2f} = 90$.

The period of oscillations of the mass is $T = 2\pi\sqrt{\frac{m}{k}}$; as previously explained, this is the same as that of an undamped spring-mass system. So the required time taken for the mass to come to rest is given by:

$$\frac{nT}{2} = \frac{\pi A}{2f}\sqrt{mk} \approx \boxed{25.8\text{ s}}$$

.

Alternative solution: The problem can also be solved using forces, by considering the shifts in "equilibrium positions" of the mass over the course of its motion.



Define the origin as the position of the mass when the spring is relaxed. Suppose the mass is released from rest from its starting position on the right. During the course of its leftward motion, two horizontal forces act on the mass: a spring force pointing left, and friction directed towards the right.

This situation is analogous to that of a mass oscillating from a vertical spring in the presence of gravity, except that the gravitational force acting on the spring is replaced by the frictional force. In the case of a vertically oscillating mass, the equilibrium position of the mass is located at distance $\frac{mg}{k}$ below the natural length of the spring. By analogy, the "equilibrium position" of the mass during its leftward motion is located a distance of $\frac{f}{k}$ to the right of the origin. In other words, when the mass moves towards the left, it undergoes simple harmonic motion with its equilibrium position at the red dot shown in the diagram above.

Eventually, the mass comes to rest and begins to travel rightwards. This is similar to before, but the spring force now points rightwards, and more importantly, friction is now constantly directed to the left. As such, the moment the mass begins travelling rightwards, its equilibrium position suddenly changes from the red dot to the blue dot.

These aforementioned shifts of "equilibrium positions" that depend on the direction of motion of the mass can be seen from the two differential equations written at the start of the previous solution. The equation that applies, and correspondingly the equilibrium values of x, depends on the sign of $\dot{x}$.

The motion of the mass during its first two half-periods is shown in the diagram. Notice that the equilibrium position taken on by the mass during each half-period falls in the centre of the path traced by the mass during that half-period. Because the equilibrium positions switch back and forth, the amplitude of oscillations decreases after each half-period. Physically, this indicates the mass' loss of energy due to friction.

The decrease in amplitude after each half-oscillation can be easily geometrically deduced to be $\frac{2f}{k}$. The mass will keep oscillating back and forth until it comes to rest at the origin. Thus, the total number of half-oscillations that the mass makes is given by $n = \frac{A}{\left(\frac{2f}{k}\right)}$.

The period of oscillations of the mass is $T = 2\pi\sqrt{\frac{m}{k}}$. So the required time taken for the mass to come to rest is given by:

$$\frac{nT}{2} = \frac{\pi A}{2f}\sqrt{mk} \approx \boxed{25.8 \text{ s}}$$

As an extension to the problem, consider what happens if the initial extension A was slightly different such that the mass no longer comes to a permanent rest at the origin. In that case, static friction plays a more important role that cannot be neglected.

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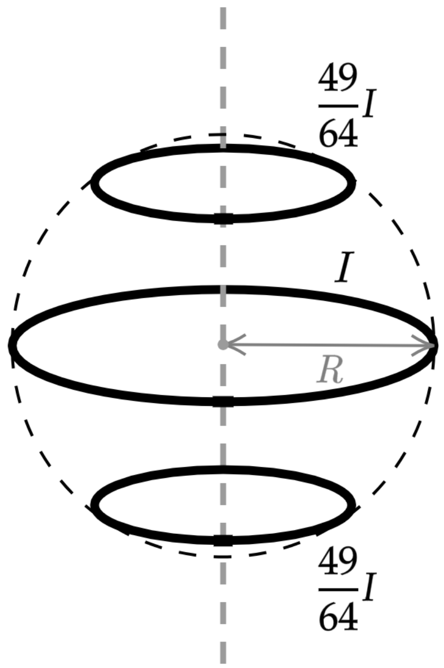
Problem 46: Super Uniform Field (5 points)

James sets up a magnetic field by using 3 coils. The planes of the coils are parallel to one another, and the centres of the coils are collinear (in other words, the coils share a common axis). The central coil has a radius of $R = 20$ cm, and carries a current of $I = 1.0$ A, while the other 2 coils are smaller and identical in size. They each carry a current of $49I/64$, and are equidistant from the central coil. Furthermore, a sphere of radius R can be drawn such that all 3 coils lie on its surface. The first, second, third, fourth and fifth derivatives of the magnetic field strength with respect to position along the common axis, taken at the centre of the central coil, are all zero. Find the magnetic field strength at the centre of the central coil.

Leave your answer to 2 significant figures in units of μT.



Solution: Let the smaller coils have radii r, and perpendicular distance x from the central coil. Since the 3 coils lie along the surface of a sphere, we have the constraint:

$$x = \sqrt{R^2 - r^2}$$

We define the z axis to be the common axis of the 3 coils, with $z = 0$ at the centre of the central coil, and the centres of the other 2 coils at $z = \pm x$. Now recall that for a single current loop of radius a, the magnetic field strength along its axis is given by $B = \frac{\mu_0 I}{2} \frac{a^2}{(y^2+a^2)^{\frac{3}{2}}}$, where y is the distance from the centre of the loop. The magnetic field $B(z)$ along the common axis of the 3 coils is then:

$$B(z) = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{\frac{3}{2}}} + \frac{\mu_0(\frac{49}{64}I)r^2}{2[(z+x)^2 + r^2]^{\frac{3}{2}}} + \frac{\mu_0(\frac{49}{64}I)r^2}{2[(z-x)^2 + r^2]^{\frac{3}{2}}}$$

In order to find $B(0)$, it suffices to find r in terms of R. We use the information provided in the question of the derivatives of $B(z)$ at $z = 0$.

$\frac{\partial B}{\partial z}|_{z=0} = 0$ for all r.

$\frac{\partial^2 B}{\partial z^2}|_{z=0} = 0$ yields:

$$\frac{49\mu_0 I r^2}{128}\left[\frac{30(R^2 - r^2)}{R^7} - \frac{6}{R^5}\right] = \frac{3\mu_0 I}{2R^3}$$

Solving for r, we obtain $r = \sqrt{\frac{8}{35}}R$ and $r = \sqrt{\frac{4}{7}}R$.

One of these solutions must be excluded, hence we continue to check the higher derivatives.

$\frac{\partial^3 B}{\partial z^3}|_{z=0} = 0$ for all r.[2]

$\frac{\partial^4 B}{\partial z^4}|_{z=0} = 0$ yields:

$$\frac{49\mu_0 I r^2}{128}\left[\frac{90}{R^7} - \frac{1260(R^2 - r^2)}{R^9} + \frac{1890(R^2 - r^2)^2}{R^{11}}\right] = -\frac{45\mu_0 I}{2R^5}$$

We find that $r = \sqrt{\frac{8}{35}}R$ does not satisfy this, implying that $r = \sqrt{\frac{4}{7}}R$ is the desired radius. Therefore, the magnetic field strength at the centre of the central coil is:

$$B(0) = \frac{15\mu_0 I}{16R} \approx \boxed{5.9\ \mu\mathrm{T}}$$

Note that in our solution, we have assumed that the currents in all 3 coils flow in the same direction. If the currents in the 2 smaller coils travel in the direction opposite to that of the central coil, it is mathematically impossible for both the second and fourth derivatives of $B(z)$ at $z = 0$ to be zero.

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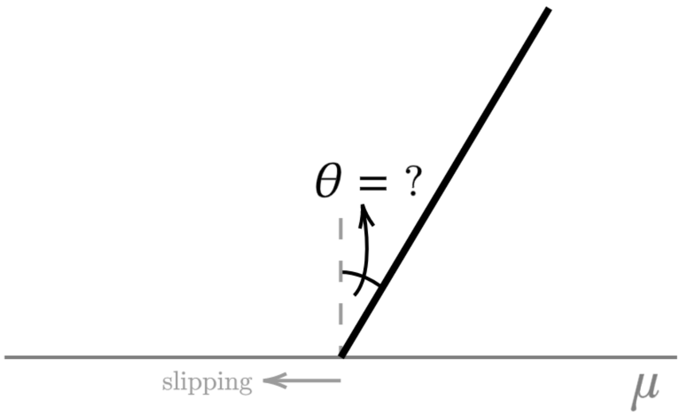
[2]Note that all odd derivatives of $B(z)$ are zero at $z = 0$ for all values of r. $B(z)$ is an even function that is symmetrical about $z = 0$. Since differentiating an even function yields an odd function and vice versa, it follows that all even derivatives of $B(z)$ are even functions, and therefore they are all symmetrical about $z = 0$. Thus, $z = 0$ must be a stationary point of the even derivatives of $B(z)$. Hence, the odd derivatives of $B(z)$ must all be zero at $z = 0$.

Problem 47: Slipping Stick (6 points)

A uniform stick is initially held vertical on a rough horizontal surface. The coefficient of static friction between the stick and the surface is $\mu = 0.250$. The stick is released from rest and falls. What is the angle θ between the stick and the vertical when the bottom end of the stick first starts slipping backwards across the surface?

Leave your answer to 3 significant figures in units of degrees.



Solution: This is an extremely difficult question.

Let us first assume that the bottom end of the stick does not slip, and determine the conditions required for such non-slip motion. Under this assumption, we can determine the stick's angular acceleration α by considering torque on the stick, with the contact point between the stick and the ground taken to be the origin. Let the angle of the stick from the vertical at a certain instant be θ.

The torque due to normal force and friction can be neglected due to our choice of origin. The only torque thus comes from gravity, which produces torque $mg\frac{l}{2}\sin\theta$. Since the stick is uniform, the moment of inertia about its end is $I = \frac{1}{3}ml^2$. Hence, using rotational Newton's Second Law, we can determine α:

$$mg\frac{l}{2}\sin\theta = \frac{1}{3}ml^2\alpha \implies \alpha = \frac{3g}{2l}\sin\theta$$

This enables us to determine the magnitude of tangential acceleration a_t of the stick's centre:

$$a_t = \frac{l}{2}\alpha = \frac{3g}{4}\sin\theta$$

Another property of the stick's centre that is of interest to us is the magnitude of its radial acceleration, a_r. Since the stick's centre is in circular motion (where the centre of the circle is its contact point with the ground, and the radius of this circle is $\frac{l}{2}$):

$$a_r = \frac{l}{2}\omega^2$$

where ω is the angular velocity of the stick. To express this fully in terms of θ, we need to find a relation between ω and θ. This may be done by conserving energy. Since the decrease in stick's GPE goes into increase in its rotational KE:

$$mg\frac{l}{2}(1-\cos\theta) = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2 \implies \omega = \sqrt{\frac{3g(1-\cos\theta)}{l}}$$

This enables us to re-express a_r purely in terms of θ:

$$a_r = \frac{3g}{2}(1-\cos\theta)$$

With a_r and a_t, we can also find a_x and a_y, the horizontal and vertical components respectively of the acceleration of the stick's CM. The rightward direction is taken to be positive for a_x, while the downward direction is taken to be positive for a_y. This can be done using trigonometry:

$$\begin{aligned} a_x &= a_t\cos\theta - a_r\sin\theta = \frac{9g}{4}\sin\theta\cos\theta - \frac{3g}{2}\sin\theta \\ a_y &= a_t\sin\theta + a_r\cos\theta = \frac{9g}{4}\sin^2\theta + \frac{3g}{2}\cos\theta - \frac{3g}{2} \end{aligned}$$

We can then relate the acceleration of the CM to the normal force N and friction force f acting on the stick, using translational Newton's Second Law:

$$\begin{aligned} mg - N = ma_y &\implies N = \frac{5mg}{2} - \frac{3mg}{2}\cos\theta - \frac{9mg}{4}\sin^2\theta \\ f = ma_x &\implies f \ = \frac{9mg}{4}\sin\theta\cos\theta - \frac{3mg}{2}\sin\theta \end{aligned}$$

Since we are operating under the assumption that the bottom end did not slip, the inequality $f \leq \mu N$ must be satisfied. Therefore, we have the following condition:

$$\frac{f}{N} = \frac{9\sin\theta\cos\theta - 6\sin\theta}{10 - 6\cos\theta - 9\sin^2\theta} \leq \mu$$

Graphically, it can be seen that the first value of θ when this inequality is defied is $\theta = \boxed{19.7°}$. This indicates that the non-slip condition is no longer satisfied, and the bottom of the stick has begun to slip.

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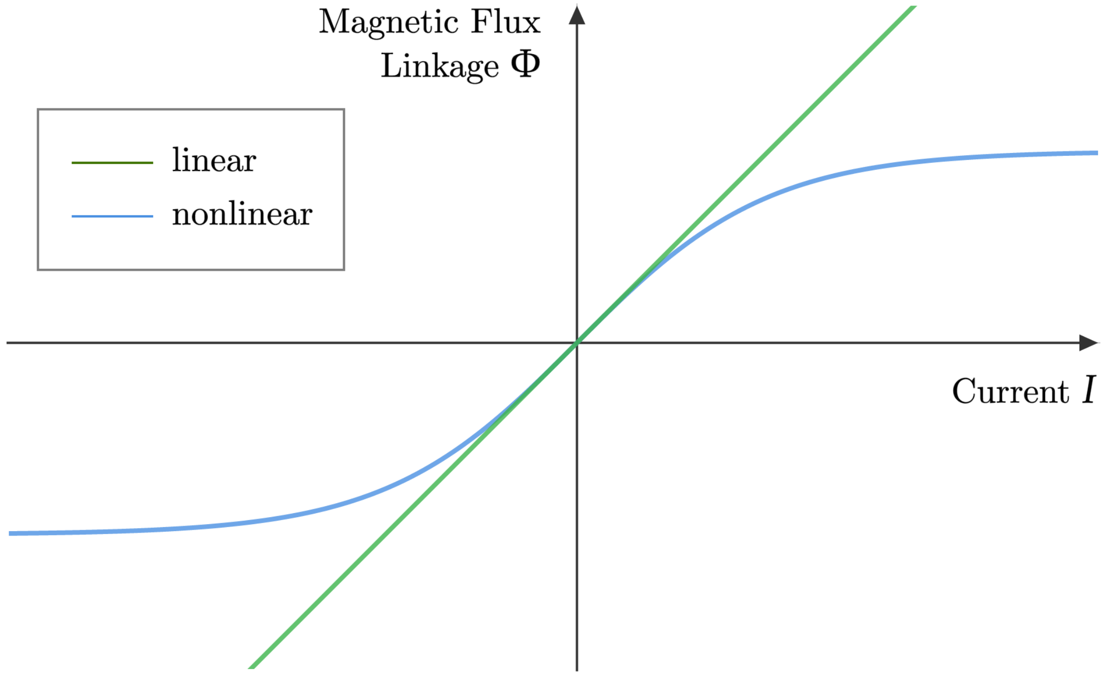
Problem 48: Nonlinear Inductors (5 points)

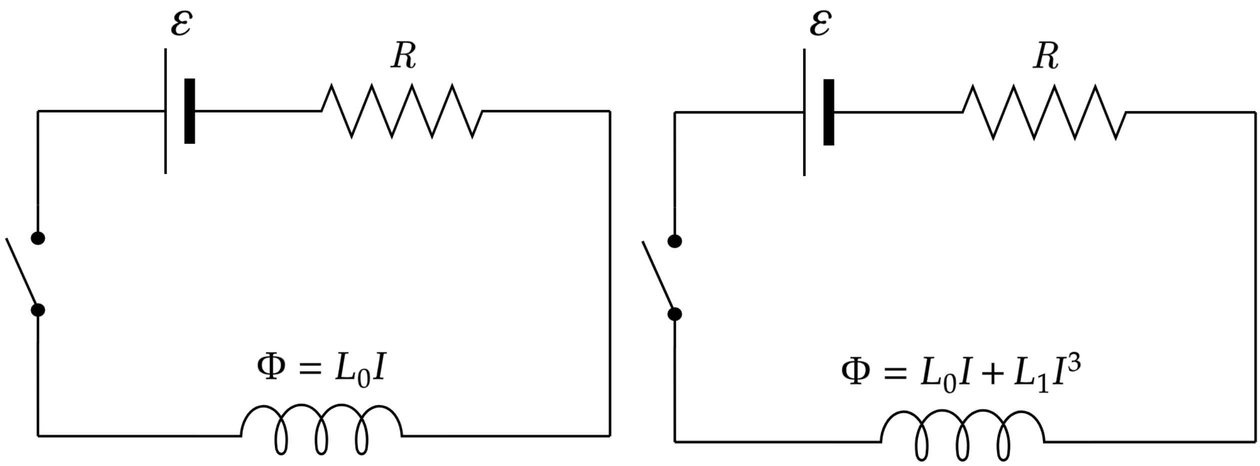
An inductor models the relationship between the magnetic flux linkage Φ and the current I through a coil. Generally, the magnetic flux linkage is modelled to be directly proportional to current in the coil, i.e. $\Phi(I) = LI$, where the inductance L is a constant independent of current. We shall call these *linear inductors*. In practice, however, many of these inductors are built with magnetic cores that exhibit saturation and other effects that cause the flux to "taper off" from proportionality when the current grows large, and these inductors are therefore described as *nonlinear*.



One way to capture these effects is to expand $\Phi(I)$ as an odd power series in I:

$$\Phi(I) = L_0 I + L_1 I^3 + L_2 I^5 + \ldots \tag{1}$$

where $L_0, L_1, L_2, \ldots$ are constants. For small I, all terms but the first become negligible and we recover the direct proportionality relationship $\Phi(I) \approx L_0 I$. However, for slightly larger currents, we will keep the first two terms in Eq. (1) and consider a nonlinear inductor with the flux-current relationship $\Phi(I) = L_0 I + L_1 I^3$. We will then compare its behavior to a linear inductor with the ordinary flux-current relationship $\Phi(I) = L_0 I$.



An RL circuit consists of a resistor, inductor, voltage source and an open switch in series, where the resistor has resistance $R = 1.00\ \Omega$ and the voltage source drives with emf $\mathcal{E} = 1.00$ V. Suppose that two such circuits are set up using the two different inductors mentioned previously. The switch is closed and the time taken for the current to reach half its maximum possible value, t_0 and t_1, is measured for the linear and nonlinear inductor circuits respectively. What is the relative change in this time, $\frac{t_1 - t_0}{t_0}$, when switching from the linear inductor to the nonlinear one? Take $L_0 = 1.00$ H, $L_1 = -1.00 \times 10^{-2}$ H A^{-2}. Note that L_1 is negative.

Leave your answer to 3 significant figures.

Solution: First, let us note that the maximum possible current in both circuits is the equilibrium current that the circuit arrives at after a long time. At that point, since the current is steady, there is no voltage induced across both inductors, so they behave as simple wires. The current is therefore given by Ohm's law to be $I_m = \frac{\mathcal{E}}{R}$.

When the switch is closed, for both circuits, Kirchhoff's voltage law tells us that $\mathcal{E} + \mathcal{E}_L - RI = 0$. Faraday's law of induction tells us that the back emf from the inductor is given by $\mathcal{E}_L = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$.

For the linear inductor, its voltage is simply $\mathcal{E}_{L_0} = -\frac{\mathrm{d}}{\mathrm{d}t} L_0 I = -L_0 \frac{\mathrm{d}I}{\mathrm{d}t}$. Thus, the equation $\mathcal{E} + \mathcal{E}_L - RI = 0$ becomes $\mathcal{E} - L_0 \frac{\mathrm{d}I}{\mathrm{d}t} - RI = 0$, which can be manipulated to give:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathcal{E} - RI}{L_0}$$

To simplify our notation, let us introduce a dimensionless current $i := \frac{I}{I_m}$, such that the maximum current is simply $i_m = 1$. The former equation becomes:

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{R}{L_0}(1 - i)$$

Separating variables gives us:

$$\frac{L_0}{R}\int_0^i \frac{1}{1-i'}\,\mathrm{d}i' = \int_0^t \mathrm{d}t' = t$$

The lower limits reflect the fact that $i = 0$ when $t = 0$. This integral can be evaluated to give:

$$t = \frac{L_0}{R}\int_0^i \frac{\mathrm{d}i'}{1-i'} = -\frac{L_0}{R}\ln(1-i)$$

Substituting $i = \frac{1}{2}$, we obtain:

$$t_0 = \frac{L_0}{R}\ln 2$$

For the nonlinear inductor, on the other hand, the back emf is given by $\mathcal{E}_{L_1} = -\frac{\mathrm{d}}{\mathrm{d}t}(L_0 I + L_1 I^3) = -(L_0 + 3L_1 I^2)\frac{\mathrm{d}I}{\mathrm{d}t}$ by the chain rule. Therefore, the equation $\mathcal{E} + \mathcal{E}_L - RI = 0$ becomes:

$$\mathcal{E} - (L_0 + 3L_1 I^2)\frac{\mathrm{d}I}{\mathrm{d}t} - RI = 0$$

$$\implies \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathcal{E} - RI}{L_0 + 3L_1 I^2}$$

Once again, to simplify our working, we set $i := \frac{I}{I_m}$, and we also let $\beta := \frac{3L_1\mathcal{E}^2}{L_0 R^2}$. Thus, the above equation becomes:

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{R}{L_0}\frac{1-i}{1+\beta i^2}$$

$$\implies \frac{L_0}{R}\int_0^i \frac{1+\beta i'^2}{1-i'}\,\mathrm{d}i' = \int_0^t \mathrm{d}t' = t$$

This can be evaluated to give:

$$t = \frac{L_0}{R}\left(\int_0^i \frac{\mathrm{d}i'}{1-i'} + \beta\int_0^i \frac{i'^2}{1-i'}\,\mathrm{d}i\right) = -\frac{L_0}{R}\left[\ln(1-i) + \beta\left(i + \frac{i^2}{2} + \ln(1-i)\right)\right]$$

where use has been made of the integral $\int_0^a \frac{x^2}{1-x}\,\mathrm{d}x = -a - \frac{a^2}{2} - \ln(1-a)$. Once again, substituting $i = \frac{1}{2}$ gives:

$$t_1 = \frac{L_0}{R}\left[\ln 2 + \beta\left(\ln 2 - \frac{5}{8}\right)\right]$$

Therefore, our final answer is:

$$\frac{t_1 - t_0}{t_0} = \left(1 - \frac{5}{8\ln 2}\right)\beta = \left(1 - \frac{5}{8\ln 2}\right)\frac{3L_1\mathcal{E}^2}{L_0 R^2} \approx \boxed{-2.95\times 10^{-3}}$$

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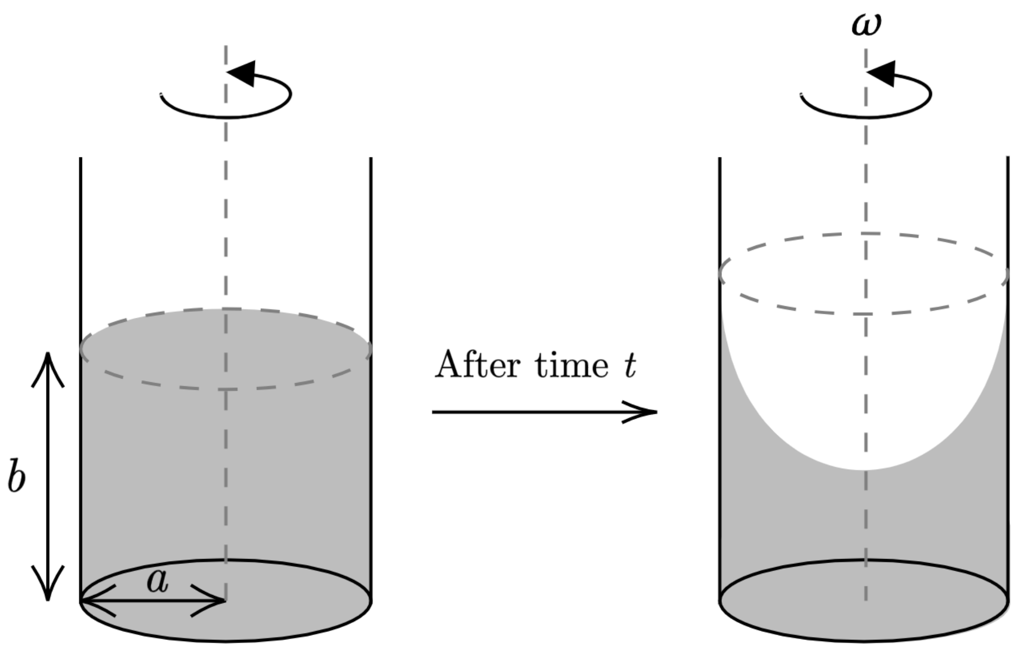
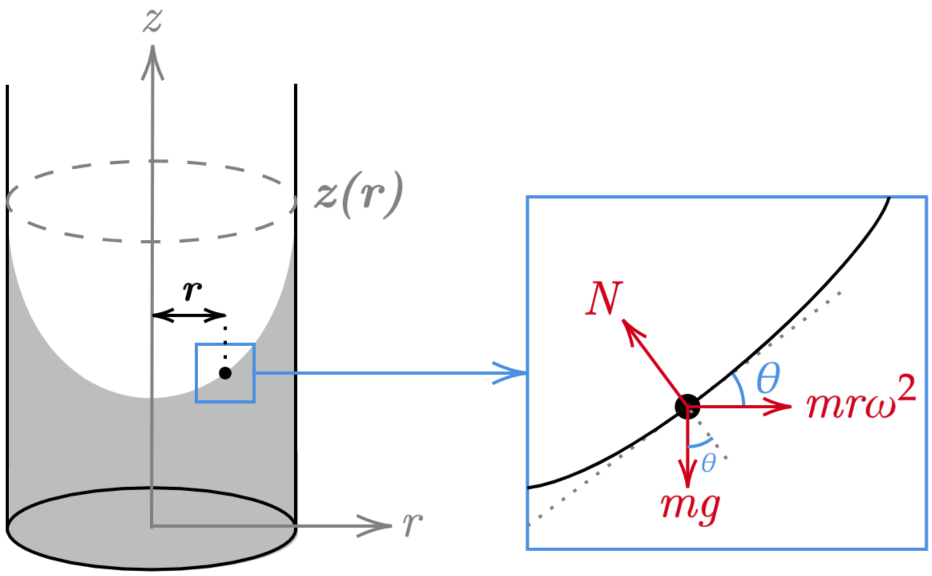
Problem 49: Spinning up a Bucket (6 points)

A tall cylindrical bucket of radius $a = 1.10$ m contains incompressible fluid of density $\rho = 100$ kg m^{-3} up to a height $b = 0.80$ m. A constant torque $\tau = 0.38$ N m then rotates the bucket and the fluid within it from rest. How much time t does it take for the whole system to reach an angular velocity of $\omega = 2.70$ rad s^{-1}? Suppose that the mass of the bucket is negligible compared to that of the fluid, that the time scale over which the angular velocity of the system changes is far longer than the time scale over which internal movement and deformations of the fluid surface reach equilibrium, and neglect any effects of surface tension.

Leave your answer to 3 significant figures in units of s.



Solution: Consider the moment at which the container rotates with angular velocity ω. The equilibrium surface is cylindrically symmetric, and let the height of this surface at radius $r \leq a$ from the centre be $z(r)$. Consider a fluid element on the surface.



Since the surface is in equilibrium, the net force tangential to the fluid surface acting on this element must be zero, and thus the corresponding components of gravity and centrifugal force must balance:

$$mg\sin\theta = m\omega^2 r\cos\theta \implies \tan\theta = \frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\omega^2 r}{g}$$

Integrating to obtain the fluid height profile $z(r)$:

$$z(r) = z(0) + \int_0^r \frac{\omega^2 r}{g}\,\mathrm{d}r = z(0) + \frac{\omega^2 r^2}{2g}$$

Since the fluid is incompressible, its volume must be conserved, and the total volume must equal the original volume $\pi a^2 b$. The volume of a cylindrical shell of fluid of thickness $\mathrm{d}r$ at radius r is $\mathrm{d}V = 2\pi r z(r)\,\mathrm{d}r$, and thus the total volume is:

$$V = \pi a^2 b = \int \mathrm{d}V = \int_0^a 2\pi r z(r)\,\mathrm{d}r = \pi a^2 z(0) + \frac{\pi\omega^2 a^4}{4g}$$

This gives us an expression for $z(0)$:

$$z(0) = b - \frac{\omega^2 a^2}{4g}$$

Thus:

$$z(r) = b + \frac{\omega^2}{2g}\left(r^2 - \frac{a^2}{2}\right)$$

Since the fluid achieves this equilibrium shape given by $z(r)$ in a far more rapid time than that which the external torque takes to significantly change ω, we can simply model the fluid as a practically rigid body whose shape, and thus moment of inertia

I, varies with ω. At a given ω, the moment of inertia of a cylindrical shell of fluid of thickness $\mathrm{d}r$ at radius r is $\mathrm{d}I = r^2\,\mathrm{d}M = \rho r^2\,\mathrm{d}V = \rho r^2 \cdot 2\pi r z(r)\,\mathrm{d}r$, and thus:

$$\begin{aligned}
I = \int \mathrm{d}I &= 2\pi\rho \int_0^a r^3 z(r)\,\mathrm{d}r \\
&= 2\pi\rho \int_0^a r^3 \left(b + \frac{\omega^2}{2g}\left(r^2 - \frac{a^2}{2}\right)\right)\,\mathrm{d}r \\
&= \frac{2\pi\rho a^4 b}{4} + \frac{2\pi\rho\omega^2}{2g}\left(\frac{a^6}{6} - \frac{a^2}{2}\cdot\frac{a^4}{4}\right) \\
&= \frac{\pi\rho a^4}{2}\left(b + \frac{\omega^2 a^2}{12g}\right)
\end{aligned}$$

We can neglect the moment of inertia of the bucket since its mass is negligible compared to the fluid. We can thus write the total angular momentum as:

$$L = I\omega = \frac{\pi\rho a^4}{2}\left(b\omega + \frac{\omega^3 a^2}{12g}\right)$$

The equation of motion $\tau = \mathrm{d}L/\mathrm{d}t$ can simply be written $L = \tau t$, and thus:

$$t = \frac{L}{\tau} = \frac{\pi\rho a^4}{2\tau}\left(b\omega + \frac{\omega^3 a^2}{12g}\right) \approx \boxed{1430\ \mathrm{s}}$$

Note that the equation $\tau = I\alpha$ is not valid because I is a function of ω and is no longer a constant.

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Problem 50: A Strong Glow (6 points)

An exceedingly high-powered lamp shines on a perfectly reflective, horizontal circular mirror of diameter $d = 0.10$ m and mass $m = 0.20$ kg from $H = 0.25$ m below the mirror's centre. Taking the lamp to be a point source which radiates uniformly in all directions with ideal efficiency, what is the minimum power of the lamp, P, required to support the weight of the mirror?

Leave your answer to 3 significant figures in units of GW.

Solution: Consider a thin ring on the mirror of radius r and thickness $\mathrm{d}r$. Define θ as the angle between a light ray incident on the ring and the vertical. From geometry, $r = H\tan\theta$.

Consider a sphere centred on the position of the lamp whose radius matches the distance between a point on the ring and the lamp, i.e. $H/\cos\theta$. We project the ring onto the surface of this sphere. All the photons that reach the ring must pass through this projection. Likewise, all the photons that pass through this projection must reach the ring. Therefore, the number of photons that pass through the projection and that reach the ring are equal. As such, the power delivered to the ring can then be found by multiplying the uniform intensity at the surface of this sphere and the area of the projected ring.

Note that calculating this power by multiplying the intensity with the area of the original non-projected ring would be erroneous since the intensity is not uniform across this ring. Therefore, there is a need to consider the aforementioned sphere.

The area $\mathrm{d}A$ of the projected ring onto sphere is:

$$\mathrm{d}A = 2\pi r\frac{H}{\cos\theta}\,\mathrm{d}\theta = 2\pi H^2\frac{\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta$$

The intensity at the surface of the sphere is $I = \frac{P}{4\pi H^2}\cos^2\theta$. As such, power $\mathrm{d}\mathcal{P}$ delivered to the ring is:

$$\mathrm{d}\mathcal{P} = I\,\mathrm{d}A = \frac{P\sin\theta}{2}\,\mathrm{d}\theta$$

Each photon experiences a change in momentum when reflected, $\Delta p = 2p\cos\theta = \frac{2E}{c}\cos\theta$. Suppose that n photons are reflected off the ring in time Δt. The total energy of the photons would be $nE = \mathrm{d}\mathcal{P}\Delta t \implies \mathrm{d}\mathcal{P} = \frac{nE}{\Delta t}$.

By Newton's Second Law, the magnitude of the total force exerted by the ring on all the photons is $\frac{n\Delta p}{\Delta t}$. Then, by Newton's Third Law, the magnitude of the force exerted on the ring would be equal. Therefore, we may compute this force $\mathrm{d}F$:

$$\mathrm{d}F = \frac{n\Delta p}{\Delta t} = \frac{2nE}{c\Delta t}\cos\theta = \frac{2\,\mathrm{d}\mathcal{P}}{c}\cos\theta = \frac{P}{c}\sin\theta\cos\theta\,\mathrm{d}\theta$$

Integrating across all rings forming the circular mirror, we obtain the total force F on the mirror:

$$F = \int \mathrm{d}F = \frac{P}{c}\int_0^{\tan^{-1}(R/H)} \sin\theta\cos\theta\,\mathrm{d}\theta = \frac{P}{2c}\frac{R^2}{R^2+H^2}$$

where $R = d/2$ is the radius of the mirror.

To support the weight of the mirror, this total force must be, at minimum, equal in magnitude to the weight of the mirror. Hence:

$$mg = \frac{P}{2c}\frac{R^2}{R^2+H^2} \implies P = 2mgc\left(1+\frac{H^2}{R^2}\right)$$

Substituting, we obtain:

$$P \approx 3.06\times 10^{10}\ \mathrm{W} = \boxed{30.6\ \mathrm{GW}}$$

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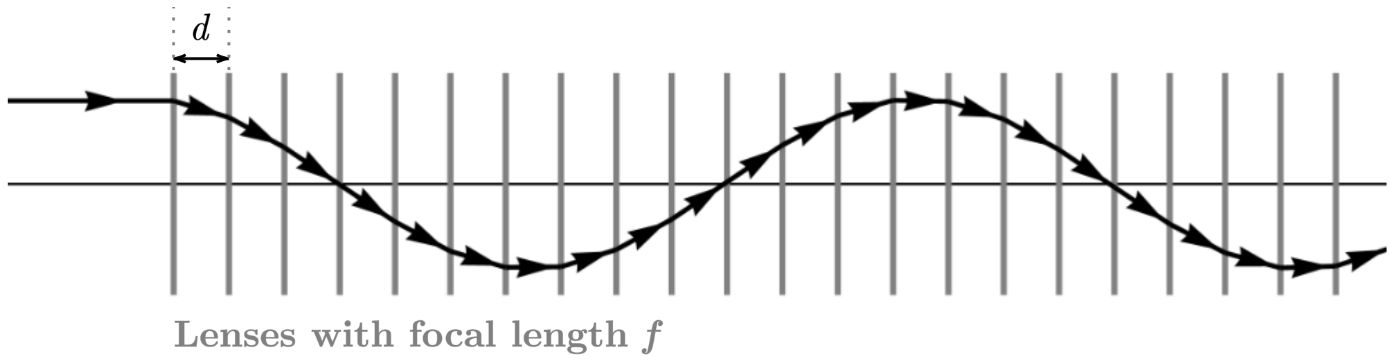
Problem 51: Compound Lens System (7 points)

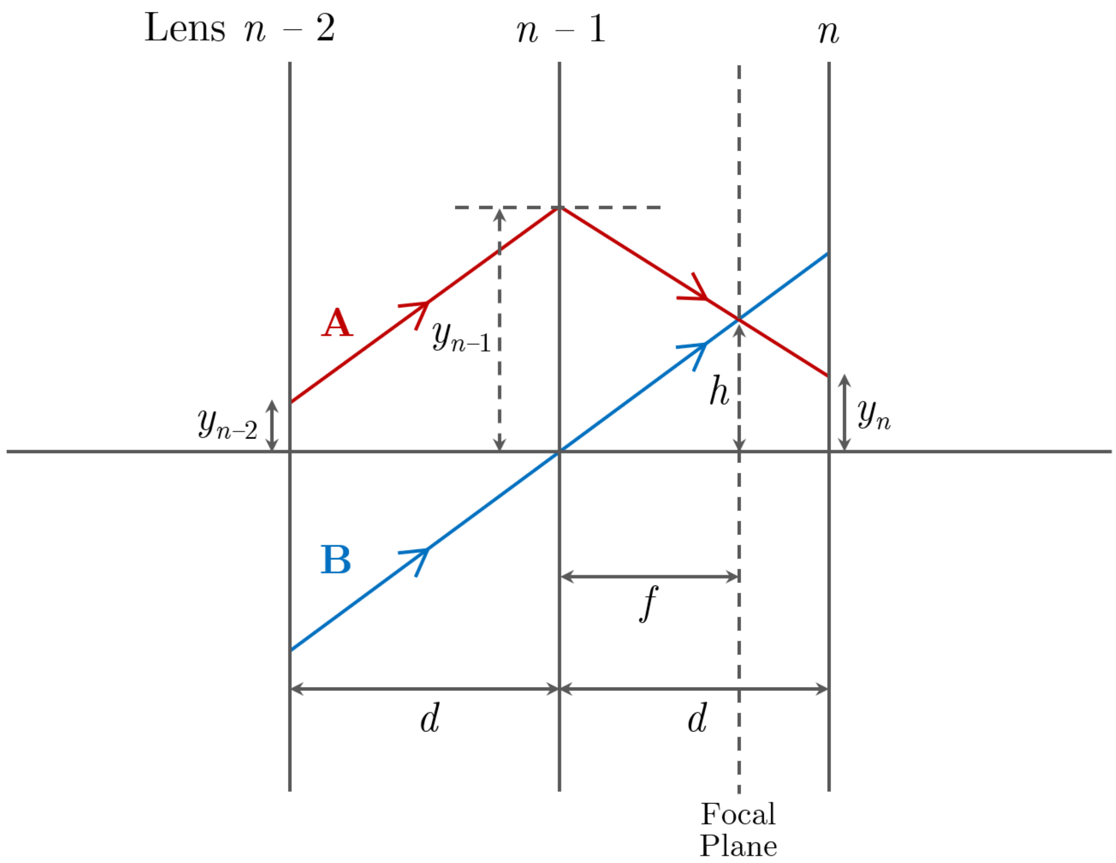
Identical thin convex lenses of focal length $f = 100$ mm are positioned along the same line, with equal separation $d = 19.8$ mm between adjacent lenses. The lenses are aligned such that they share the same horizontal principal axis. When a horizontal ray of light is shined through the first lens above its centre, the resulting path of the ray is periodic. Determine the wavelength of this path.

Leave your answer to 2 significant figures in units of mm.



Solution: In our analysis, we label the lens at the n^{th} position from the left as lens n ($n \in \mathbb{Z}^+$). Let y_n be the distance of the point where the light ray intersects the n^{th} lens above the principal axis. Note that $y_n < 0$ if the light ray hits the n^{th} lens below the principal axis.

To derive an equation governing y_n, consider the deflection of the light ray as it passes through the $(n-1)^{\text{th}}$ lens, which is labelled as ray A in the diagram below. We shall see that it is possible to determine the position y_n at which the ray hits the n^{th} lens, in terms of y_{n-1} and y_{n-2}.



Consider also ray B, an imaginary light ray drawn parallel to ray A that passes through the optical centre of lens $(n-1)$, such that it is undeflected as it goes through the lens. The gradient of this ray is equal to that of ray A before passing through the lens, $\frac{y_{n-1}-y_{n-2}}{d}$. The position of ray B on the focal plane of lens $(n-1)$ thus lies at height $h = \frac{f}{d}(y_{n-1} - y_{n-2})$ above the principal axis.

Parallel rays passing through a lens converge at the same point on the focal plane. Ray A and ray B must therefore intersect at the same point on the focal plane of lens $(n-1)$. As such, the gradient of ray A after going through the lens can be deduced to be $\frac{h-y_{n-1}}{f}$. The position y_n at which ray A intersects the n^{th} lens is therefore given by $y_n = y_{n-1} + \left(\frac{h-y_{n-1}}{f}\right) d$. Further simplification yields:

$$y_n = y_{n-1}\left(2 - \frac{d}{f}\right) - y_{n-2}$$

In order to solve this second-order linear recurrence relation, the values of y_1 and y_2 are required as initial conditions.[3] These are determined by the position and direction of the initial light ray. As given in the question, the light ray starts out parallel to the principal axis. Treating the ray to intersect the first lens at distance A from the

[3] Starting the light ray with different initial conditions leads to the resulting trajectory taking on a different amplitude and phase, but otherwise produces no significant effects.

principal axis, we have two initial conditions:

$$y_0 = A, \qquad y_1 = A\left(1 - \frac{d}{f}\right)$$

This recurrence can be solved by substituting a trial solution of the form $y_n = a\cos(kn - \phi)$, which yields the following expression for y_n:

$$y_n = \frac{A}{\cos\left(\frac{k}{2}\right)} \cos\left[k\left(n - \frac{1}{2}\right)\right]$$

where $k = \cos^{-1}\left(1 - \frac{d}{2f}\right)$. As such, the wavelength λ can be written as:

$$\lambda = \frac{2\pi d}{k} = \frac{2\pi d}{\cos^{-1}\left(1 - \frac{d}{2f}\right)} \approx \boxed{280\ \text{mm}}$$

Note that the factor of d is required since $\frac{2\pi}{k}$ gives the wavelength in terms of number of lenses traversed, but we are looking for the wavelength in terms of actual spatial distance.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 52: The Sun is a Deadly Laser (7 points)

A rigid, hollow, thin-walled, airtight sphere filled with an ideal gas lies on the equator of Earth. The sphere has a surface area of $A = 1.00\ \text{m}^2$ and perfect thermal contact with the gas. Suppose the entire sphere and gas system behaves as an ideal blackbody with heat capacity $C = 50\ \text{kJ K}^{-1}$. During the day, the intensity of light incident at the position of the sphere can be assumed to be constant at $I = 1.361\ \text{kW m}^{-2}$, while during the night, no light is incident on the sphere. As such, the pressure of the gas in the sphere varies over time. Let the minimum and maximum pressures of the gas be P_1 and P_2 respectively. Find the ratio P_1/P_2.

You may assume that the sphere is thermally isolated from the Earth by enclosing it in a transparent vacuum chamber, so that radiation is the only form of heat transfer.

Leave your answer to 3 significant figures.

Solution: Since the gas is ideal and has a constant volume, we can write $\frac{P_1}{P_2} = \frac{T_1}{T_2}$, where T_1 and T_2 are the minimum and maximum temperatures of the gas respectively. Our goal is to find T_1 and T_2, which require finding its temperature as a function of time $T(t)$.

Let the duration of a day be $t_0 = 24$ hours. During the 12 hours when the gas is not exposed to sunlight, the gas loses heat via radiation. We can write a differential equation for its temperature $T(t)$ in the night:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = -A\sigma T^4$$

During the 12 hours when the gas is exposed to sunlight, the gas continues radiating heat. Simultaneously, it also gains heat from absorbing sunlight. The rate of heat gain from sunlight is $I(\pi R^2) = \frac{IA}{4}$, so in the day:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = -A\sigma T^4 + \frac{IA}{4}$$

Defining $t = 0$ to be the beginning of the day, we thus have the following differential equations:

$$\begin{cases} C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{IA}{4} - A\sigma T^4, & 0 < t \leq \frac{t_0}{2} \\ C\frac{\mathrm{d}T}{\mathrm{d}t} = -A\sigma T^4, & \frac{t_0}{2} < t \leq t_0 \end{cases}$$

To simplify these equations, we introduce the function $x(\tau)$ and the numerical constant

κ, defined as follows:

$$x = \left(\frac{4\sigma}{I}\right)^{1/4} T(t)$$
$$\tau = \frac{\sigma^{1/4} I^{3/4} A}{2^{3/2} C} t$$
$$\kappa = \frac{\sigma^{1/4} I^{3/4} A}{2^{3/2} C} \frac{t_0}{2}$$

This enables us to re-express $T(t)$ and $\frac{\mathrm{d}T}{\mathrm{d}t}$ in terms of $x(\tau)$ and τ:

$$T(t) = \left(\frac{I}{4\sigma}\right)^{1/4} x$$
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \left(\frac{I}{4\sigma}\right)^{1/4} \cdot \frac{\mathrm{d}x}{\mathrm{d}\tau} \cdot \frac{\sigma^{1/4} I^{3/4} A}{2^{3/2} C} = \frac{IA}{4C} \cdot \frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Thus, we can rewrite the original equations involving $T(t)$:

$$\begin{cases} C \cdot \frac{IA}{4C} \cdot \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{IA}{4} - A\sigma \cdot \frac{I}{4\sigma} \cdot x^4, & 0 < \tau \leq \kappa \\ C \cdot \frac{IA}{4C} \cdot \frac{\mathrm{d}x}{\mathrm{d}\tau} = -A\sigma \cdot \frac{I}{4\sigma} \cdot x^4, & \kappa < \tau \leq 2\kappa \end{cases} \implies \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = 1 - x^4, & 0 < \tau \leq \kappa \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = -x^4, & \kappa < \tau \leq 2\kappa \end{cases}$$

At the steady state, $x(\tau)$ is periodic, i.e. $x(0) = x(2\kappa)$, and evidently the temperature T must be such that $\sigma A T^4 < IA \implies x < 1$, since any higher temperature would require a higher intensity than I to sustain in the long term.

In the day, when $0 < \tau \leq \kappa$, as the gas heats up, its temperature $x(\tau)$ strictly increases, since $\frac{\mathrm{d}x}{\mathrm{d}\tau} = 1 - x^4 > 0$. Likewise at night, when $\kappa < \tau \leq 2\kappa$, the gas cools down and its temperature $x(\tau)$ strictly decreases, since $\frac{\mathrm{d}x}{\mathrm{d}\tau} = -x^4 < 0$. As such, the gas achieves its maximum temperature $x(\kappa)$ at the end of the day and its minimum temperature $x(2\kappa) = x(0)$ at the end of the night, and the desired ratio is $\frac{P_1}{P_2} = \frac{T_1}{T_2} = \frac{x(0)}{x(\kappa)}$.

We will now solve the two differential equations by separation of variables. For the first equation, we have:

$$\kappa = \int_0^\kappa \mathrm{d}\tau = \int_{x(0)}^{x(\kappa)} \frac{\mathrm{d}x}{1 - x^4} = \frac{1}{2}\left[\tan^{-1} x + \tanh^{-1} x\right]_{x(0)}^{x(\kappa)}$$

For the second, we have:

$$\kappa = \int_\kappa^{2\kappa} \mathrm{d}\tau = \int_{x(\kappa)}^{x(2\kappa)=x(0)} \frac{\mathrm{d}x}{-x^4} = \frac{1}{3}\left[\frac{1}{x^3}\right]_{x(\kappa)}^{x(0)} = \frac{1}{3}\left[\frac{1}{x(0)^3} - \frac{1}{x(\kappa)^3}\right]$$

The second equation allows us to solve for $x(\kappa) = x(0)\left(1 - 3\kappa \cdot x(0)^3\right)^{-1/3}$. Substituting this back into the first equation gives:

$$2\kappa = \tan^{-1}\left[\frac{x(0)}{(1 - 3\kappa \cdot x(0)^3)^{1/3}}\right] - \tan^{-1} x(0) + \tanh^{-1}\left[\frac{x(0)}{(1 - 3\kappa \cdot x(0)^3)^{1/3}}\right] - \tanh^{-1} x(0)$$

We can solve this equation graphically or numerically for $x(0)$ to obtain:

$$x(0) \approx 0.619722$$

and our desired ratio is:

$$x(\kappa) = x(0)\left(1 - 3\kappa \cdot x(0)^3\right)^{-1/3} \implies \frac{x(0)}{x(\kappa)} = \left(1 - 3\kappa \cdot x(0)^3\right)^{1/3} = 0.626436... \approx \boxed{0.626}$$

Alternative solution: We may also employ computational methods to find $T(t)$.

We choose to use the first-order Euler method. To do this, we discretize $T(t)$, viewing it as a collection of discrete values T_i. From one discrete value to the next, we have a time step Δt. So T_i represents the temperature at time $i\Delta t$. We then recursively determine T_{i+1} in terms of T_i using the following relation:

$$T_{i+1} = \begin{cases} T_i - \frac{A\sigma T^4}{C}\Delta t, & t_0(n-1) \leq i\Delta t < t_0(n - \frac{1}{2}) \\ T_i + \left(\frac{IA}{4C} - \frac{A\sigma T^4}{C}\right)\Delta t, & t_0(n - \frac{1}{2}) \leq i\Delta t < t_0 n \end{cases}$$

where $n \in \mathbb{Z}^+$.

To implement this, any programming language would work. The step size Δt should be made small in order to best simulate the continuous nature of $T(t)$ with minimal error. Take note that initial conditions (i.e. the value of T_0) do not matter because the system eventually converges to a stable limit cycle. Hence, in our implementation, we must allow for sufficient time to have elapsed such that this is achieved before obtaining the T_1 and T_2 values.

Python code used is shown below. The step size `dt` can be as large as 1000 and still yield the same answer to the required number of significant figures. The value of `maxday` allows the programmer to observe as many days as is desired, up to a limit determined by the computer's memory. As such, discretion should be used in judging if the stable limit cycle has been reached.

```
# CONSTANTS #
period = float(24*60*60)       # period of day in seconds
a = 1                          # surface area
c = 50000                      # heat capacity
sunPower = 1361*a/4            # intensity * effective surface area exposed to sun
sb = 5.670374419e-8            # stefan-boltzmann constant
maxday = 10                    # number of days after which to stop simulating

# simulate temperature changes in a certain day
# T0: temperature at the start of the day
# day: which day we are looking at (varies from 1 to maxday)
def simulate(T0, day):
    dt = 0.1                   # time step
    length = int(period/dt)    # number of steps
```

```python
    Tmin = 1000                 # set Tmin to be rewritten
    Tmax = -1                   # set Tmax to be rewritten
    T = T0                      # begin at a certain starting temperature
    for i in range(length):
        pOut = sb*a*(T**4)      # power radiated due to blackbody radiation
        if i <= length/2:       # if during daytime (illuminated)
            pIn = sunPower      # sunlight is absorbed
        else:                   # if not
            pIn = 0             # no sunlight is absorbed
        du = (pIn-pOut)*dt      # change in internal energy
        dT = du/c               # change in temperature (isovolumetric)
        T = T + dT              # update temperature of the system
        if T < Tmin:            # if current temperature < lowest reached
            Tmin = T            # update Tmin, lowest reached temperature today
        if T > Tmax:            # if current temperature > highest reached
            Tmax = T            # update Tmax, highest reached temperature today
    print(day, Tmin, Tmax, Tmin/Tmax)
    if day < maxday:
        simulate(T, day+1)      # start new day (recursive)
    return

simulate(100, 1)                # simulate day 1 with starting temperature 100K
```

The table below shows the output of the code above:

Day	T_1 (K)	T_2 (K)	$\frac{T_1}{T_2}$
1	100.00066915925116	269.49718966810974	0.37106386631490906
2	171.55287190726256	275.29232812409566	0.6231661923754386
3	172.47473640366087	275.33820923761584	0.626410467625348
4	172.4812772403134	275.3385589593545	0.6264334276034876
5	172.4812776506867	275.3385616242639	0.6264334230308807
6	172.48127765381676	275.33856164456614	0.6264334229960582
7	172.48127765384075	275.3385616447208	0.6264334229957935
8	172.48127765384103	275.3385616447224	0.6264334229957909
9	172.48127765384103	275.33856164472246	0.6264334229957907
10	172.48127765384103	275.33856164472246	0.6264334229957907

After day 3, T_1, T_2 and $\frac{T_1}{T_2}$ begin to converge to fixed values, indicating that a stable limit cycle is reached. As such, $\frac{P_1}{P_2} = \frac{T_1}{T_2} \approx \boxed{0.626}$.

Setter: Galen Lee, galen.lee@sgphysicsleague.org

Half Hour Rush M1: Passing IPPT (3 points)

Rick and Morty are pre-enlistees. As part of their IPPT 2.4 km running assessment, they run around a circular track. They begin at the same starting point and start running simultaneously in the same direction. To complete each round, Rick takes $T_1 = 1$ min 30 sec, while Morty takes $T_2 = 2$ min 20 sec. How long does it take for Rick to lap Morty after they start running?

Leave your answer to 3 significant figures in units of s.

Solution: Rick and Morty's angular velocities can be written as $\omega_1 = \frac{2\pi}{T_1}$ and $\omega_2 = \frac{2\pi}{T_2}$ respectively. Their relative angular velocity ω_r can thus be written as $\omega_r = \omega_1 - \omega_2 = 2\pi\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$. Rick laps Morty when there is an angular displacement of 2π between them, which occurs in time interval T, where:

$$T = \frac{2\pi}{\omega_r} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \boxed{252 \text{ s}}$$

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Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Half Hour Rush M2: Ferry to Tekong (4 points)

A newly enlisted recruit travels by ferry from mainland Singapore to Pulau Tekong. At the start of the journey, the ferry accelerates forward uniformly. There is a marking on the floor of the ferry at the original position of the recruit. He jumps vertically upward relative to the ferry, such that his body's centre of mass rises by a maximum height $h = 0.50$ m above its original position. When he lands, he finds that he is now at distance $x = 0.15$ m behind the marking. What is the acceleration, a, of the ferry?

Leave your answer to 2 significant figures in units of $\mathrm{m\ s^{-2}}$.

Solution: We consider the reference frame of a stationary observer. Let the velocity of the recruit at the start of his jump have forward component u_x and vertical component u_y. Let the duration of the recruit's jump be t, forward displacement of the marking on the ferry floor after time t be s_f, and that of the recruit be s_r.

The initial forward component of the recruit's velocity would be equal to that of the ferry, since they initially travel together. However, once the recruit is in the air, gravity is the only force acting on him, thus he experiences no acceleration in the forward direction. This is in contrast to the ferry, which accelerates forward at a. We can write expressions for s_f and s_r:

$$\begin{aligned} s_f &= u_x t + \frac{1}{2}at^2 \\ s_r &= u_x t \end{aligned}$$

The distance that the recruit lands behind the marking is thus $x = s_f - s_r = \frac{1}{2}at^2$.

To determine the time t that the recruit spends in the air, we note that the recruit's centre of mass rises to a maximum height h, which gives us $u_y = \sqrt{2gh}$. The time of flight can then be found by solving for t:

$$u_y t - \frac{1}{2}gt^2 = 0 \implies t = \frac{2u_y}{g} = 2\sqrt{\frac{2h}{g}}$$

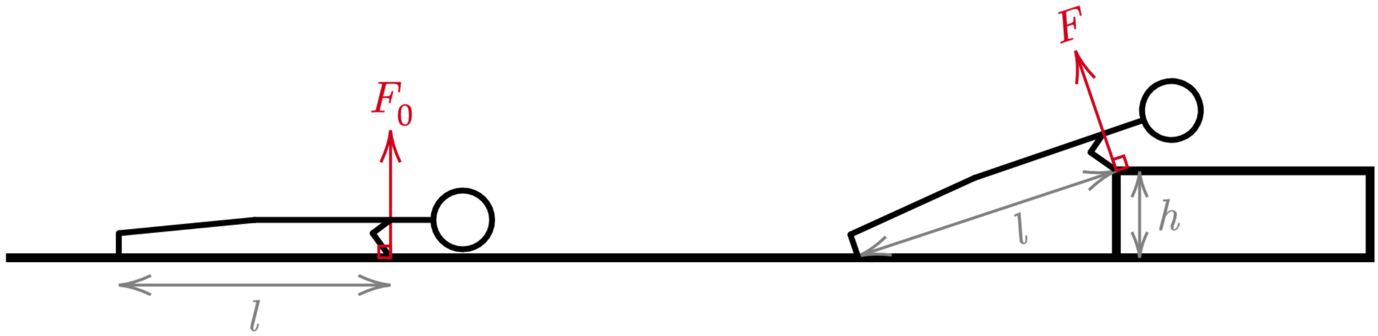
This enables us to complete our expression for $x = \frac{1}{2}at^2 = \frac{4ah}{g}$. Consequently, $a = \frac{gx}{4h} \approx \boxed{0.74\ \mathrm{m\ s^{-2}}}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

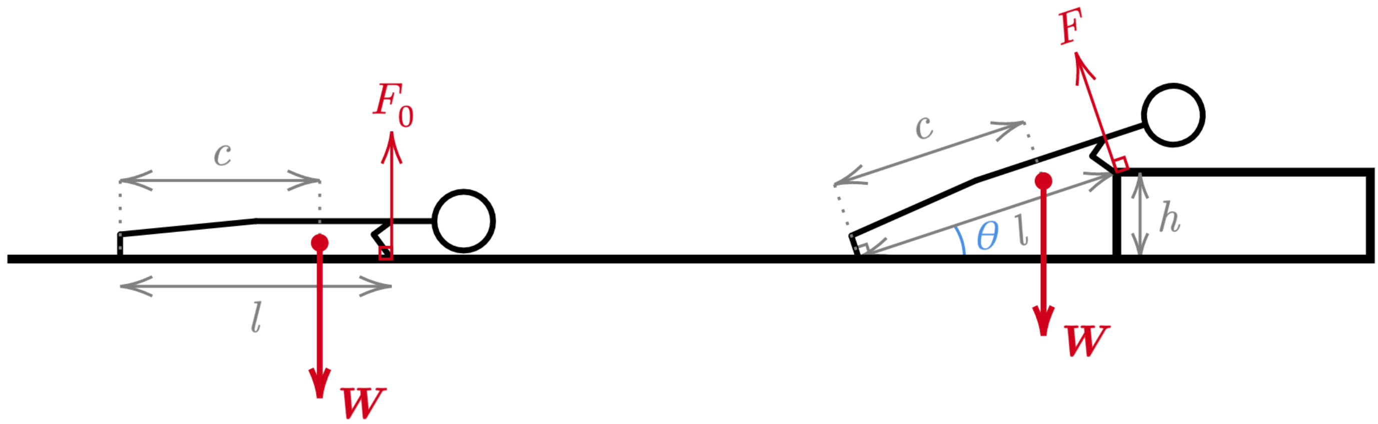
Half Hour Rush M3: Inclined Push-ups (4 points)

An inclined push-up is a push-up where one places their hands on a surface that is elevated relative to that on which their feet are resting, while in a normal push-up the hands and feet are on level ground. We will model a person as a rigid rod with two points of contact with the floor: one at the feet, and another at the hands, as shown. Suppose that the feet-to-hand distance is $l = 1.50$ m. We will assume that the forces on the hands are directed perpendicular to the person's body. If the combined force on the hands is $F_0/W = 64\%$ of body weight in a normal push-up, what will F/W be when the hands are elevated by height $h = 50$ cm above the level of the feet?

Leave your answer to 2 significant figures as a percentage of body weight. (For example, if you think the final answer should be $F/W = 0.51 = 51\%$*, input your answer as* 51*)*



Solution: As shown in the diagram, there are only three forces acting on the person: their weight W, and the forces at the two points of contact. Let the force on the hand be F, and let the centre of mass of the person be at distance c from the feet.



Letting the angle of the body from the horizontal be θ, we can deduce from geometry that:

$$\cos\theta = \frac{\sqrt{l^2 - h^2}}{l} = \sqrt{1 - \left(\frac{h}{l}\right)^2}$$

Since the person is in static equilibrium, the net torque about the feet must be zero:

$$W\cos\theta \times c = F \times l \implies \frac{F}{W} = \frac{c}{l}\cos\theta = \frac{c}{l}\sqrt{1-\left(\frac{h}{l}\right)^2}$$

In a normal push-up, $\theta = 0$, so $F_0/W = c/l = 0.64$. Therefore, in an inclined push-up where $h = 0.50\text{m}$:

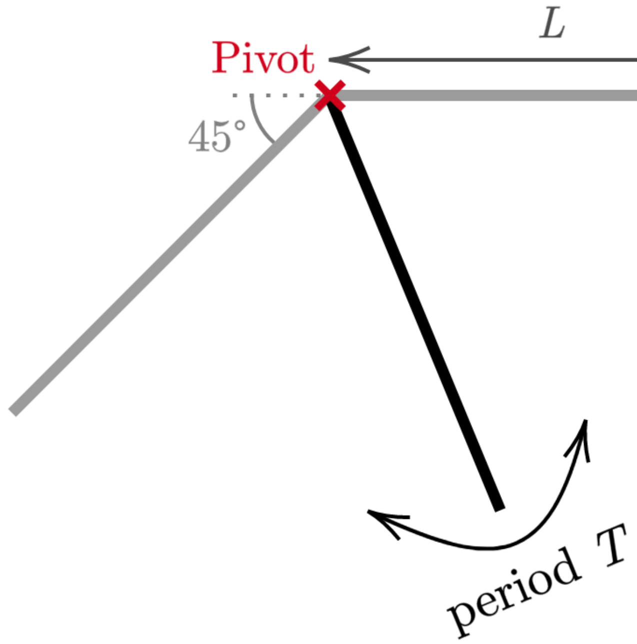
$$\frac{F}{W} = \frac{c}{l}\sqrt{1-\left(\frac{h}{l}\right)^2} \approx \boxed{60\%}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

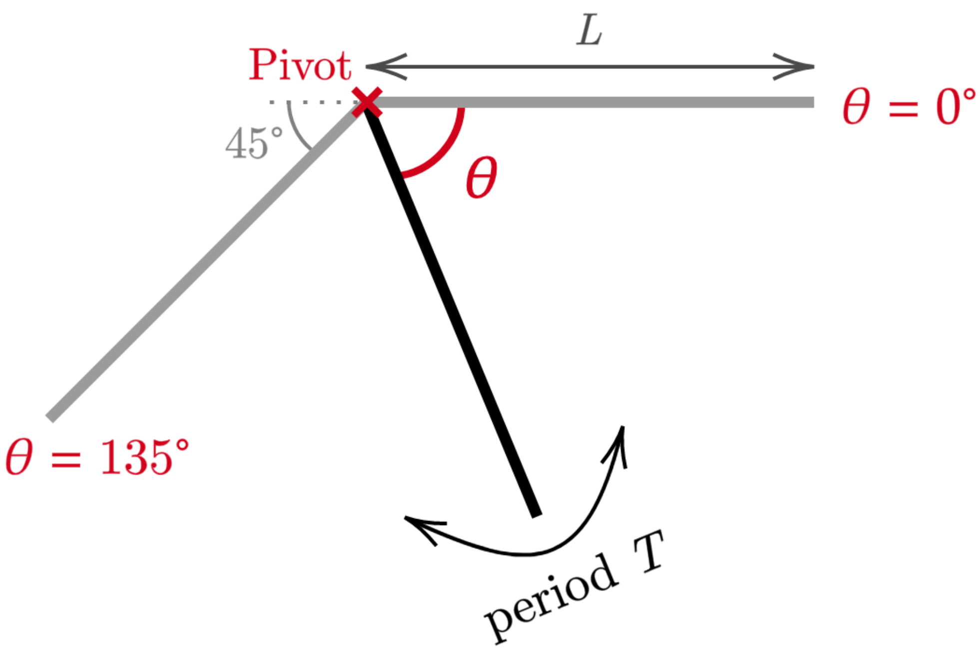
Half Hour Rush M4: Marching in the SAF (4 points)

In the Singapore Armed Forces, marching is done using the 90-45 rule. Soldiers swing their arms from a horizontal position in front of their body (90° to the vertical), to a diagonal position behind their body (45° to the vertical) and back repeatedly. To model the effect of the swinging motion on blood pressure, we treat the blood vessel in the arm as a sealed isolated narrow cylindrical tube completely filled with blood. As such, we may neglect atmospheric pressure and interactions with the rest of the body. The tube has length $L = 65$ cm and swings about its fixed end sinusoidally, taking $T = 1$ s to complete one oscillation. Given that the density of blood is $\rho = 1060$ kg m^{-3}, find the maximum blood pressure in the arm during the whole swinging process. You may neglect gravity.

Leave your answer to 3 significant figures in units of kPa.



Solution: Let $\theta(t)$ be the angle the arm makes with the horizontal at a time t. $\theta(t)$ evidently ranges from 0 to $\frac{3\pi}{4}$.



Suppose that the tube has cross sectional area A. Now, let $P(r)$ be a function describing the pressure along the tube, with r being the distance along the tube from the centre of rotation (hence the tube spans from $r = 0$ to $r = L$). We consider an infinitesimal portion of fluid in the tube from r to $r + \mathrm{d}r$. As this portion is in circular motion, the pressure gradient must supply a centripetal force. Newton's Second Law in the radial direction yields:

$$AP(r + \mathrm{d}r) - AP(r) = (\rho A\,\mathrm{d}r)r\dot{\theta}^2$$

Since θ oscillates sinusoidally, we have $\theta = \frac{3\pi}{8} + \frac{3\pi}{8}\sin\omega t$, where $\omega = \frac{2\pi}{T}$.

Combining the above two, we have:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{9\pi^2}{64}\rho r\omega^2\cos^2\omega t$$

Since we are neglecting atmospheric pressure, we have $P = 0$ at $r = 0$. Integrating with this condition:

$$P = \frac{9\pi^2\rho\omega^2 r^2\cos^2\omega t}{128}$$

Maximum pressure occurs at $r = L$, $\cos^2\omega t = 1$, so:

$$P_{\text{max}} = \frac{9\pi^2\rho\omega^2 L^2}{128}$$

Substituting in the values, we obtain the final answer as $P_{\text{max}} \approx \boxed{12.3\text{ kPa}}$.

Setter: Brian Siew, brian.siew@sgphysicsleague.org

Half Hour Rush E1: Plate Levitation 1 (3 points)

Jim wishes to prank his co-workers into believing that he has telekinetic powers. To do this, he wants to make an object levitate mid-air via electrostatic means.

He chooses to use a square metal plate of mass $m = 0.150$ kg. He charges the plate until it acquires positive charge $q = +1.0\ \mu$C. Then, he orients it such that it is horizontal, and secretly applies an upward electric field E, causing it to levitate upon release.

Determine the minimum E that Jim needs to apply for his plan to work.

Leave your answer to 2 significant figures in units of MN C^{-1}.

Solution: The electric field results in an upward electric force qE on the plate. For the plate to levitate, this electric force must be greater than or equal to the downward weight mg of the plate. As such:

$$qE \geq mg \implies E \geq \frac{mg}{q} \approx \boxed{1.5\text{ MN C}^{-1}}$$

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Half Hour Rush E2: Plate Levitation 2 (4 points)

Jim wishes to prank his co-workers into believing that he has telekinetic powers. To do this, he wants to make an object levitate mid-air via electrostatic means. Thanks to your calculation earlier, Jim knows that his prank can certainly work.

He chooses to use a square metal plate of side length $l = 0.20$ m. He charges the plate until it acquires positive charge $q = +1.0$ μC. Then, he orients it such that it is horizontal, and secretly applies an upward electric field $E = 6.0$ MN C^{-1}, causing it to levitate upon release.

As the plate levitates at equilibrium, determine the overall charge that resides across the entire area of the lower surface of the plate. You may neglect edge effects.

Leave your answer to 2 significant figures in units of μC.

Solution: Because the plate is conducting, the charges on the plate redistribute themselves across the top and bottom surfaces of the plate so that the electric field inside the plate is zero. Let the surface charge density for the top surface be σ_0, and for the bottom surface be σ_1. Since the net charge on the plate is q:

$$(\sigma_0 + \sigma_1)l^2 = q$$

Now invoking the condition that electric field inside the plate is zero, this means that the electric field due to the induced charges on the plate must be directed downward, and its magnitude must cancel that of the external field:

$$\frac{\sigma_0 - \sigma_1}{2\varepsilon_0} = E$$

Solving for σ_0 and σ_1:

$$\sigma_0 = \frac{q}{2l^2} + \varepsilon_0 E$$
$$\sigma_1 = \frac{q}{2l^2} - \varepsilon_0 E$$

Correspondingly, we can now determine the charge q_1 on the bottom surface of the plate:

$$q_1 = \sigma_1 l^2 = \frac{q}{2} - \varepsilon_0 E l^2 \approx \boxed{-1.6\ \mu\text{C}}$$

which is negative. This might seem surprising, but considering how strong the electric field is, it is certainly possible to draw so many positive charges to the top surface, to the extent that the bottom surface becomes negatively charged.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

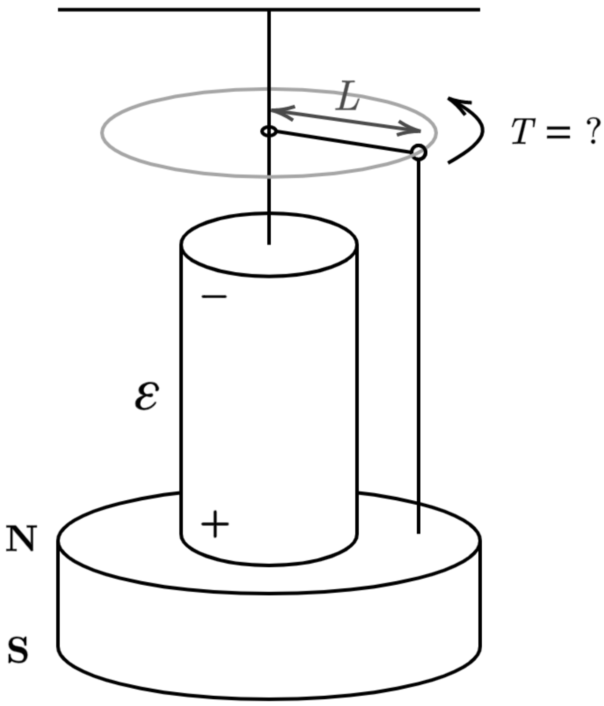
Half Hour Rush E3: An Inconspicuous Motor (4 points)

Jim's co-workers eventually got bored of the levitating plate. Jim feels that a little more motion is needed to reignite his co-workers' interests. So, he designs an inconspicuous motor.

The motor is constructed from a battery, wire, and magnet, as shown in the figure. The L-shaped wire is constrained to a vertical plane by a conducting vertical shaft and an insulating ring. The magnet produces a uniform vertical magnetic field of strength $B = 0.1$ T. The battery has voltage $U = 4.8$ V, and the wire has resistance $R = 0.52\ \Omega$. The horizontal section of the wire has length $L = 0.10$ m. Assume that the magnet, battery and shaft have no resistance.

A resistive torque $\tau_{\text{resistive}} = -k\omega$ acts on the wire, where ω is the angular frequency of the wire and damping coefficient $k = 0.060$ kg m s^{-1}. As a result, the wire will eventually rotate with a constant period T. Find T.

Leave your answer to 3 significant figures in units of s.



Solution: When the system is assembled, a current will flow through the wire. Using Ohm's Law, the magnitude of the current is $I = \frac{U}{R}$.

The flow of current through the horizontal section of the wire results in a magnetic force that acts perpendicularly to the wire. This magnetic force leads to a driving torque that causes the wire to rotate.

Consider an infinitesimal portion of the horizontal section of wire of length $\mathrm{d}l$ at distance l from the vertical shaft. The magnetic force acting on this portion is given by $\mathrm{d}F = BI\,\mathrm{d}l$. The torque on this portion due to the magnetic force is therefore $\mathrm{d}\tau = BIl\,\mathrm{d}l$.

The net torque acting on the wire due to the battery, τ_{battery}, can be found through integration:

$$\tau_{\text{battery}} = \int_0^L BIl\,\mathrm{d}l = \int_0^L \frac{BUl}{R}\,\mathrm{d}l = \frac{BU}{2R}[l^2]_0^L = \frac{BUL^2}{2R}$$

At steady state, the wire no longer experiences any angular acceleration, thus the net torque on the wire is zero. The wire then rotates at a constant angular frequency ω_f.

$$\begin{aligned}\tau_{\text{battery}} + \tau_{\text{resistive}} &= 0 \\ \frac{BUL^2}{2R} - k\omega_f &= 0 \implies \omega_f = \frac{BUL^2}{2Rk}\end{aligned}$$

From here, the period of the wire T can be determined:

$$T = \frac{2\pi}{\omega_f} = \frac{4\pi Rk}{BUL^2} \approx \boxed{81.7\text{ s}}$$

Setter: Luo Zeyuan, zeyuan.luo@sgphysicsleague.org

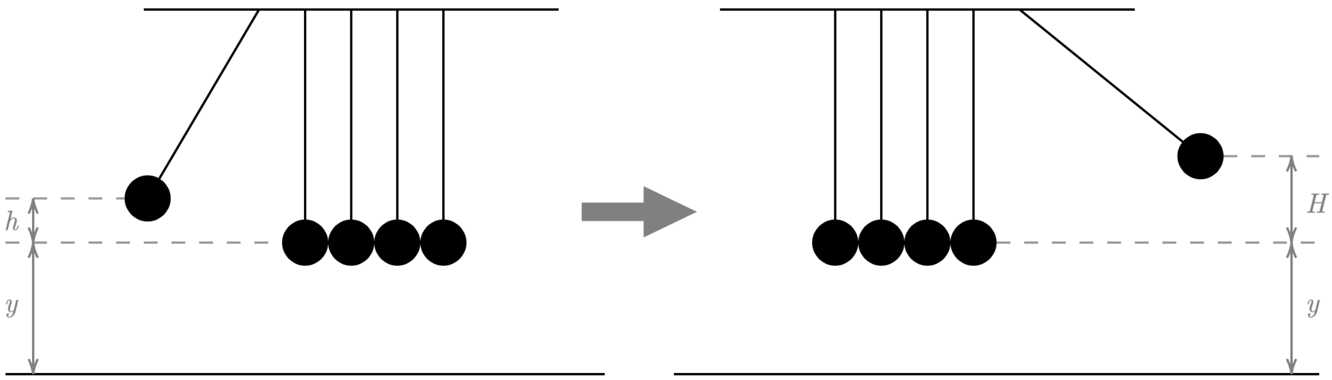
Half Hour Rush E4: Phony Newton's Cradle (4 points)

Jim's co-worker Dwight remains unconvinced of his telekinetic powers. Jim now opts to deceive Dwight using a phony Newton's Cradle. He designs a Newton's Cradle with balls of insulating material, which have identical size and mass $m = 100$ g.

However, he secretly charges the leftmost ball with $Q = +3.0\ \mu$C, while all other balls remain electrically neutral. His demonstration is done on a large, flat and horizontal metal table, which is grounded. There is a vertical distance $y = 20.0$ cm between the balls' resting positions and the tabletop.

When he displaces the leftmost ball by vertical height $h = 5.0$ cm (with the string remaining taut) and releases it from rest, the rightmost ball rises to a larger maximum height $H > h$. Find H. You may assume that all collisions are perfectly elastic and that the balls are point masses. Neglect any effects of polarisation or magnetism.

Leave your answer to 2 significant figures in units of cm.



Solution: The large metal table is a large conducting plate. Initially, the image charge $-Q$ is located at vertical distance $y+h$ below the tabletop, directly below the leftmost ball. The interaction between the charged leftmost ball and the image charge thus gives rise to electric potential energy E_i:

$$E_i = -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{2(y+h)}$$

Note the factor of $\frac{1}{2}$, which arises from the interaction between a real charge and an image charge. Upon collision, when the leftmost ball is now in a resting position, the image charge $-Q$ is now at vertical distance y below the tabletop. This leads to an electric potential energy E_f:

$$E_f = -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{2y}$$

The leftmost ball was released from rest, so there was no initial kinetic energy within the system. At the instant when the rightmost ball reaches its maximum height H, all balls are stationary, so the system also possesses no kinetic energy. Invoking the law of conservation of energy on the system of balls:

$$mgh + E_i = mgH + E_f$$

$$H = h + \frac{Q^2}{16\pi\varepsilon_0 mg}\left(\frac{1}{y} - \frac{1}{y+h}\right) \approx \boxed{7.1 \text{ cm}}$$

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Half Hour Rush X1: Magnifying Glass (3 points)

When an object of length $l = 1.0$ cm is placed in front of a magnifying glass, its image has length $L = 4.0$ cm. The magnifying glass is now moved such that its distance from the object is halved. What is the new length of the image formed? Model the magnifying glass as an ideal thin converging lens.

Leave your answer to 2 significant figures in units of cm.

Solution: Let the distance between the object and the magnifying glass be u, the distance between the image and the magnifying glass be v, and the focal length of the magnifying glass be f.

Using the thin lens equation, we can express v in terms of u and f:

$$v = \frac{1}{\frac{1}{f} - \frac{1}{u}}$$

For a magnifying glass to produce its desired virtual image, it is necessary that $u < f$, so $v < 0$. As such, the linear magnification of the magnifying glass (which is defined as the ratio of image size to object size) $M = \left|\frac{v}{u}\right| = -\frac{v}{u}$, which can be written as:

$$M = \frac{1}{1 - \frac{u}{f}}$$

Initially, the magnification $M = \frac{L}{l} = 4$, which means that $\frac{u}{f} = \frac{3}{4}$. Now, when the object distance u is halved, $\frac{u}{f}$ takes on a new value $\frac{u}{f} = \frac{3}{8}$. Recomputing M with this new value of $\frac{u}{f}$, $M = 1.6$. Hence, the new length of the object $L = Ml = \boxed{1.6 \text{ cm}}$.

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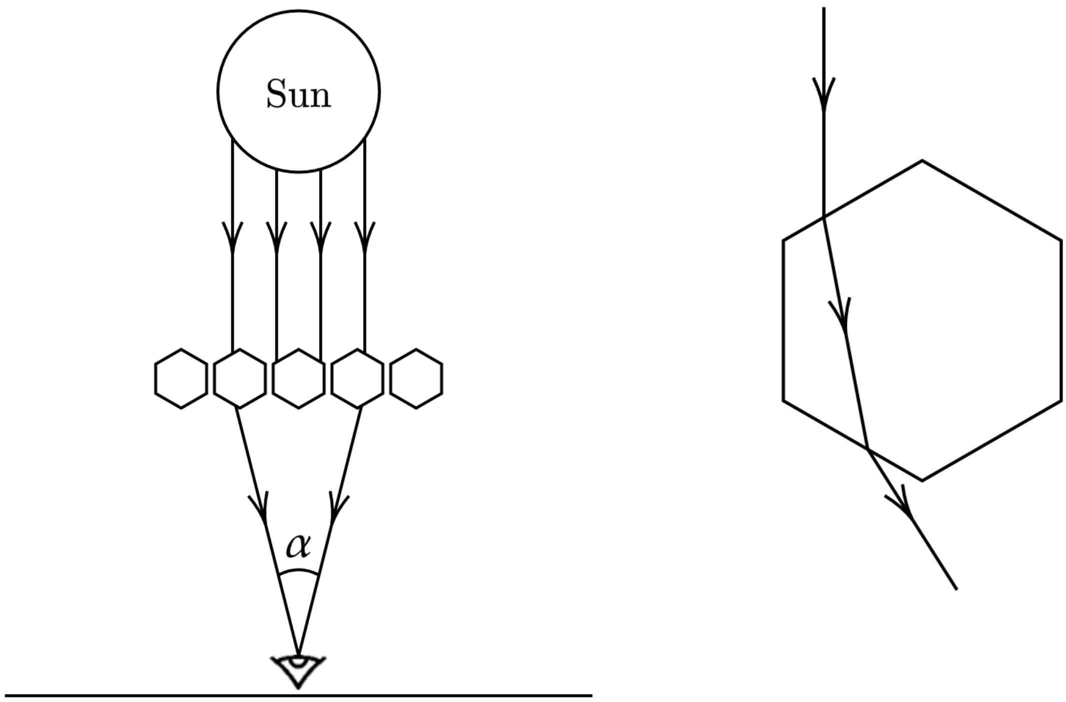
Half Hour Rush X2: Lights over Nuremberg (4 points)

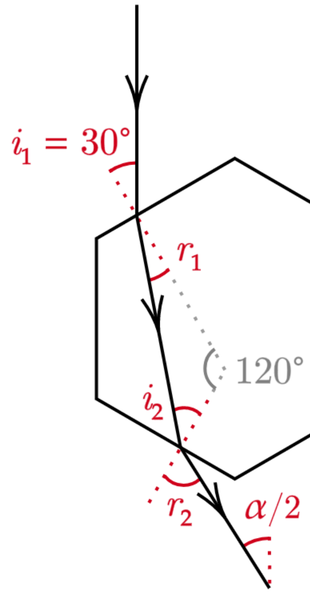
In April 1561, strange geometric objects were observed over the city of Nuremberg in Germany, seemingly engaged in aerial battle. Many explanations for this strange occurrence have been put forth, ranging from warnings from divine powers, to a space war between extra-terrestrials. However, most scientists today believe that the sightings are best explained by an optical phenomenon known as a *sun-dog*, consisting of a circular ring of light around the Sun. One theory suggests that these rings are formed when light from the Sun is refracted by regular hexagonal ice crystals suspended within the atmosphere.

For this problem, assume that the Sun is directly overhead and that all the ice crystals adopt the configuration shown below. Find the angle α subtended by the ring. Take the refractive index of ice to be $\eta_{\text{ice}} = 1.31$, and the refractive index of air to be $\eta_{\text{air}} = 1.00$.

Leave your answer to 5 significant figures in units of degrees.



Solution: This question can be solved by considering the refraction undergone by the light in the ice crystals. Since there is a bit of geometry involved, it will be useful to construct a diagram and define the variables before performing the calculations.



Let us the consider the point at which the light ray enters the crystal. We define the angle of incidence and refraction at that point to be i_1 and r_1 respectively. Due to the hexagonal geometry of the ice crystal, $i_1 = 30°$. Applying Snell's Law, we get:

$$\frac{\sin i_1}{\sin r_1} = \frac{\eta_{\text{ice}}}{\eta_{\text{air}}} = \eta_{\text{ice}} \implies r_1 = \sin^{-1}\left(\frac{\sin i_1}{\eta_{\text{ice}}}\right)$$

Now consider the point at which the light ray exits the crystal. Let the angles of incidence and refraction here be i_2 and r_2 respectively. By considering the geometry of the light ray in the figure above, we find:

$$i_2 = 180° - 120° - r_1$$

To find r_2, we apply Snell's Law and get:

$$\frac{\sin i_2}{\sin r_2} = \frac{\eta_{\text{air}}}{\eta_{\text{ice}}} = \frac{1}{\eta_{\text{ice}}}$$

We then manipulate the equation to make r_2 the subject, as follows:

$$r_2 = \sin^{-1}(\eta_{\text{ice}} \sin i_2) = \sin^{-1}\left[\eta_{\text{ice}} \sin(60° - r_1)\right]$$

It is clear that $\alpha/2$ represents the deviation of the emergent ray from the vertical. By geometry, this deviation is also given by $r_2 - i_1$. Therefore, we may form an equation to obtain the value of α:

$$\frac{\alpha}{2} = r_2 - i_1 \implies \alpha \approx \boxed{45.994°}$$

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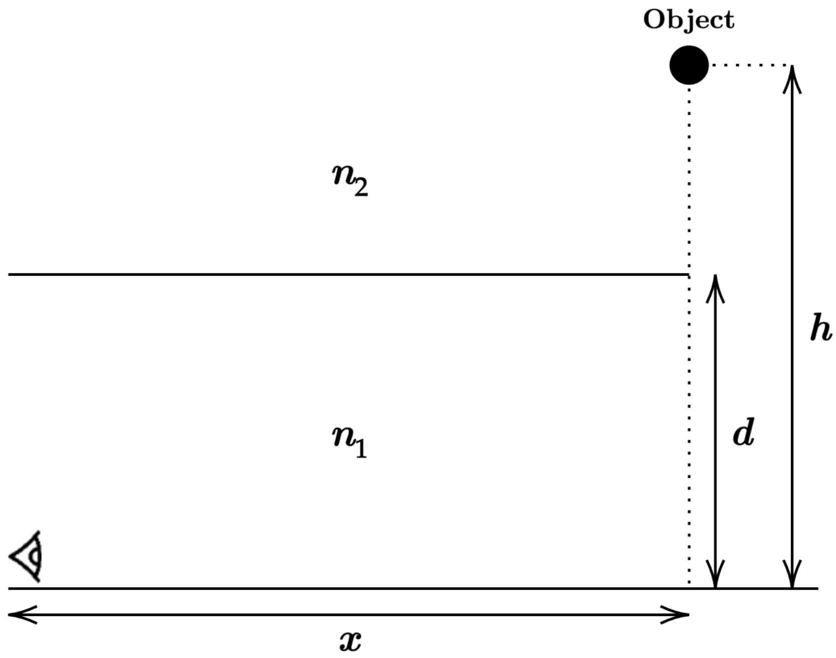
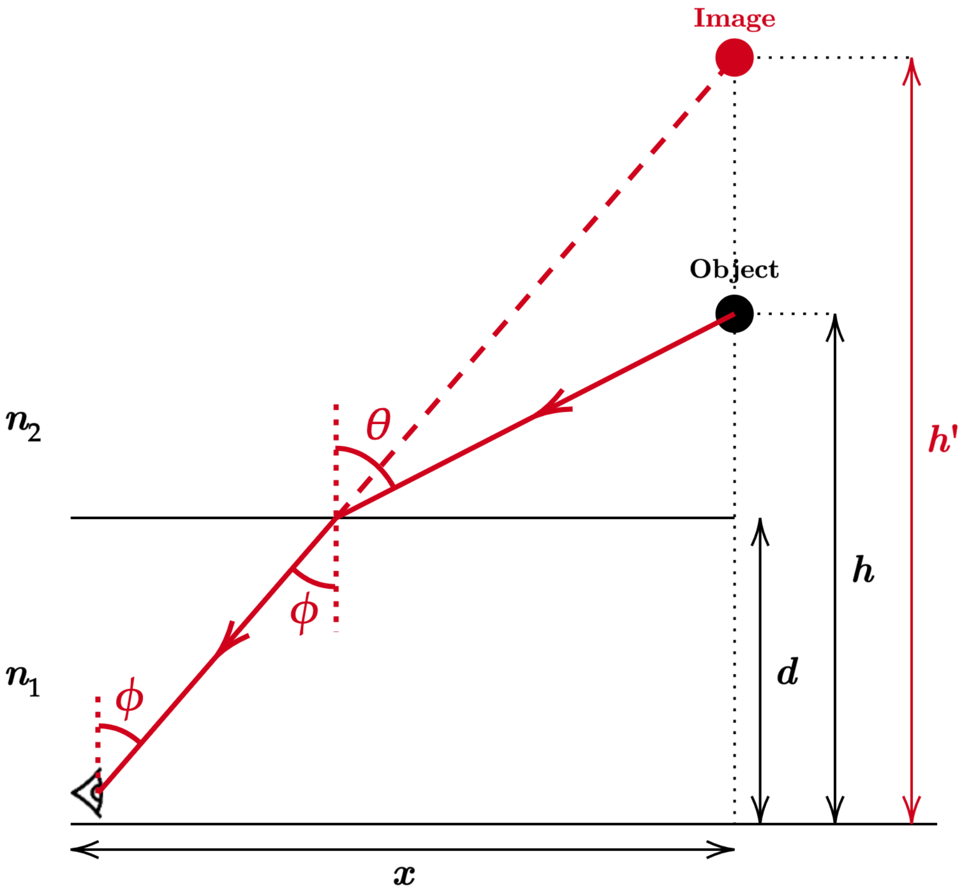
Half Hour Rush X3: Mirage (4 points)

In polar regions, air tends to be colder near the ground, causing the refractive index of air to vary with height. This results in an optical phenomenon known as a mirage. As a simplified model of this effect, let us suppose the air within distance $d = 3.00$ m from the horizontal ground has uniform refractive index $n_1 = 1.30$, while all the air above has refractive index $n_2 = 1.05$. If an object is placed at height $h = 5.00$ m above the ground and horizontal distance $x = 10.0$ m away from an observer, how high above the ground does the observer perceive the object to be?

Leave your answer to 2 significant figures in units of m.



Solution: When light travelling from the object to the observer reaches the interface between the air of two different refractive indices, it undergoes refraction, bending towards the normal. Let θ be the angle of the incident light ray from the normal, and ϕ be the angle of the refracted light ray from the normal, as illustrated below.



By Snell's Law, the two angles are related:

$$\frac{n_1}{n_2} = \frac{\sin\theta}{\sin\phi}$$

The object has to be at horizontal distance x from the observer, giving a geometric constraint to the light ray's path:

$$(h-d)\tan\theta + d\tan\phi = x$$

Putting these two equations together, we can form an equation for ϕ:

$$(h-d)\frac{\sin\phi}{\sqrt{\frac{n_2^2}{n_1^2} - \sin^2\phi}} + d\left(\frac{\sin\phi}{\cos\phi}\right) = x$$

This can be solved numerically, giving $\phi \approx 50.41°$. The perceived height h' is thus given by:

$$h' = \frac{x}{\tan\phi} \approx \boxed{8.3 \text{ m}}$$

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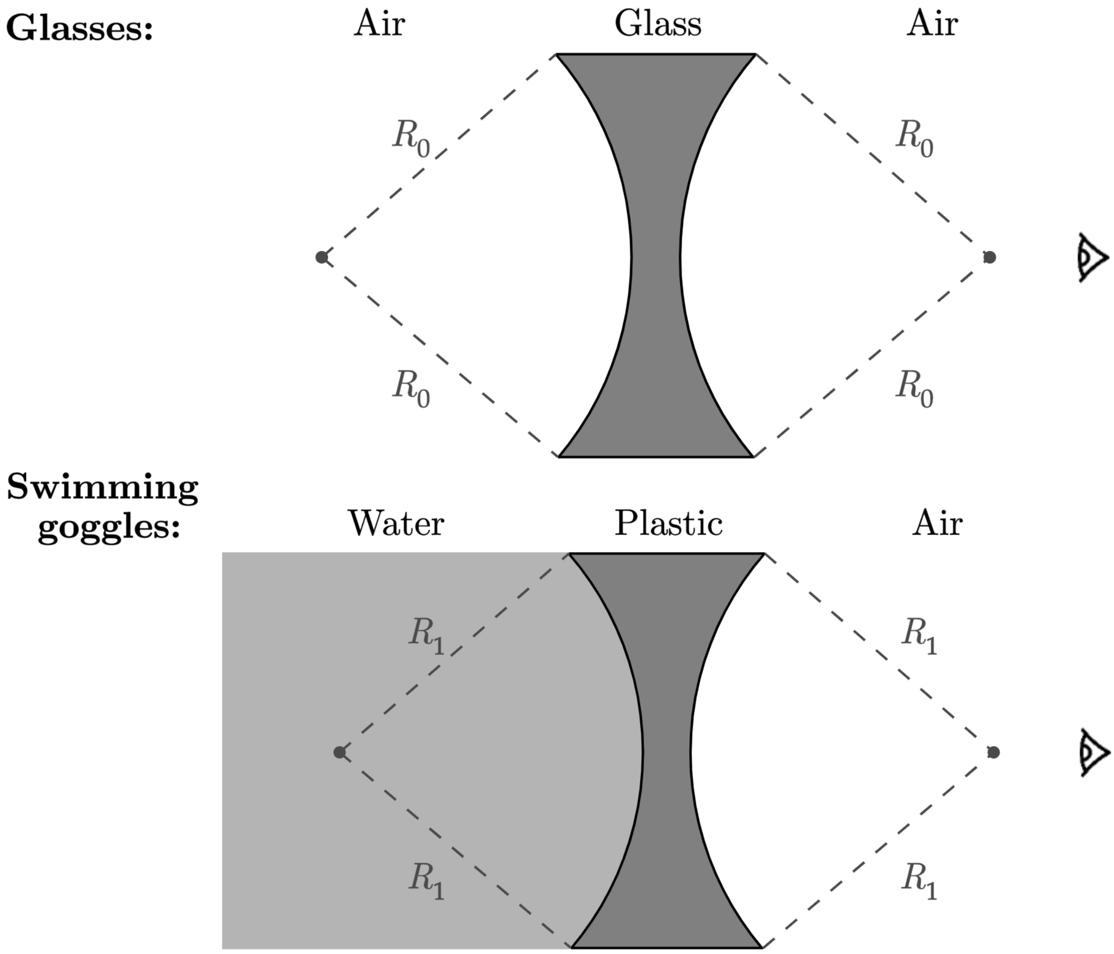
Half Hour Rush X4: Shortsighted Swimmer (4 points)

A biconcave thin lens is used to correct myopic vision and can be described with a radius of curvature R on both faces. In glasses, light from the surroundings travels through the media air $\rightarrow$ glass $\rightarrow$ air before reaching the eye, while in swimming goggles, light travels through the media water $\rightarrow$ plastic $\rightarrow$ air. A swimmer has glasses and goggles, both of which suit his myopia perfectly. Find the ratio R_1/R_0, where R_1 is the radius of curvature of the swimming goggles and R_0 is that of the glasses.

Take the refractive indices of glass and plastic to be $\eta = 1.5$ and the refractive index of water to be $\eta_{\text{water}} = 1.33$. Incident light may be assumed to consist of paraxial, parallel rays. The eye's position relative to the lens is the same in both cases, and lies along the lenses' principal axes.

Leave your answer to 2 significant figures.



Solution: The surface power of such a surface (for parallel, paraxial incident rays) is given by $P(R) = \frac{\eta_2 - \eta_1}{R}$, where light passes from a medium of refractive index η_1 to one of refractive index η_2. Note that R is positive for convex surfaces, and negative for concave surfaces, relative to the direction of light travel.

The air $\rightarrow$ glass and water $\rightarrow$ plastic surfaces, with respective powers P_1 and P_1', are concave, while the glass $\rightarrow$ air and plastic $\rightarrow$ air surfaces, with respective powers P_2 and P_2', are convex. This gives:

$$P_1 = \frac{\eta - \eta_{\text{air}}}{-R_0} \text{ and } P_1' = \frac{\eta - \eta_{\text{water}}}{-R_1}$$

$$P_2 = \frac{\eta_{\text{air}} - \eta}{R_0} \text{ and } P_2' = \frac{\eta_{\text{air}} - \eta}{R_1}$$

where $\eta_{\text{air}} = 1$.

The optical power of the lens is given by the sum of the surface powers. Hence, the optical power of the glasses and goggles would be $P = P_1 + P_2$ and $P' = P_1' + P_2'$ respectively. Since both lenses are designed with the same desired optical correction, $P = P'$. Hence:

$$P_1 + P_2 = P_1' + P_2' \implies \frac{2(\eta_{\text{air}} - \eta)}{R_0} = \frac{\eta_{\text{water}} + \eta_{\text{air}} - 2\eta}{R_1}$$

$$\therefore \frac{R_1}{R_0} = \boxed{0.67}$$

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