2^{nd} *Singapore Physics League*

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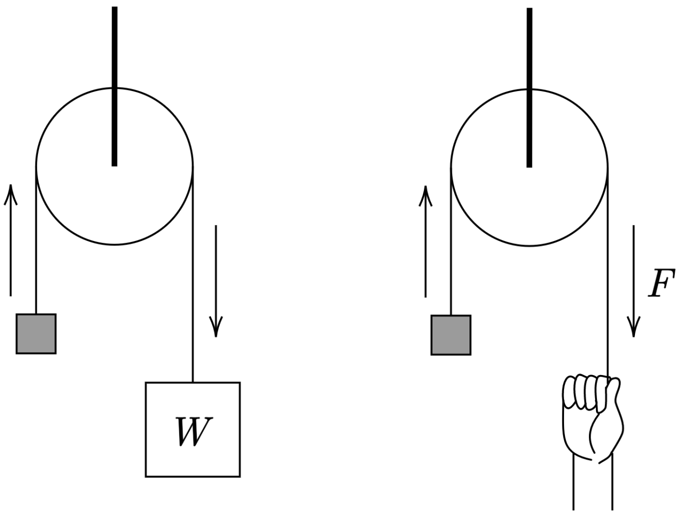
Problem 1: Pulley Pull (2 points)

A block is connected to a light inextensible string that loops over a massless pulley and is pulled by weight W. As a result, the block travels with acceleration a.

Now, instead of pulling on the string with weight W, we directly exert downward force F on the string. For the block to travel with the same acceleration a, how must F and W compare?



(1) $F = W$

(2) $F > W$

(3) $F < W$

Solution: In both cases, the block has the same acceleration, implying the tensions in the string must be equal in order to provide the same net force on the block. Let this tension be T.

For the first case, since the weight accelerates downward, there must be a net downward force acting on it. This means that $T < W$.

As for the second case, $T = F$ since the string is light, and tension is uniform.

Hence, the correct relation between F and W is $\boxed{(3)\ F < W}$.

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Problem A: No Speeding! (3 points)

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

Along a stretch of road of length $L = 1.2$ km, a maximum speed limit of $v_s = 100$ km h^{-1} is imposed. Cars must strictly adhere to this at every point along their journey.

To enforce this, two speed sensors are installed, one at the start of the road and one at the end, each measuring the instantaneous speeds of cars passing by. But does this really help? Even if a car registers speeds lower than v_s at both sensors, we can only conclude that it was travelling within the speed limit at the start and end of the road. We still wouldn't know whether the car complied with the speed limit across its entire journey.

In view of this, clever engineers decided to reconfigure the sensors to now measure the time interval T between a car passing the first sensor and the same car passing the second sensor. If $T < T_s$, we can be absolutely sure that the car must have been speeding. Find the maximum value of T_s.

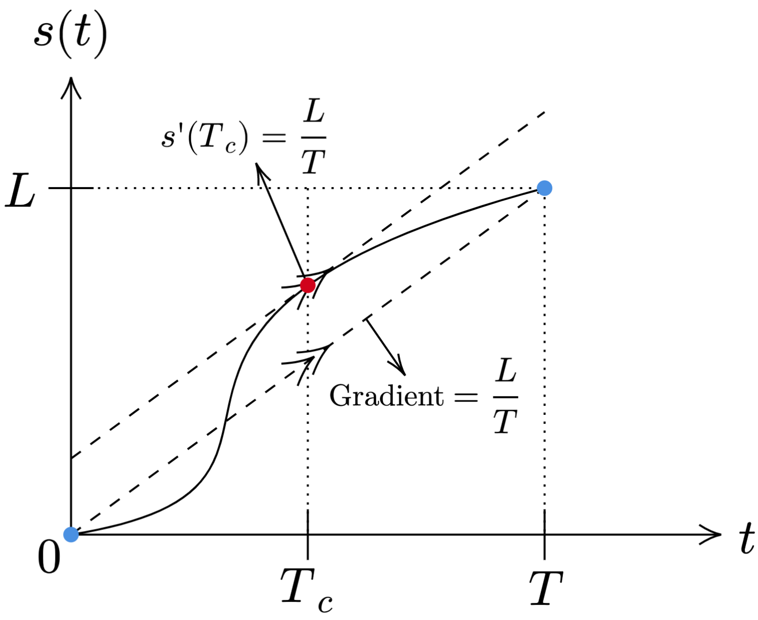
Leave your answer to 3 significant figures in units of s.

Solution: Consider the displacement $s(t)$ of the car at time t after the car passes the start of the road. We set $s(0) = 0$ at the start of the road. Since the car reaches the end of the road in time T, we have $s(T) = L$.

By the mean value theorem, there must exist a time T_c between 0 and T where the derivative of the graph obeys the following relation:

$$s'(T_c) = \frac{s(T) - s(0)}{T - 0} = \frac{L}{T}$$

Here's a diagrammatic illustration of this fact:



So we know, with absolute certainty, that there must have been a point in the car's journey where its instantaneous velocity was L/T. Hence, the speed limit was definitely violated if:

$$\frac{L}{T} \geq v_s \implies T \leq \frac{L}{v_s}$$

Therefore, $T_s = L/v_s = 0.012\text{ h} = \boxed{43.2\text{ s}}$.

Alternative solution: If the car were to travel at the speed limit v_s the entire way, it'll take time L/v_s to traverse the whole road. If it were to go any faster, it would be speeding, and it would take $T < L/v_s$, hence $T_s = L/v_s$. Nonetheless, the mean value theorem provides a more rigorous illustration of this.

The context of this question hopefully gives you a better understanding of the rationale for average rather than instantaneous speed measurement used in some speed limit enforcement systems (full article).

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Problem B: Airdrop (3 points)

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

Two identical crates are each loaded with different masses and sealed. When the loaded crates are dropped separately from positions high above the ground, the heavier crate reaches terminal velocity $U = 8.0 \text{ m s}^{-1}$, while the lighter crate reaches terminal velocity $u = 2.0 \text{ m s}^{-1}$.

The two crates are now joined together by a light rope and, with the heavier crate dangling by the rope below the lighter one, are dropped once more from a position high above the ground. What is the new terminal velocity of the joined crates?

Assume the air drag to be linear, and the length of the rope to be much greater than the size of the crates.

Leave your answer to 2 significant figures in units of m s^{-1}.

Solution: Since the air drag is linear, the force on a crate travelling at velocity v can be written as $-bv$, where b is a proportionality constant. The value of b, which depends on properties like the cross-sectional area and material of the falling body, must be common between both crates, since the crates themselves are identical.

Let M and m be the masses of the heavier and lighter crate respectively. At terminal velocity, net force on each crate is zero. In the first case when the crates were dropped individually, we may write the following force balance statements for each crate:

$$Mg = bU \implies M = \frac{bU}{g}$$
$$mg = bu \implies m = \frac{bu}{g}$$

When the crates are dropped together, the rope exerts some tension T on each crate. Tension acts upwards on the heavier crate, and downwards on the lighter crate. Let the final terminal velocity of the system be v. Balancing forces on each crate:

$$\begin{aligned} Mg - T &= bv \\ mg + T &= bv \\ \implies v &= \frac{(M+m)g}{2b} \end{aligned}$$

Substituting the expressions for M and m in terms of U and u, we may calculate the value of v:

$$v = \frac{U+u}{2} = \boxed{5.0 \text{ m s}^{-1}}$$

Evidently, the value of v lies between u and U. This is somewhat intuitive; the heavier crate wants to fall fast, but the lighter crate wants to fall slowly. These two effects oppose one another and the joined crates end up falling at an intermediate speed. Interestingly, this v turns out to be exactly the average of u and U.

Alternative solution: As a shorter method, we can treat both crates as a combined system. This way, tension need not be considered as it is an internal force. The only external forces are the weights Mg, mg and air drag forces acting on the crates that total up to $2bv$. Acceleration is zero when the net external force on the system is zero, hence:

$$Mg + mg = 2bv \implies v = \frac{(M+m)g}{2b} = \frac{U+u}{2} = \boxed{5.0 \text{ m s}^{-1}}$$

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Problem C: Messing with Springs

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

Two ideal light springs have the same unstretched length $L = 10.0$ cm, but different spring constants $k_1 = 40$ N cm^{-1} and $k_2 = 80$ N cm^{-1}.

(a) Each spring is stretched by the same distance of $x = 3.0$ cm. Find the ratio E_1/E_2 of the energy stored in each spring.

Leave your answer to 3 significant figures. (2 points)

(b) Each spring is now stretched with the same force of $F = 100$ N. Find the ratio E_1/E_2 of the energy stored in each spring.

Leave your answer to 3 significant figures. (2 points)

(c) Each spring is now stretched to store the same energy of $E = 50$ J. Find the ratio x_1/x_2 of the extension of each spring.

Leave your answer to 3 significant figures. (2 points)

Solution: The basic property of ideal springs is that its elastic force F is proportional to its extension x, as given by Hooke's Law $F = kx$.

The stored energy is the area under the force-extension graph, and this graph is a straight line through the origin. Thus the area is a triangle, $E = \frac{1}{2}(kx)(x) = \frac{1}{2}kx^2$.

(a) Since $k_1 = \frac{1}{2}k_2$, and x is the same for both springs, we have $E_1/E_2 = \boxed{0.500}$.

(b) If the force is the same for both springs, then $x \propto k^{-1}$. Since $k_1/k_2 = \frac{1}{2}$, we have $x_1/x_2 = 2$.

We have $E_1 = \frac{1}{2}k_1x_1^2$ and $E_2 = \frac{1}{2}k_2x_2^2$, so the ratio is:

$$\frac{E_1}{E_2} = \frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^2 = \frac{1}{2} \times 2^2 = \boxed{2.00}$$

(c) Since the energy stored is the same for both springs, we can set $\frac{1}{2}k_1x_1^2 = \frac{1}{2}k_2x_2^2$. Hence, we have:

$$\frac{x_1^2}{x_2^2} = \frac{k_2}{k_1} = 2 \implies \frac{x_1}{x_2} = \sqrt{2} \approx \boxed{1.41}$$

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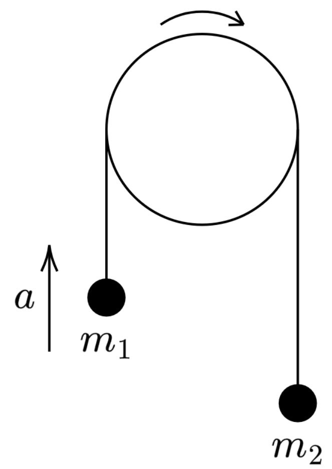
Problem D: A Simple Pulley

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

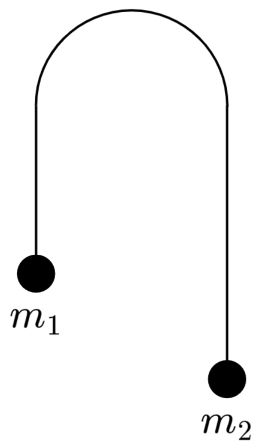
Two point masses, $m_1 = 1.0$ kg and $m_2 = 2.0$ kg, are attached to the ends of a light inextensible string that is threaded over a light frictionless pulley.



(a) Find the upward acceleration a of mass m_1.

Leave your answer to 2 significant figures in units of m s^{-2}. (2 points)

In the remaining parts, consider the combined system of the two point masses and the string that connects them, as drawn below.



(b) Find the downward acceleration a_{CM} of the centre of mass of this system.

Leave your answer to 2 significant figures in units of m s^{-2}. (2 points)

(c) By considering the external forces acting on this system, along with your answer in (b), find the net contact force N exerted on the string by the pulley.

Leave your answer to 2 significant figures in units of N. (2 points)

Solution: The mass m_1 experiences a net upward acceleration as the tension T from the string is greater than its weight. The mass m_2 experiences a net downward acceleration as its weight is greater than T. Note that the string is inextensible so the motions are linked – both have accelerations of magnitude a.

(a) Applying Newton's Second Law for each mass, we have:

$$\begin{aligned} T - m_1 g &= m_1 a \\ m_2 g - T &= m_2 a \end{aligned}$$

We can solve by eliminating T and get:

$$a = \frac{m_2 - m_1}{m_1 + m_2} g \approx \boxed{3.3 \text{ m s}^{-2}}$$

(b) Since the string is approximated to have no mass, we do not need to account for it when calculating a_{CM}.

Let us suppose that m_1 and m_2 have vertical positions y_1 and y_2 respectively. The position y_{CM} of their centre of mass is given by:

$$y_{\mathrm{CM}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

By differentiating twice, we have the acceleration a_{CM} of the centre of mass:

$$a_{\mathrm{CM}} = \frac{m_1 \ddot{y}_1 + m_2 \ddot{y}_2}{m_1 + m_2}$$

Taking the downward direction to be positive, $\ddot{y}_2 = a$ and $\ddot{y}_1 = -a$. Thus a_{CM} is given by:

$$a_{\mathrm{CM}} = \frac{m_2 a - m_1 a}{m_1 + m_2} \approx \boxed{1.1 \text{ m s}^{-2}}$$

(c) The only external forces acting on the system are the weights $m_1 g$ and $m_2 g$, and the required contact force N exerted by the pulley on the string. Applying Newton's Second Law to this combined system[1]:

$$m_1 g + m_2 g - N = (m_1 + m_2) a_{\mathrm{CM}}$$

Solving for N gives:

$$N = (m_1 + m_2) g - (m_2 - m_1) a = \frac{4 m_1 m_2 g}{m_1 + m_2} \approx \boxed{26 \text{ N}}$$

[1] When using Newton's Second Law $\Sigma F = ma$ on a system that comprises multiple objects, the net force ΣF considers only the external forces on the system, m considers the total mass of the objects within the system, while a refers to the acceleration of the system's centre of mass.

In fact, this expression for N exactly equals $2T$. Using Newton's Third Law, this also means the pulley would experience a downward force of $2T$ from the string – a common textbook assumption that we have just proven!

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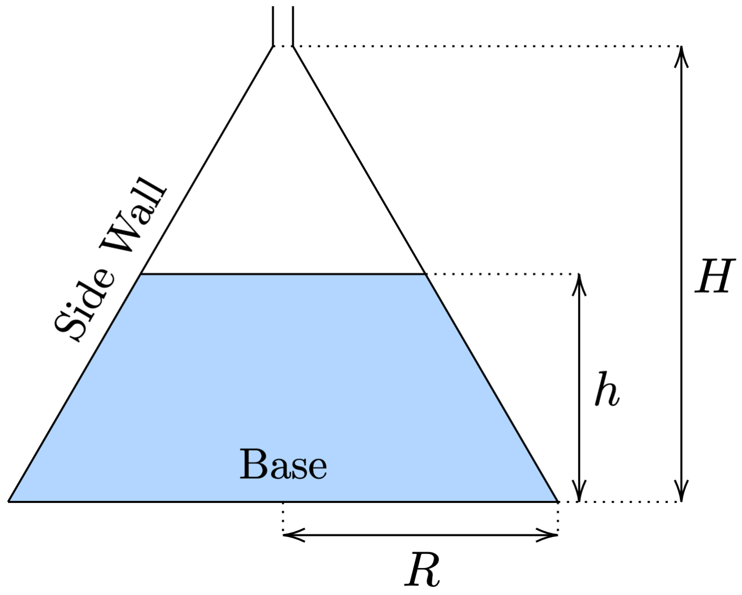
Problem E: Conical Flask

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

A conical flask has the shape of a perfect cone with radius $R = 10.0$ cm and height $H = 15.0$ cm, except that there is a small opening (of negligible size) at the top. Water is poured into the flask to a depth of $h = 5.0$ cm.



(a) Find the weight W of the water.

Leave your answer to 2 significant figures in units of N. (2 points)

The water experiences contact forces, both from the circular base of the conical flask, as well as from the side walls of the conical flask.

(b) Find the net contact force F_b exerted by the circular base of the conical flask on the water.

Leave your answer to 2 significant figures in units of N. (2 points)

(c) Find the net contact force F_s exerted by the side walls of the conical flask on the water.

Leave your answer to 2 significant figures in units of N. (2 points)

Solution:

(a) By similar triangles, the radius r of the circular upper surface of the water is:

$$r = \frac{H - h}{H} R$$

The volume of water can be calculated from the difference of two cones – the cone with height H and base radius R, subtracted by the cone with height $H-h$ and base radius r. The volume of a cone is given by $\frac{1}{3}\pi \times (\text{base radius})^2 \times (\text{height})$, so the volume V of the water is:

$$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi \left(\frac{H-h}{H}R\right)^2 (H-h)$$

The weight W can thus be calculated as follows:

$$W = V\rho g \approx \boxed{11 \text{ N}}$$

(b) Let the hydrostatic pressure at the base of the conical flask be p. This pressure is caused by the height h of fluid above, so:

$$p = \rho g h$$

At equilibrium, this pressure is uniform over the entire base area of the flask. Hence, the downward force exerted by the water on the base of the flask is:

$$F_b = p(\pi R^2) = \rho g h \pi R^2 \approx \boxed{15 \text{ N}}$$

By Newton's Third Law, this is equal to the upward contact force exerted by the base on the water. This is greater than the weight of the water!

(c) Evidently, since $F_b > W$, there must be an additional downward force acting on the water so that it can remain at equilibrium. This force F_s is provided by the side walls. Balancing forces yields:

$$F_s = F_b - W \approx \boxed{4.5 \text{ N}}$$

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Problem 2: Freefall Photography (3 points)

A ball that is initially at height h above the ground is released from rest. Between its time of release and its time of landing, a camera is calibrated to capture one photograph of the ball at a random time. Find the probability that in the captured photograph, the height of the ball above the ground is smaller than $h/2$.

Leave your answer to 3 significant figures.

Your answer should range between 0 and 1.

Solution: The time taken for the ball to reach the ground t_{total} can be determined by kinematics:

$$h = \frac{1}{2}gt_{\text{total}}^2 \implies t_{\text{total}} = \sqrt{\frac{2h}{g}}$$

The time interval for which the ball is above height $h/2$ is equal to the time taken for the ball to exactly reach height $h/2$. Denoting this time as t_1, it can be similarly determined:

$$\frac{h}{2} = \frac{1}{2}gt_1^2 \implies t_1 = \sqrt{\frac{h}{g}}$$

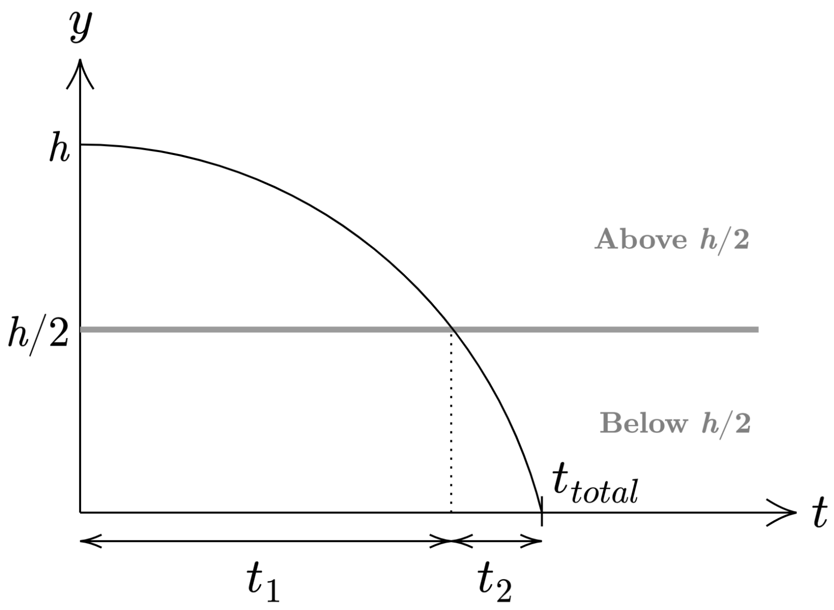
The time interval t_2 for which the ball is below height $h/2$ can thus be found:

$$t_2 = t_{\text{total}} - t_1 = \sqrt{\frac{h}{g}}(\sqrt{2} - 1)$$

The probability for which the ball is below height $h/2$ in the photograph is therefore given by the ratio t_2/t_{total}:

$$\frac{t_2}{t_{\text{total}}} = 1 - \frac{1}{\sqrt{2}} \approx \boxed{0.293}$$

This probability turns out to be smaller than half. This is because the ball spends lesser time in the second half of its journey than in the first half, since it has picked up speed by then. This can be better understood using the graph of the ball's vertical height y against time as drawn below.



Evidently, the time interval for which $y < h/2$ is shorter than for $y > h/2$. So a photograph snapped at random is less likely to capture $y < h/2$ compared to $y > h/2$, making the probability for the ball to be below $h/2$ smaller than half.

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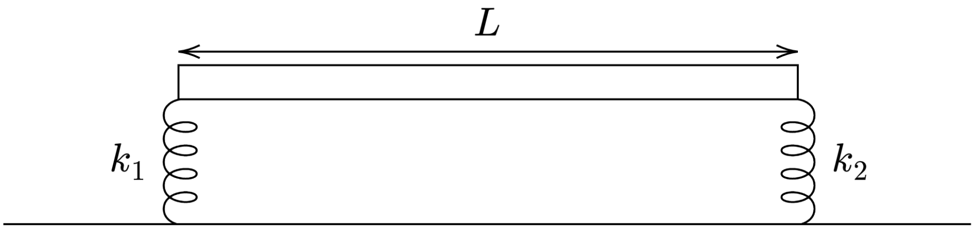
Problem 3: Springs in Parallel

Two ideal light springs have the same unstretched length, but different spring constants $k_1 = 30$ N m^{-1} and $k_2 = 50$ N m^{-1}. The springs are vertical, with their bottom ends fixed onto a horizontal table, and their top ends respectively attached to the left and right ends of a horizontal board of length $L = 1.00$ m and uniform density, as shown below. Neglect gravity.



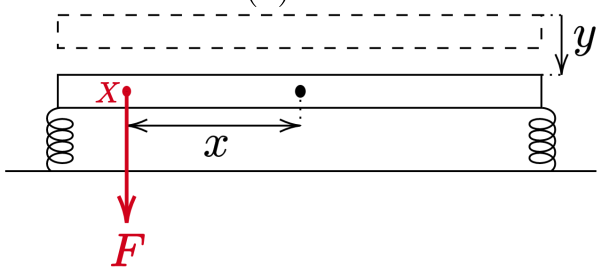
A vertical force F is now applied on the board, causing the board to have displacement y at equilibrium. Textbooks commonly claim that:

$$F = k_{\text{eff}}\, y, \quad \text{where } k_{\text{eff}} = k_1 + k_2$$

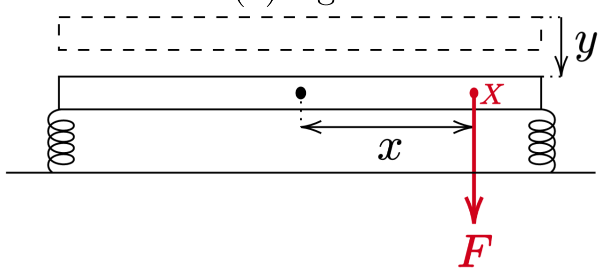
However, this will only be true under certain conditions. In particular, force F must be applied at a specific point X on the board for this relation to be valid.

(a) Point X must be to the ________ of the centre of the board. (1 point)

(1) left



(2) right



(b) What must the horizontal distance x between the centre of the board and point X be?

Leave your answer to 2 significant figures in units of m. (3 points)

Solution: Generally, since $k_2 y > k_1 y$, the board will rotate when displaced. When this happens, the springs will have different extensions, and will deviate from their vertical positions, resulting in a complex equilibrium state that will not be simply given by the claimed relation. Hence, for the board to be displaced without rotating, the force must be applied at the specific point X.

(a) Suppose that force F and thus displacement y are both directed downwards. Considering the torque about the board's centre, the spring on the right exerts a greater upward force than the spring on the left, so their combined effect is an anticlockwise torque. As such, F needs to exert a clockwise torque in order to prevent the board from rotating. So point X must be on the $\boxed{\text{(2) right}}$ of the centre of the board. The same analysis holds if F and y are both upwards.

(b) For there to be no rotation, a torque balance about the board's centre gives:

$$k_2 y \frac{L}{2} = k_1 y \frac{L}{2} + Fx \implies x = \frac{(k_2 - k_1)y}{F}\frac{L}{2}$$

Simultaneously, a force balance gives $y = F/k_{\text{eff}} = F/(k_1+k_2)$, as claimed above. Hence, x can be determined to be:

$$x = \frac{k_2 - k_1}{k_1 + k_2}\frac{L}{2} \approx \boxed{0.13\text{ m}}$$

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Problem 4: Dice Roll (3 points)

A uniform cube rests on a surface that is initially horizontal. We slowly increase the angle of the surface from the horizontal, keeping the cube's bottom face parallel to the surface. At some point, the cube starts moving relative to the surface. When that happens, we see that the cube rolls down rather than slides down the surface.

Find the minimum coefficient of static friction μ between the cube and the surface.

Leave your answer to 3 significant figures.

Solution: Let us define the angle θ as the angle of the surface from the horizontal. We seek to find the threshold angle for the cube to rotate assuming it does not slide.

Considering the torque on the cube about its lowest corner, we note that the weight, which acts at the cube's centre of gravity, tends to restore the cube back towards the surface when $\theta < 45°$. Once $\theta > 45°$, the centre of gravity has gone past the cube's lowest corner, so the torque causes the cube to tip over. Hence, the threshold for the cube to rotate occurs when $\theta = 45°$.

This requires that the cube does not slide down the ramp at any point before $\theta = 45°$. Hence, friction must have been sufficiently large to prevent the cube from sliding for all $\theta < 45°$.

Let the weight of the block be $W = mg$. The static friction f must balance the component of the weight along the surface for the block to remain at equilibrium, so $f = mg\sin\theta$. The normal contact force N also balances the component of weight perpendicular to the surface, and has a magnitude $N = mg\cos\theta$.

Since $f \leq \mu N$:

$$mg\sin\theta \leq \mu mg\cos\theta \implies \mu \geq \tan\theta$$

For this condition to be satisfied for all $\theta < 45°$, we need $\mu \geq \boxed{1.00}$.

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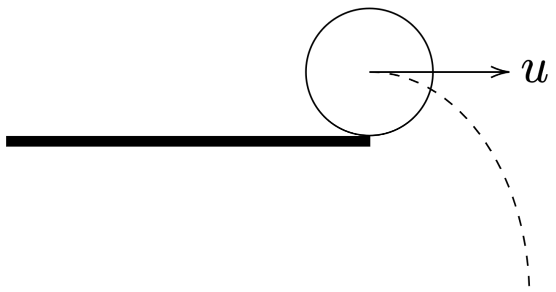
Problem 5: Parabolic Fall (3 points)

A spherical ball of radius $r = 0.04$ m rests exactly on the rightmost end of a horizontal platform. The ball is given a quick push that causes it to instantaneously acquire speed u towards the right. The subsequent trajectory of the ball's falling motion traces a perfect parabola if and only if $u > u_{\min}$. Find $u_{\min}$. Neglect friction and air resistance.

Leave your answer to 2 significant figures in units of m s^{-1}.



Solution: Take the rightmost end of the platform to be the origin. Let the coordinates of the ball's centre be $(x(t), y(t))$, where $t = 0$ is the time when the push is applied. Using standard kinematics equations, we can write expressions for $x(t)$ and $y(t)$:

$$x(t) = ut$$
$$y(t) = r - \frac{1}{2}gt^2$$

These expressions for $x(t)$ and $y(t)$ are predicated on the assumption that after getting pushed, the ball never comes in contact with the platform, such that it is always in free fall. Mathematically, this requires that $x^2 + y^2 > r^2$ for all $t > 0$.

Expanding the expression for $x^2 + y^2$:

$$x^2 + y^2 = \frac{1}{4}g^2t^4 + (u^2 - gr)t^2 + r^2$$

This expression will only be larger than r^2 for all $t > 0$ provided that $u^2 - gr > 0$. Hence, $u_{\min} = \sqrt{gr} \approx \boxed{0.63 \text{ m s}^{-1}}$.

Alternative solution: It is useful to consider what happens if $u < u_{\min}$. In this case, the ball will not instantly lose contact with the platform after being pushed. It will execute circular motion around the rightmost end of the platform for a brief period before finally falling off. The trajectory will thus look like an arc and a parabola joined together, rather than a perfect parabola.

Thus, for $u < u_{\text{min}}$, the initial motion of the ball's centre would be a circle of radius r centred at the platform's end. For the ball's centre to initially execute circular motion, the normal force N on the ball must satisfy:

$$mg - N = \frac{mu^2}{r} \implies N = mg - \frac{mu^2}{r}$$

For this circular motion to be possible, it is necessary that $N > 0$, which implies that $u < \sqrt{gr}$ is required for circular motion. Conversely, for the ball to travel in a perfect parabola, this circular motion must not take place. This gives us the condition $N < 0 \implies u > \sqrt{gr}$.

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Problem 6: Twisted Stick (4 points)

A stick is placed in a transparent rectangular container, and fixed at angle $\theta = 25°$ from the vertical. The container is partially filled with water of refractive index $n = 1.33$, such that some (but not all) parts of the stick are underwater. If you view the stick from near the top, what is the acute angle ϕ that you perceive the underwater portion of the stick to make from the vertical?

Leave your answer to 2 significant figures in units of degrees.

Solution: Denote the point on the stick at the water surface as P. Consider the underwater point on the stick located at a horizontal distance x away from P. The real depth h of this point (the actual vertical distance from the water surface to this point) is given by:

$$h = \frac{x}{\tan\theta}$$

As a result of refraction, the apparent depth h' of this point (the vertical distance that the point is perceived to make with the water surface) is given by:

$$h' = \frac{h}{n}$$

Note that this equation only holds true for small viewing angles. Since it is specified by the question that you're viewing from near the top, this assumption is valid.

h' is thus dependent on x as given in the following relation:

$$h' = \frac{x}{n\tan\theta}$$

Hence, the apparent angle ϕ of the underwater portion of the stick can be determined by trigonometry:

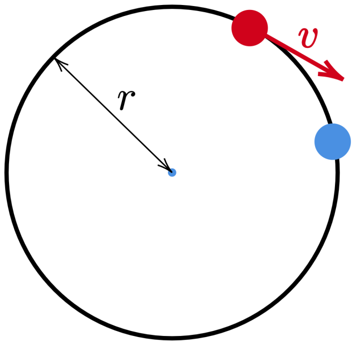
$$\tan\phi = \frac{x}{h'} \implies \phi = \tan^{-1}(n\tan\theta) \approx \boxed{32°}$$

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 7: Boing Boing (3 points)

Two small identical beads are threaded on a fixed frictionless horizontal circular hoop of radius $r = 0.2$ m. One bead is given an initial kick such that it slides with speed $v = 0.8$ m s^{-1} around the hoop. It then collides with the other stationary bead.

Determine the number of collisions that take place between the two beads within time $T = 50$ s after the first collision. (Do not count the first collision.) Assume that all collisions are instantaneous and perfectly elastic.



Leave your answer as an integer.

Solution: When the moving bead first collides with the stationary bead, the moving bead turns stationary and the stationary bead starts moving at speed v. (Since the two beads have the same mass, conservation of energy and momentum yields this outcome.)

The next collision takes place when the now-moving bead covers a full circumference back to the now-stationary bead. This takes exactly time $\tau = 2\pi r/v$. When the next collision occurs, the same switch happens – the moving bead is brought to rest, while the stationary bead is brought to speed v. And the cycle continues.

Essentially, a collision takes place every time τ. Hence, within time T, the subsequent number of collisions that occurs is given by $\left\lfloor \frac{T}{\tau} \right\rfloor = \boxed{31}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 8: Light but Powerful

Chris the child prodigy is attempting to build a solar cell. His attempt consists of two vacuum-sealed, parallel conducting plates with a small separation. When sunlight is incident upon one of these plates, charges are liberated and travel to the other plate where they accumulate, thus creating a potential difference between the plates.

(a) Chris uses platinum, with a work function of $\Phi_{\text{Pt}} = 6.35$ eV, to make the plates. What is the maximum wavelength, λ_{max}, that would cause a potential difference to form between the plates?

Leave your answer to 2 significant figures in units of nm. (2 points)

(b) Chris then uses a different material, with a work function $\Phi = 2.00$ eV, and light of wavelength $\lambda = 400$ nm. After a long time, what is the final voltage generated by the solar cell?

Leave your answer to 2 significant figures in units of V. (2 points)

Solution: The key to solving both parts comes from the equation for the photoelectric effect: $E = E_k + \Phi$, where E is the energy of the incident photon, given by $E = hf = \frac{hc}{\lambda}$ and E_k is the kinetic energy of the most energetic ejected electron.

(a) If $E < \Phi_{\text{Pt}}$, no electron will be ejected. Therefore, $E \geq \Phi_{\text{Pt}}$. The equality case yields the minimum photon energy necessary, and hence the maximum wavelength:

$$E = \Phi_{\text{Pt}} \implies \frac{hc}{\lambda_{\text{max}}} = \Phi_{\text{Pt}} \implies \lambda_{\text{max}} = \frac{hc}{\Phi_{\text{Pt}}} \approx \boxed{200 \text{ nm}}$$

(b) The transfer of each electron increases the positive charge on the sunlight-exposed plate, while increasing the magnitude of negative charge on the other plate. The potential difference between the plates thereby increases. At a certain potential difference, all ejected electrons have insufficient kinetic energy and will come to a stop due to the force exerted by the field between the plates. At this point, any further ejected electrons will return to the positively-charged plate, thus the potential difference will not increase further. This potential difference is known as the stopping potential, V_s.

The maximum kinetic energy of an ejected electron, E_k, is found by $E_k = E - \Phi = \frac{hc}{\lambda} - \Phi$. The stopping potential is achieved when $E_k = eV$, where e is the elementary charge. By equating both expressions for E_k, we find:

$$eV = \frac{hc}{\lambda} - \Phi \implies V = \frac{\frac{hc}{\lambda} - \Phi}{e} \approx \boxed{1.1 \text{ V}}$$

Apparently real solar cells work in a very similar (but better) way, which is honestly quite fascinating. Time to build my own solar cell.

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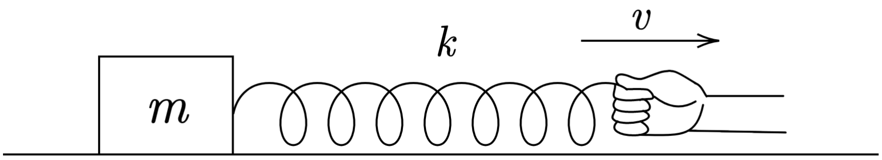
Problem 9: Catch-Up Spring (3 points)

A mass $m = 0.10$ kg rests attached to the left end of an ideal light spring of relaxed length $l = 0.15$ m and spring constant $k = 40$ N m^{-1}. You grab the right end of the spring, and start pulling it with constant velocity v towards the right. The ground is horizontal and frictionless.



If v is sufficiently large, it is possible for the mass to "catch up" and make contact with your hand at some point during its subsequent motion. Find the minimum value of v required for this to happen.

Leave your answer to 2 significant figures in units of m s^{-1}.

Solution: We'll work in the reference frame that moves with constant velocity v towards the right, i.e. the reference frame that moves along with the hand. In this frame, things are much simpler: the right end of the spring is always stationary, while the mass has initial velocity v towards the left. As this frame is inertial, Newton's laws equivalently apply here.

We need to determine the minimum value of v such that the mass can eventually reach the position of the right end of the spring. In other words, the initial kinetic energy of the mass must be sufficient to compress the spring by at least a length l. By conserving energy, we have:

$$\frac{1}{2}mv^2 \geq \frac{1}{2}kl^2 \implies v \geq l\sqrt{\frac{k}{m}} = \boxed{3.0 \text{ m s}^{-1}}$$

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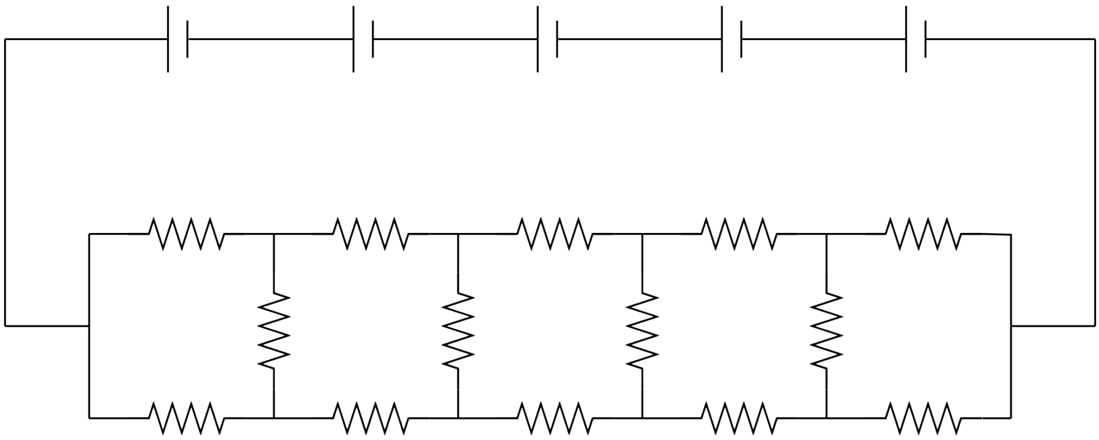
Problem 10: Resistor Squares

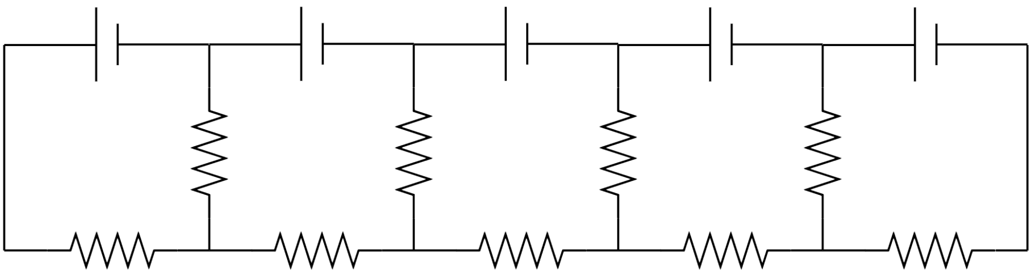
(a) 5 identical cells, each with emf $\varepsilon = 12$ V and negligible internal resistance, are connected to a network of 5 squares comprising 14 identical resistors each with resistance $R = 8.0\ \Omega$, as shown below.



What is the total power supplied by the cells?

Leave your answer to 2 significant figures in units of W. (3 points)

(b) The circuit is now rearranged, with each of the 5 cells replacing each of the 5 resistors on the top branch, as shown below.



What is the total power supplied by the cells?

Leave your answer to 2 significant figures in units of W. (3 points)

Solution:

(a) There is symmetry between the top branch and the bottom branch of the resistor network. As such, there will be no potential difference across each resistor on the 4 vertical branches, thus no current flowing through the vertical branches. We may hence disregard the vertical branches of the network, reducing the circuit to 5 resistors in parallel to another 5 resistors.

Analysing one of these horizontal branches, the voltage across the entire branch is 5ε, so the resulting current $I = 5\varepsilon/5R = \varepsilon/R$. The power supplied to the 5 resistors on the branch is thus given by $P = I^2(5R) = 5\varepsilon^2/R$.

Summing both branches, the total power is $2P = 10\varepsilon^2/R = \boxed{180\ \text{W}}$.

(b) Again, the 4 vertical branches of the network can be neglected. To see this, let us imagine removing them from the circuit. Notice that the potential at a point between the i-th and $(i+1)$-th cell is equal to the potential at a point between the i-th and $(i+1)$-th resistor. Therefore, if we now add the vertical branches back to the circuit, there will be no potential difference across each vertical branch, and correspondingly, no current. So the presence of the vertical resistor branches have no effect on the circuit, and we may treat them to be non-existent.

This reduces the circuit to 5 resistors in series connected to 5 cells. The current is $I = 5\varepsilon/5R = \varepsilon/R$, so the power supplied is $P = I^2(5R) = 5\varepsilon^2/R = \boxed{90\ \text{W}}$.

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Problem 11: Self-Propelled Flashlight (3 points)

Theoretically, it is possible for a flashlight to propel itself when turned on. For a flashlight of mass $m = 0.8$ kg on a frictionless horizontal surface shining light horizontally, determine the minimum operating power P required for the flashlight to self-propel with acceleration $a = 1.0 \text{ m s}^{-2}$.

Leave your answer to 2 significant figures in units of MW.

Solution: When the flashlight is turned on, it continuously produces photons. All of these photons carry momentum. By conservation of momentum, this results in the flashlight gaining momentum in the direction opposite to the direction of light emission, causing the flashlight to accelerate.

To determine the minimum value of P, we may assume that the flashlight is 100% efficient, such that all of the power being supplied to the flashlight is used to produce photons.

Within a time interval Δt, the total energy supplied to the flashlight is $P\Delta t$. If the energy of a single photon is E, we can write an expression for the number of photons n emitted within time Δt:

$$n = \frac{P\Delta t}{E}$$

The momentum carried by each photon is E/c. By conservation of momentum, the change in momentum Δp of the flashlight within time Δt is given by the total momentum carried by all of the photons emitted within time Δt, which is:

$$\Delta p = \frac{nE}{c} = \frac{P}{c}\Delta t$$

By Newton's Second Law, the force F exerted by the photons on the flashlight can be written as:

$$F = \frac{\Delta p}{\Delta t} = \frac{P}{c}$$

The value of F required for the flashlight to have acceleration a, in the absence of other forces, is ma. Hence, the required P can be determined:

$$P = Fc = mac = 2.4 \times 10^8 \text{ W} = \boxed{240 \text{ MW}}$$

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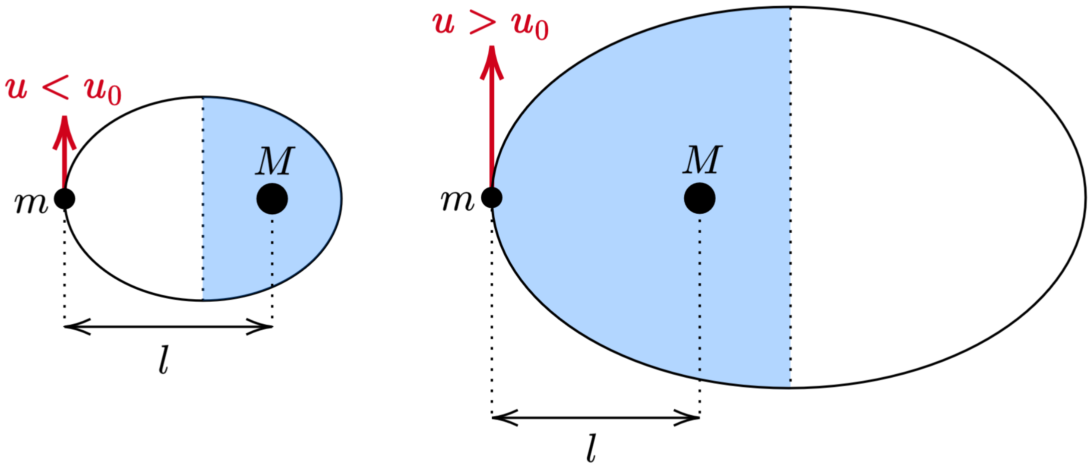
Problem 12: Ellipse Halves (3 points)

At distance $l = 4 \times 10^7$ m to the left of a stationary planet with mass $M = 10^{22}$ kg, there is a small projectile of mass $m \ll M$. It is launched tangentially with velocity u, such that the resulting trajectory of its orbit takes the shape of an ellipse.

It is observed that for $u < u_0$, the planet is always located in the right half of the elliptical orbit, whereas for $u > u_0$, the planet always lies in the left half of the elliptical orbit. Determine the value of u_0.

Leave your answer to 2 significant figures in units of m s^{-1}.



Solution: By Kepler's First Law, the planet must be positioned at one of the foci of the projectile's elliptical orbit.

It is useful to consider what happens at the exact boundary between the two regimes of $u < u_0$ and $u > u_0$, that is, when $u = u_0$. Noting that the positions of ellipse's foci change in a continuous manner as u is varied continuously, the planet will be located exactly at the boundary between the left half and the right half of the elliptical orbit. In other words, the planet is at the centre of the ellipse when $u = u_0$.

This implies that at $u = u_0$, one focus of the ellipse is at its centre. This is only possible if the ellipse is a circle. As such, at $u = u_0$, the trajectory of the projectile is circular, with radius l.

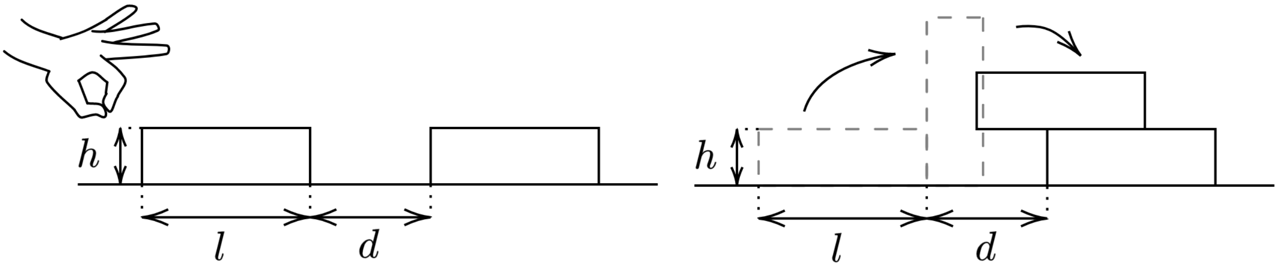
With this observation, u_0 can be found by noting that the gravitational force provides the centripetal force for the projectile's circular motion:

$$\frac{mu_0^2}{l} = \frac{GMm}{l^2} \implies u_0 = \sqrt{\frac{GM}{l}} \approx \boxed{130 \text{ m s}^{-1}}$$

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 13: Country Erasers

A game of "Country Erasers" is played with two identical square-shaped erasers of length $l = 3.0$ cm, thickness $h = 0.5$ cm and uniform mass $m = 20$ g. They initially lie flat beside each other on a horizontal table, with their ends a distance $d > h$ apart. In a winning move, a player flicks one eraser towards the other, such that it rotates vertically and eventually lands resting flat on the other eraser, as shown below. Assume that neither of the erasers slip at any point of contact with the table or with each other throughout their motion.



(a) Find the minimum energy E_0 delivered by the flick such that this winning move can be performed.

Leave your answer to 2 significant figures in units of mJ. (2 points)

(b) Find the minimum energy E_1 dissipated in the process after the eraser is flicked.

Leave your answer to 2 significant figures in units of mJ. (2 points)

(c) Find the maximum value of d such that this winning move is physically possible.

Leave your answer to 2 significant figures in units of cm. (3 points)

Solution:

(a) The energy transferred by the flick needs to be large enough to be converted into sufficient GPE for the eraser to reach a vertical position. As long as the eraser is able to get past this vertical position, where its CM attains maximum height, it will certainly fall onto the other eraser and fulfill the winning condition.

Initially, the eraser's CM is at height $h/2$ above the table. At the vertical position, the eraser's CM is at height $l/2$ above the table. The change in GPE is $mg(l - h)/2$, so the required E_0 is given by:

$$E_0 > \frac{mg(l-h)}{2} \approx \boxed{2.5 \text{ mJ}}$$

(b) At its ending position, the eraser's CM is at height $3h/2$ above the table. As such, from start to end, the eraser's CM had risen by height h, so the change in energy ΔE of the eraser is given by $\Delta E = mgh \approx 1$ mJ.

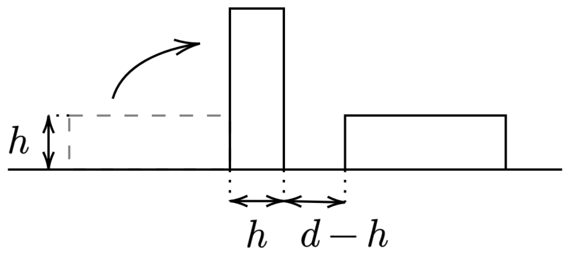
Notice that $\Delta E < E_0$. This implies that the final change in energy is less than the energy that our flick puts into the system. Hence, energy must have been dissipated in the process. This energy dissipated E_1 is given by:

$$E_1 = E_0 - \Delta E > \frac{mg(l-3h)}{2} \approx \boxed{1.5\ \text{mJ}}$$

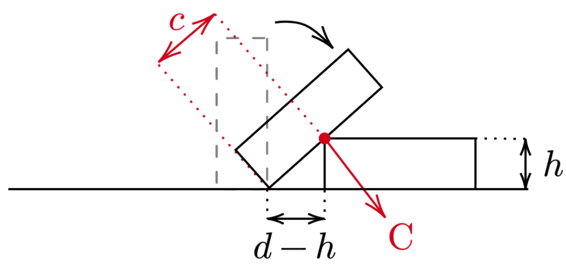
The main source of this energy dissipation is when the eraser collides with the other eraser. At the instant before the collision, the eraser still has some kinetic energy. All of this kinetic energy must be dissipated for it to come to rest.

(c) For the eraser to rest flat on the other at equilibrium, its CM must be directly above the other eraser and cannot stick out beyond it – otherwise, the eraser will tip over and land on the table. This means that the maximum length of the eraser that protrudes beyond the other eraser is $l/2$.

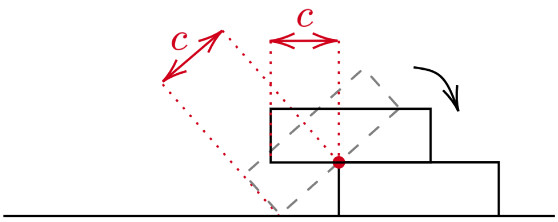
Let us analyse every part of the eraser's motion. First, consider how it gets from its starting position to its vertical position. Because it rotates about its corner without any slipping, the distance from its end to the other eraser's end decreases from d to $d-h$.



Subsequently, as it continues rotating about its corner without slipping, a point C on the eraser strikes the corner of the other eraser. This point C is located at distance $c = \sqrt{h^2 + (d-h)^2}$ away from the eraser's bottom corner.



Thereafter, as the eraser gets from its slant position to its flat horizontal position, it now rotates about point C. Since point C doesn't slip, the final length of the eraser that protrudes beyond the other eraser is exactly c.



Hence, the condition for a flat final equilibrium is:

$$\begin{aligned} c < \frac{l}{2} &\implies d^2 - 2dh + 2h^2 - \frac{l^2}{4} < 0 \\ &\implies d < h + \sqrt{\frac{l^2}{4} - h^2} \approx \boxed{1.9 \text{ cm}} \end{aligned}$$

You may have noticed a slight inaccuracy in the solution.[2]

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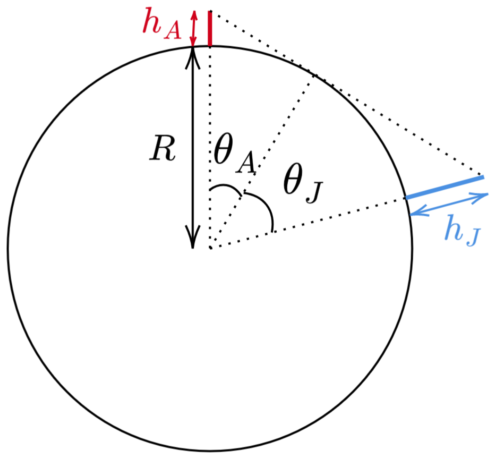
[2]The answers presented here for (a) and (b) are actually slightly approximated – their exact expressions should be (a) $mg(\sqrt{l^2+h^2}-h)/2$, (b) $mg(\sqrt{l^2+h^2}-3h)/2$. The eraser's CM is not actually highest when the eraser is at a vertical position, but rather, when the diagonal of the eraser is vertical! Hence, the maximum height reached by the eraser's CM above the table is $(\sqrt{l^2+h^2})/2$ rather than $l/2$. However, considering that h is relatively small compared to l, both approaches produce the same numerical answer to 2 significant figures.

Problem 14: Simp (4 points)

Amy has a crush on Jake. Jake runs in a constant direction with speed $v_J = 7$ m s^{-1} relative to the ground. Amy chases him with a speed of $v_A = 2$ m s^{-1}. Jake's and Amy's eye levels are at height $h_J = 2.0$ m and $h_A = 1.5$ m from the ground respectively. Assume that Earth is a perfect sphere with radius $R = 6400$ km, and that there are no obstructions. Given that they start running from the same point at the same time, find t, the time taken for Jake to disappear from Amy's view.

Leave your answer to 3 significant figures in units of min.

Solution: When Jake disappears, the line joining Amy's and Jake's eyes will be tangent to the Earth. Define θ_A and θ_J as shown below:



Since the radius is perpendicular to the tangent line, we have:

$$\cos(\theta_A) = \frac{R}{R + h_A}$$
$$\cos(\theta_J) = \frac{R}{R + h_J}$$

Thus, the distance between Amy and Jake along the surface of the Earth when Jake disappears is:

$$R(\theta_A + \theta_J) = R\left[\cos^{-1}\left(\frac{R}{R + h_A}\right) + \cos^{-1}\left(\frac{R}{R + h_J}\right)\right] \approx 9441 \text{ m}$$

The relative speed between Amy and Jake is $v_A - v_J = 5$ m s^{-1}. Hence:

$$t = \frac{9441}{5} \text{ s} \approx \boxed{31.5 \text{ min}}$$

If your calculator cannot perform $\cos^{-1}$, a small angle approximation will also suffice. We have: $\cos\theta \approx 1 - \frac{\theta^2}{2} = \frac{R}{R+h}$ which gives $\theta = \sqrt{\frac{2h}{R+h}}$. This value of θ gives practically the same answer as above.

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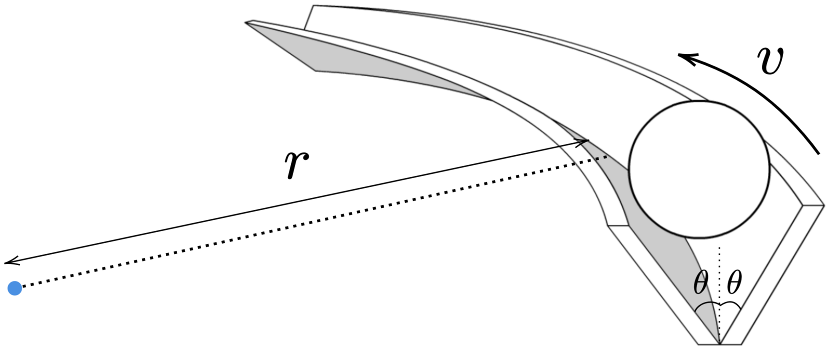
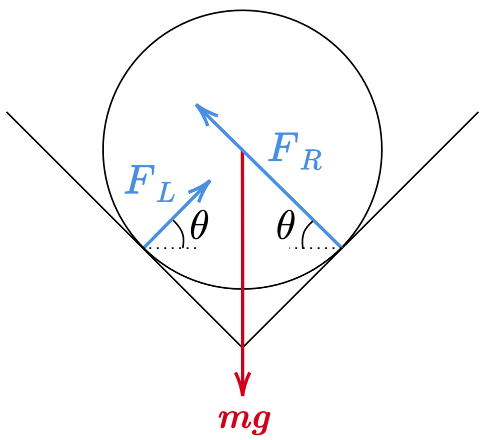
Problem 15: Bending Groove (4 points)

A sphere slides without rolling through a fixed frictionless V-shaped groove. Each side of the groove makes angle $\theta = 30°$ with the vertical. The path traced by the groove is horizontal and bends with radius of curvature $r = 0.15$ m. Determine the maximum speed v of the sphere such that it can traverse the bend without straying from the path of the groove.

Leave your answer to 2 significant figures in units of m s^{-1}.



Solution: Assume without loss of generality that the groove bends towards the left. Let F_L and F_R be the magnitudes of the forces by the left and right sides of the groove on the sphere respectively. Since the groove is frictionless, the forces exerted by the groove on the sphere must be directed normal to the groove's surface (toward the centre of the sphere). As such, F_L and F_R are both directed at angle θ above the horizontal. Qualitatively speaking, it is necessary for $F_R > F_L$ so that the net force on the sphere is leftward to supply the centripetal force for it to traverse the bend. The free-body diagram of the sphere is drawn below:



Since the groove is horizontal, the sphere will not have any vertical motion, so its net force in the vertical direction must be zero:

$$(F_L + F_R)\sin\theta = mg$$

The net force on the sphere in the horizontal direction supplies its centripetal acceleration:

$$(F_R - F_L)\cos\theta = \frac{mv^2}{r}$$

These two equations enable us to solve for F_L and F_R:

$$F_L = \frac{m}{2}\left(\frac{g}{\sin\theta} - \frac{v^2}{r\cos\theta}\right)$$
$$F_R = \frac{m}{2}\left(\frac{g}{\sin\theta} + \frac{v^2}{r\cos\theta}\right)$$

For the sphere to remain in contact with the groove, it is necessary for both F_L and F_R to be greater than 0. From the expressions above, $F_R > 0$ is satisfied for all v, but $F_L > 0$ only holds for a certain range of v:

$$v < \sqrt{\frac{gr\cos\theta}{\sin\theta}} \approx \boxed{1.6\ \text{m s}^{-1}}$$

Should v exceed this value, the sphere will lose contact with the left wall and travel outwards. It will no longer follow the path traced by the groove; the radius of curvature of its trajectory will exceed that of the groove.

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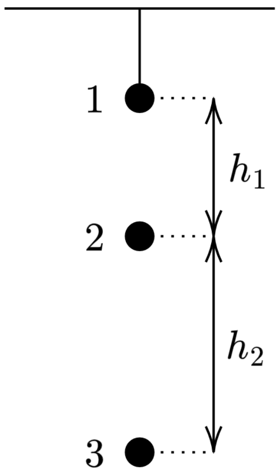
Problem 16: Decorating the Future (4 points)

Three identical small spherical ornaments are used to make a futuristic decoration. All three ornaments have charges of equal magnitude and are perfectly insulating.

The first ornament is suspended from the ceiling by a light string. The second ornament is placed a distance h_1 directly below it, and the third is placed a distance h_2 directly below the second. Given that the system is in equilibrium, find h_1/h_2.

Leave your answer to 3 significant figures.



Solution: It is fairly clear that the ornaments must have alternating charges – if the first ornament was positively charged, then the second must be negatively charged and the third positive. If the second ornament was the same charge as the first, the repulsive electrostatic force would push it downward. Similarly, if the third ornament had the same charge as the second, the repulsive force exerted by the second ornament would be stronger than any attractive force exerted by the first ornament on the third.

Let q be the magnitude of each ornament's charge, m be the mass of each ornament, and $k = \frac{1}{4\pi\epsilon_0}$. We can form two simultaneous equations by balancing the forces on the second and third ornaments respectively:

$$\begin{aligned}
\frac{kq^2}{h_1^2} &= mg + \frac{kq^2}{h_2^2} &&\implies mg = kq^2\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) \\
\frac{kq^2}{h_2^2} &= mg + \frac{kq^2}{(h_1+h_2)^2} &&\implies mg = kq^2\left(\frac{1}{h_1^2} - \frac{1}{(h_1+h_2)^2}\right)
\end{aligned}$$

It follows that:

$$\frac{1}{h_1^2} - \frac{1}{h_2^2} = \frac{1}{h_1^2} - \frac{1}{(h_1+h_2)^2} \implies \frac{1}{h_1^2} + \frac{1}{(h_1+h_2)^2} = \frac{2}{h_2^2}$$

Let $n = h_1/h_2 \implies h_1 = nh_2$. We substitute this into the above equation to obtain:

$$\frac{1}{n^2h_2^2} + \frac{1}{(nh_2 + h_2)^2} = \frac{2}{h_2^2} \implies \frac{1}{n^2} + \frac{1}{(n+1)^2} = 2$$

At this point, the remainder of the work is purely mathematical. You may solve the above equation algebraically or use a computational tool of your choice, which would give you $n \approx \boxed{0.771}$ as the only real positive solution.

It's worth noting that the equilibrium of the set-up is unstable. Consider a vertical perturbation to the third charge. If it is moved downward slightly, the attraction from the second charge weakens more than the repulsion from the first charge. The net force acting on the third charge will hence point downwards. In other words, it will not experience a restoring force back to equilibrium, and the set-up will fall apart.

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Problem 17: Useless Pipe (4 points)

A light U-shaped pipe lies on a horizontal floor. The pipe comprises two straight sections that are affixed to a vertical wall and connected by a semicircular section with radius of curvature $r = 1.00$ m. The pipe has a circular cross-section with an inner diameter of $d = 0.200$ m. Water flows in one end at a constant rate of $Q = 0.100\ \mathrm{m^3\ s^{-1}}$ and out the other. What is the total force F that the semicircular section of the pipe exerts on the straight sections of the pipe?

Leave your answer to 2 significant figures in units of N.

Solution: We begin by deriving the following measurements: The cross-sectional area of the pipe is $A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$. The velocity of water entering the pipe is thus $v = \frac{Q}{A} = \frac{4Q}{\pi d^2}$.

As the water passes through the pipe, it reverses direction after traversing the semicircular portion. A mass Δm of water passing through the curved section of the pipe would therefore experience a total change of momentum Δp given by:

$$\Delta p = 2mv = \frac{8mQ}{\pi d^2}$$

Within time Δt, the mass Δm of water that flows through the pipe is given by $\Delta m = \rho_w Q\,\Delta t$, where ρ_w is the density of water. The force F exerted by the curved portion in changing the momentum of the water is thus:

$$F = \frac{\Delta p}{\Delta t} = \frac{8\rho_w Q^2}{\pi d^2}$$

By Newton's Third Law, the water exerts a force of magnitude F on the curved part of the pipe. Since the curved section of the pipe is stationary, net force acting on it must be zero. Thus, the straight parts of the pipe exert a total force of magnitude F on the curved part also. Invoking Newton's Third Law yet again, the curved part must consequently exert a total force $F \approx \boxed{640\ \mathrm{N}}$ on the straight parts of the pipe.

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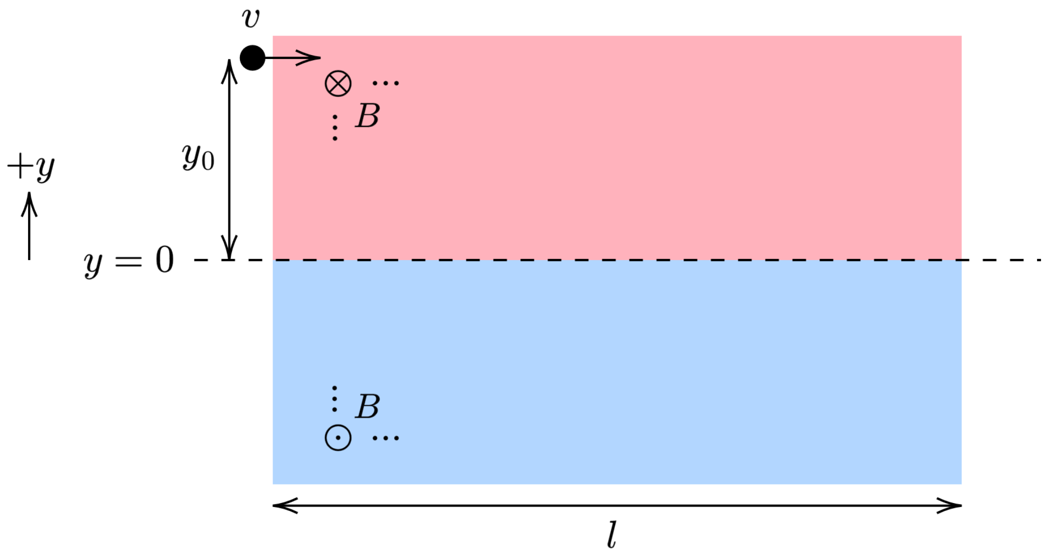
Problem 18: A Swimming Electron (4 points)

Consider a magnetic field which extends infinitely along the vertical y-axis, but has a finite length $l = 900$ nm along the horizontal x-axis. The magnetic field has uniform magnitude B, and points into the page for $y > 0$, but out of the page for $y < 0$.

An electron is initially located at y-coordinate $y_0 = +170$ nm. It is propelled horizontally into the magnetic field with a velocity $v = y_0 eB/m_e$, where e is the elementary charge, and m_e is the mass of an electron.



Find the y-coordinate of the electron, y_1, when it exits the field. Neglect gravity.

Leave your answer to 2 significant figures in units of nm.

Solution: First, we begin by describing the general motion of the electron within the magnetic field. We shall denote the regions where $y > 0$ and $y < 0$ as the top field and the bottom field respectively.

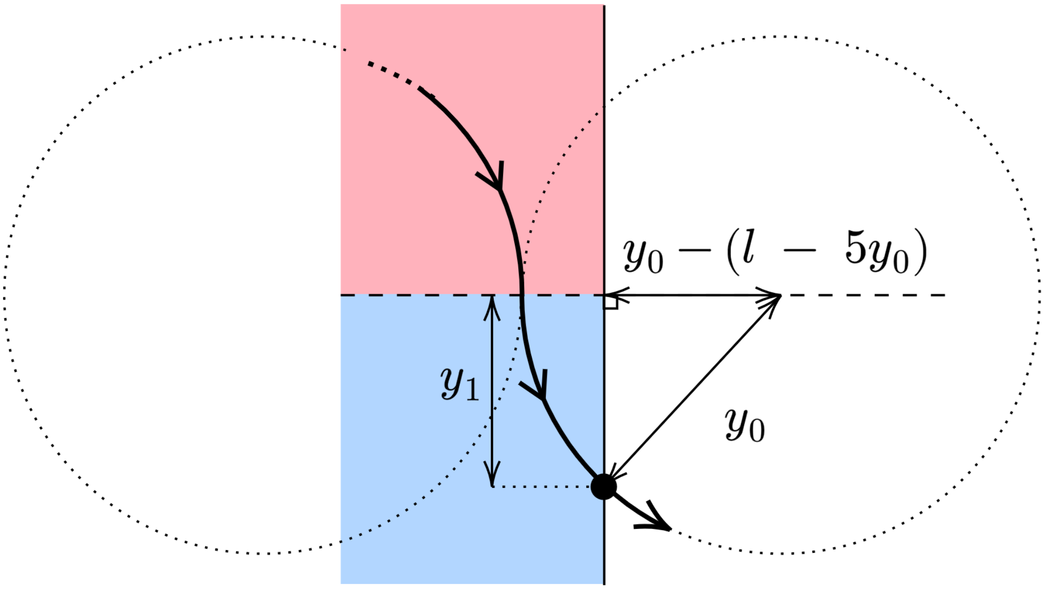
Generally, a charge moving perpendicular to a magnetic field undergoes uniform circular motion. Let us determine the radius r of the circular path in this case. Since the magnetic force supplies the centripetal force, we can write:

$$evB = \frac{m_e v^2}{r} \implies r = y_0$$

Therefore, as the electron enters the top field, it will move towards the bottom field in a circular path with radius y_0 and with constant speed v. It will then enter the bottom field vertically. Thereafter, it will once again follow a circular path back towards the top field with the same radius and speed. Consequently, we would observe the electron alternating between the top and bottom fields until it exits the field entirely after travelling a distance of l along the x-axis.

In general, the electron will be in the following sequence of fields for each horizontal distance of y_0 travelled: top $\to$ bottom $\to$ bottom $\to$ top $\to$ top $\to$ bottom $\to$ bottom $\to$ top $\to$ top...

Observe that l/y_0 is between 5 and 6. This implies that the final stage of the electron's motion will be an incomplete circular arc in the bottom field. In other words, $y_1 < 0$.



From the geometry of the set-up, we can apply Pythagoras' Theorem to derive y_1:

$$y_1 = -\sqrt{y_0^2 - (y_0 - (l - 5y_0))^2} \approx \boxed{-120 \text{ nm}}$$

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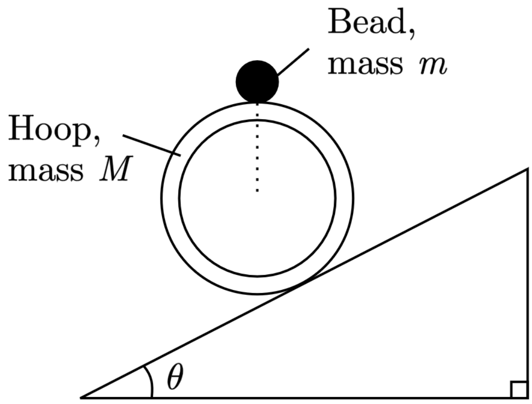
Problem 19: Defying Gravity

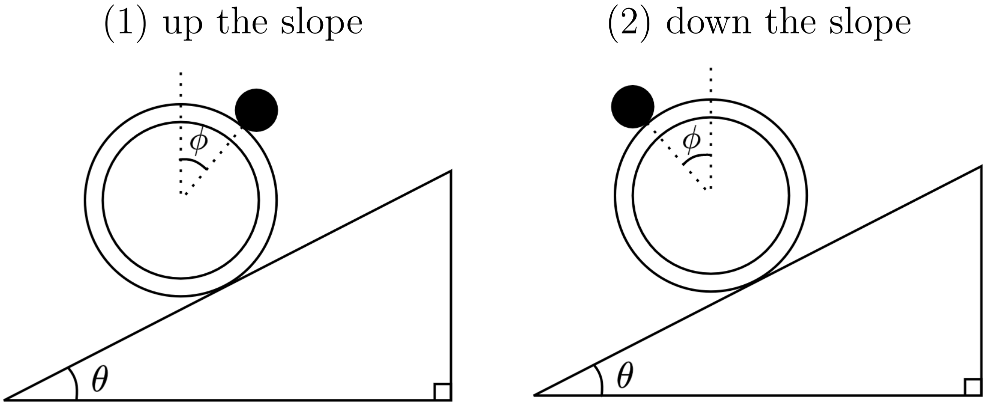
A thin uniform hoop of mass $M = 50$ g has a small bead of mass $m = 150$ g glued to a point on its rim. The hoop is placed on a slope inclined at angle $\theta = 30°$ above the horizontal, and is initially oriented such that the bead is directly above the hoop's centre.



Then, the hoop's position is slightly adjusted by rotating it ____________ by some angle $0° < \phi < 180°$. This way, when the hoop is released from rest, it begins rolling **up** the slope. Assume that the slope is sufficiently rough such that the hoop never slips.

(a) Which of the following options correctly fills in the blank above? (You may refer to the diagram below to visualise the physical setup illustrated by each option.)

(1 point)

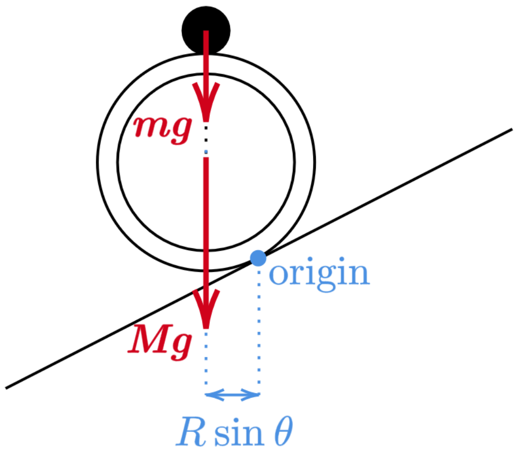


(b) The hoop rolls up upon release from rest if and only if $\phi_1 < \phi < \phi_2$, where ϕ_1 and ϕ_2 are constants to be determined. Find $\phi_2 - \phi_1$.

Leave your answer to 2 significant figures in units of degrees. (3 points)

Solution:

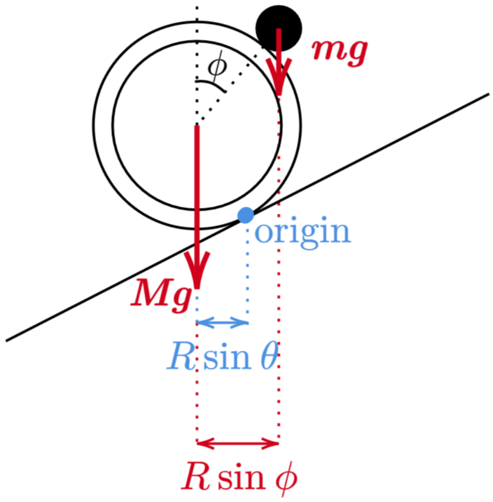
(a) Consider the forces acting on the hoop-and-bead system: the weight of the hoop, the weight of the bead, normal force and friction. Let us consider the torque acting on the system with the contact point between the hoop and the slope taken to be the origin. Our choice of origin means the torque due to normal force and friction may be neglected.



Refer to the diagram above. Initially, when the bead was directly above the hoop's centre, the torques due to the hoop's weight and the bead's weight are both directed anticlockwise. Hence the net torque on the system is anticlockwise, causing the hoop to roll down upon release.

However, if the system is rotated such that the bead is now displaced sufficiently to the right, the torque due to the bead's weight is clockwise while the torque due to the hoop's weight is anticlockwise. If the conditions are set up right, it is possible for the torque due to the bead's weight to exceed the torque due to the hoop's weight, such that the net torque is clockwise and the system rolls up upon release. As such, during the adjustment process, the hoop should be rolled $\boxed{\text{up the slope}}$.

(b) Let τ_h and τ_b be the torques due to the hoop's weight and the bead's weight respectively. The line of action of the hoop's weight is at perpendicular distance $R\sin\theta$ to the left of the origin, while the line of action of the bead's weight is at perpendicular distance $R(\sin\phi - \sin\theta)$ to the right of the origin.



Hence, we may write the following expressions for τ_h and τ_b:

$$\tau_h = MgR\sin\theta$$
$$\tau_b = mgR(\sin\phi - \sin\theta)$$

As previously discussed, the condition for roll-up is $\tau_b > \tau_h$, which gives us a condition for ϕ:

$$\sin\phi > \left(1 + \frac{M}{m}\right)\sin\theta$$

Solving for ϕ, this gives:

$$\sin^{-1}\left[\left(1 + \frac{M}{m}\right)\sin\theta\right] < \phi < 180^\circ - \sin^{-1}\left[\left(1 + \frac{M}{m}\right)\sin\theta\right]$$

ϕ_1 is the lower bound while ϕ_2 is the upper bound in the above inequality. Hence, we may calculate $\phi_2 - \phi_1$:

$$\phi_2 - \phi_1 = 180^\circ - 2\sin^{-1}\left[\left(1 + \frac{M}{m}\right)\sin\theta\right] \approx \boxed{96^\circ}$$

Alternative solution: We can similarly analyse the same problem from an energy perspective.

(a) Qualitatively, a system always tends towards the direction of decreasing potential energy. As the hoop rolls up the slope, the hoop's centre ascends, but if the bead also descends, it is possible for the system's potential energy to have a net decrease rather than increase. This way, rolling up can be energetically favoured!

For this to be possible, the hoop must have been rolled $\boxed{\text{up the slope}}$ during the adjustment process so that subsequent upward rotation causes the bead to descend.

(b) To work this out quantitatively, consider the vertical coordinate y with the upward direction taken to be positive, and set the reference point $y = 0$ to be the vertical level of the hoop's centre when $\phi = 0$. For any angle ϕ, the hoop's centre is located at $y_h = R\phi \sin\theta$, while the bead is at $y_b = R\phi \sin\theta + R\cos\phi$. The potential energy $V(\phi)$ of the system can thus be written as:

$$V(\phi) = Mgy_h + mgy_b = (M + m)gR\phi \sin\theta + mgR\cos\phi$$

Its derivative $V'(\phi)$ can be determined:

$$V'(\phi) = (M + m)gR\sin\theta - mgR\sin\phi$$

Since increasing ϕ corresponds to rolling the hoop up, $V'(\phi) < 0$ indicates that rolling the hoop up decreases potential energy. Hence, when $V'(\phi) < 0$, the hoop will spontaneously roll up. This gives us the condition for ϕ:

$$\sin\phi > \left(1 + \frac{M}{m}\right)\sin\theta$$

This inequality is the same as the one previously derived using forces, and solving for the range of ϕ will subsequently give us the same answer.

Note that this method assumes that there is no work done against friction. Since the hoop never slips, static friction indeed does no work, therefore this method is valid.

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Problem 20: Water Cube (4 points)

What is the work required to deform a spherical water droplet of radius $r = 1.0$ cm into the shape of a cube? Take the surface tension of water as $\gamma = 7.2 \times 10^{-2}$ N m^{-1}. Neglect gravity.

Leave your answer to 2 significant figures in units of μJ.

Solution: By the definition of surface tension, the energy associated with surface area A of a fluid is given by $E = \gamma A$. The energy associated with the surface of the original spherical droplet E_i is thus given by:

$$E_i = 4\pi r^2 \gamma$$

Since water is an incompressible fluid, the volume occupied by the original spherical droplet must be equal to the volume occupied by the final cube of water. Let the side length of this cube be l. We may express l in terms of r by applying this volume conservation principle:

$$l^3 = \frac{4}{3}\pi r^3 \implies l = r\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}$$

We can then determine the energy associated with the surface of the final water cube E_f:

$$E_f = 6l^2\gamma = 6\left(\frac{4}{3}\pi\right)^{\frac{2}{3}} r^2\gamma$$

The required work is thus given by:

$$E_f - E_i = r^2\gamma\left[6\left(\frac{4}{3}\pi\right)^{\frac{2}{3}} - 4\pi\right] \approx \boxed{22\ \mu\text{J}}$$

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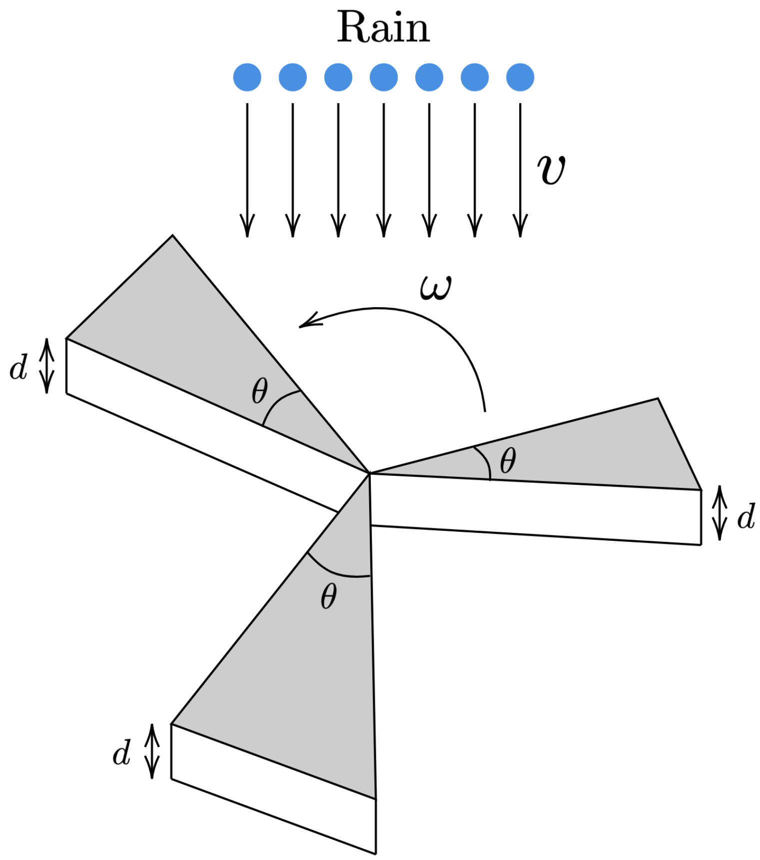
Problem 21: Ceiling Fan (3 points)

A ceiling fan comprises three horizontal triangular rigid blades arranged symmetrically, as drawn below. It rotates with a constant angular speed ω. Each blade has thickness $d = 0.1$ m and subtends an angle $\theta = \pi/6$ rad. Raindrops from above fall through the ceiling fan with **constant** velocity $v = 8$ m s^{-1} vertically downwards. What is the minimum ω required for every raindrop to be hit by the fan? Assume that the fan rotates about a fixed vertical axis without wobbling.

Leave your answer to 3 significant figures in units of rad s^{-1}.



Solution: The time interval t that each raindrop spends in the region between the fan blades is given by:

$$t = \frac{d}{v}$$

A raindrop hits the fan if and only if it collides with a blade during this short time t. Consider the raindrop that is least likely to get hit by the fan; that is, the raindrop at the furthest angle away from the nearest approaching blade. This raindrop is located at angle $(2\pi/3 - \theta)$ from the near end of the closest approaching blade. For the fan blade to traverse this angle $(2\pi/3 - \theta)$ within time t:

$$\omega t \geq \frac{2\pi}{3} - \theta \implies \omega \geq \left(\frac{2\pi}{3} - \theta\right)\frac{v}{d} \approx \boxed{126 \text{ rad s}^{-1}}$$

If the fan is able to hit this furthest raindrop, then it will hit all other raindrops. Hence, beyond this ω, the fan hits every raindrop. Evidently, this requires a very fast rotation (20 revolutions per second), so it's not easy to achieve this in practice.

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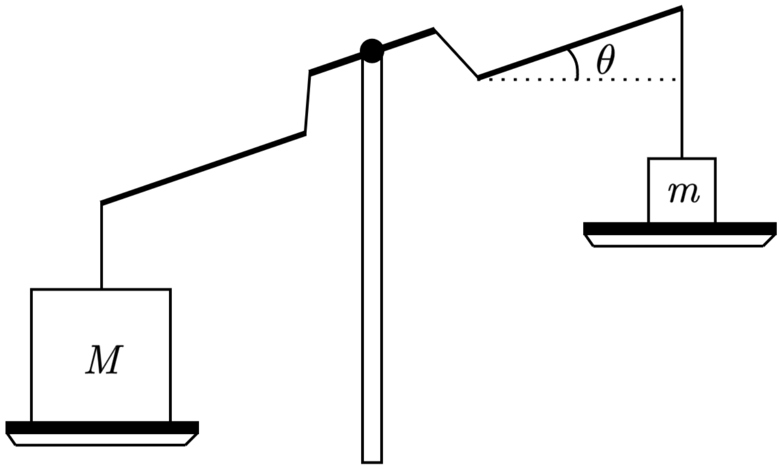
Problem 22: Balance Scale (4 points)

The first instrument invented for measuring mass was the balance scale, as drawn below. It comprises a beam of negligible weight (as shown in the diagram below) whose point of symmetry is held by a frictionless pivot, with two identical pans attached to its ends. When the two pans contain different masses, the beam deflects towards the heavier end.

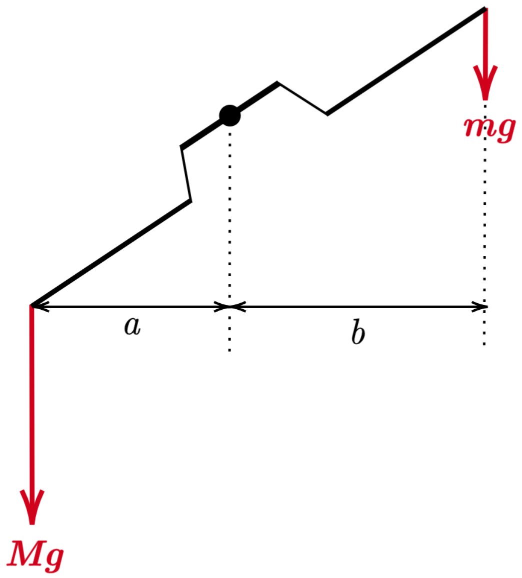


Let M and m be the masses of each pan and its respective contents. When $M/m = 2$, the equilibrium angle of the beam from the horizontal is $\theta = 25°$. If this ratio is doubled to $M/m = 4$, what is the new value of θ?

Leave your answer to 2 significant figures in degrees.

Solution: Let us assume that $M > m$. Let the length of the line connecting the two ends of the beam be $2l$. Considering the torques exerted on the beam by the masses, it seems like the net torque should be $(M - m)gl\cos\theta$. For all $\theta < 90°$, this net torque will be non-zero. Hence, it appears that it is impossible for the beam to be at equilibrium!

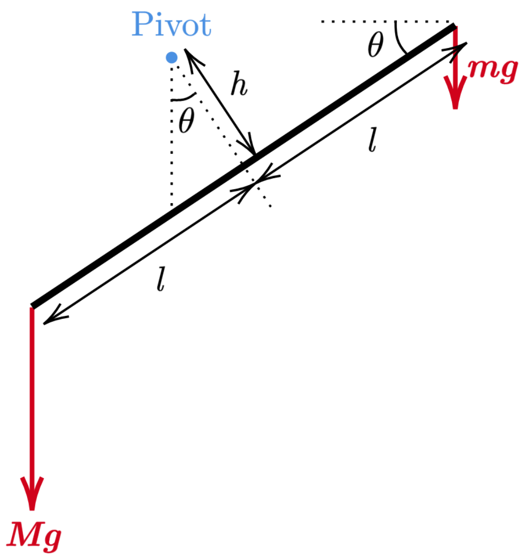
Of course, that is not true. The trick here is that, as hinted by the diagram, the beam is shaped in a peculiar manner, such that its pivot lies slightly above the line connecting the two ends of the beam. As such, when the beam rotates, the horizontal distances between the pivot and each end of the beam differ. This is illustrated below, where a is the horizontal distance between the heavier end of the beam and the pivot, and b is the horizontal distance between the lighter end of the beam and the pivot.



Evidently, $b > a$, making it possible for the torque mgb due to the weight of m to equate to the torque Mga due to the weight of M. Hence, equilibrium is achieved when:

$$Ma = mb$$

To relate a and b to θ, let the perpendicular distance between the line connecting the beam's ends and the pivot be h. Since the whole beam rotates as a rigid body, the distance h will never change.



Geometrically, a and b are:

$$a = l\cos\theta - h\sin\theta$$
$$b = l\cos\theta + h\sin\theta$$

Hence, we can find the equilibrium angle θ:

$$\tan\theta = \frac{M-m}{M+m}\frac{l}{h} = \frac{(M/m)-1}{(M/m)+1}\frac{l}{h}$$

It is given that when $M/m = 2$, $\theta = 25°$. Substituting these values, this means that $l/h = 3\tan 25°$.

Hence, when $M/m = 4$, $\tan\theta = (9\tan 25°)/5 \implies \theta \approx \boxed{40°}$.

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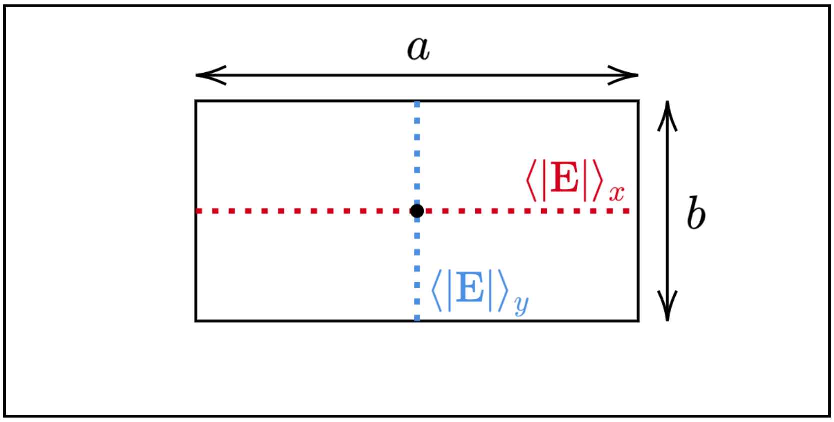
Problem 23: Rectangular Cavity (3 points)

Consider a thin charged rectangular plate made of conducting material. A rectangular cavity of horizontal length $a = 12$ cm and vertical length $b = 5$ cm is made at the centre of the plate.

A probe measures the magnitude $|\mathbf{E}|$ of the electric field within the cavity at every point along the horizontal through the cavity's centre. The average magnitude measured is $\langle|\mathbf{E}|\rangle_x$. Then, the probe measures the magnitude of the electric field within the cavity at every point along the vertical through the cavity's centre. The average magnitude measured is $\langle|\mathbf{E}|\rangle_y$. Determine the ratio $\langle|\mathbf{E}|\rangle_y/\langle|\mathbf{E}|\rangle_x$.

Leave your answer to 2 significant figures.



Solution: Let the centre of the cavity be point O. Let A be the point in the conductor at distance $a/2$ to the right of O, and B be the point in the conductor at distance $b/2$ above O.

The potential difference V_{OA} between points O and A is given by:

$$V_{OA} = \langle|\mathbf{E}|\rangle_x \frac{a}{2}$$

Similarly, the potential difference V_{OB} between points O and B is given by:

$$V_{OB} = \langle|\mathbf{E}|\rangle_y \frac{b}{2}$$

All points on a conductor are equipotential at equilibrium. As such, points A and B must have the same potential. This implies:

$$V_{OA} = V_{OB} \implies \langle|\mathbf{E}|\rangle_x \frac{a}{2} = \langle|\mathbf{E}|\rangle_y \frac{b}{2}$$

The ratio of the average fields is thus given by $\frac{\langle|\mathbf{E}|\rangle_y}{\langle|\mathbf{E}|\rangle_x} = \frac{a}{b} = \boxed{2.4}$.

Note that this solution only works because the electric field along line OA is horizontal at all points, while the electric field along line OB is vertical at all points, due to symmetry. As such, in the computation of V_{OA} and V_{OB}, the components of the electric field along lines OA and OB respectively correctly reflect the actual magnitude of the electric fields there.

Additionally, the $\langle|\mathbf{E}|\rangle_x$ used in calculating V_{OA} should, strictly speaking, be the average field purely along line OA. But, as specified by the question, $\langle|\mathbf{E}|\rangle_x$ refers to the average field along the whole horizontal inside the cavity (i.e. from point A to its reflection across the central vertical). However, due to symmetry about the central vertical, the field at any distance x to the left of O has the same magnitude as the field at distance x to the right of O. Hence, these two averages are actually equal, so the calculation is valid. The same logic applies to the $\langle|\mathbf{E}|\rangle_y$ used in calculating V_{OB}.

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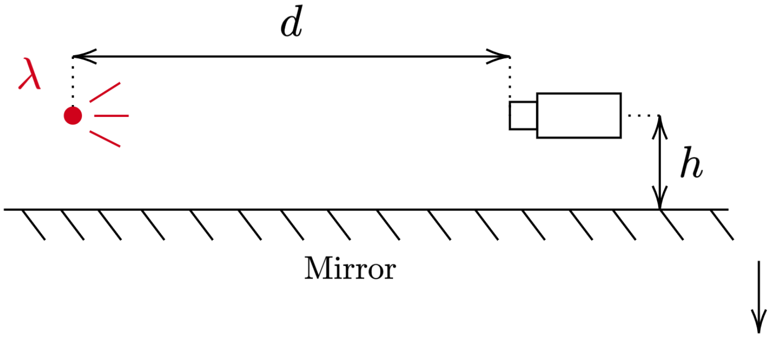
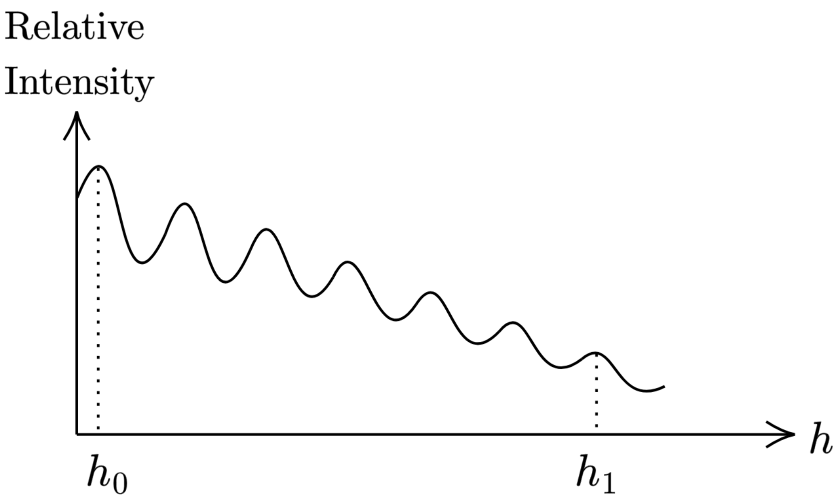
Problem 24: Uncomfortably Close Mirror (4 points)

A point source that emits monochromatic light of unknown wavelength λ and a photometer are placed a distance $d = 30$ cm apart and at some common perpendicular distance h from an adjacent mirror, as shown in the diagram below.



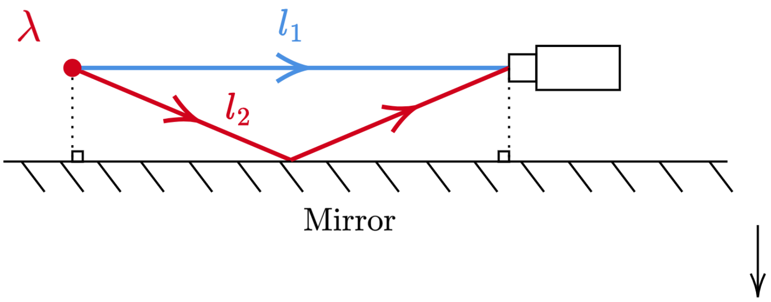
The mirror is then slowly moved from a distance of $h_0 = 1.0$ mm to a distance of $h_1 = 1.2$ mm from the source and photometer, while the resulting light intensity is recorded by the photometer. The graph of relative intensity recorded by the photometer is plotted below.



Deduce the wavelength λ of the source.

Leave your answer to 2 significant figures in units of nm.

Solution: Light from the light source reaches the photometer in two separate paths: in a straight line through the air, and in a bent path that reflects off the mirror.



From the diagram above, we can compute the former optical path length l_1 as

$$l_1 = d$$

and the latter optical path length l_2 as

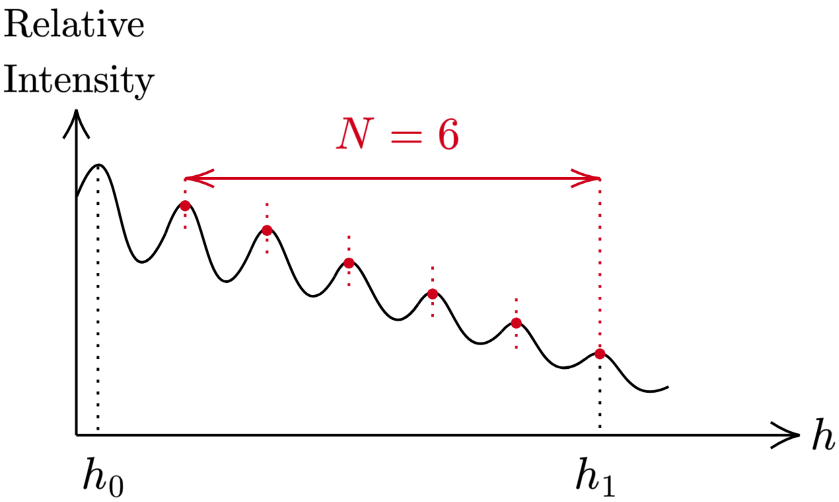
$$l_2 = \sqrt{\left(\frac{d}{2}\right)^2 + h^2} + \alpha + \sqrt{\left(\frac{d}{2}\right)^2 + h^2} = \sqrt{d^2 + 4h^2} + \alpha$$

where α is a constant to account for the phase shift when reflecting off the mirror.

The path length difference Δ is thus:

$$\Delta = l_2 - l_1 = \sqrt{d^2 + 4h^2} + \alpha - d$$

From the graph of relative intensity, we see that the light intensity at the photometer goes through exactly $N = 6$ cycles of intensity maxima and minima as h is increased from h_0 to h_1. Furthermore, h_0 and h_1 both correspond to intensity maxima themselves.



Intensity maxima occur when the optical path difference is an integer multiple of the wavelength. This implies that the optical path difference $\Delta(h)$, as a function of h, changes by precisely $N\lambda$ as h is increased from h_0 to h_1, and thus:

$$\Delta(h_1) - \Delta(h_0) = \left(\sqrt{d^2 + 4h_1^2} + \alpha - d\right) - \left(\sqrt{d^2 + 4h_0^2} + \alpha - d\right) = N\lambda$$

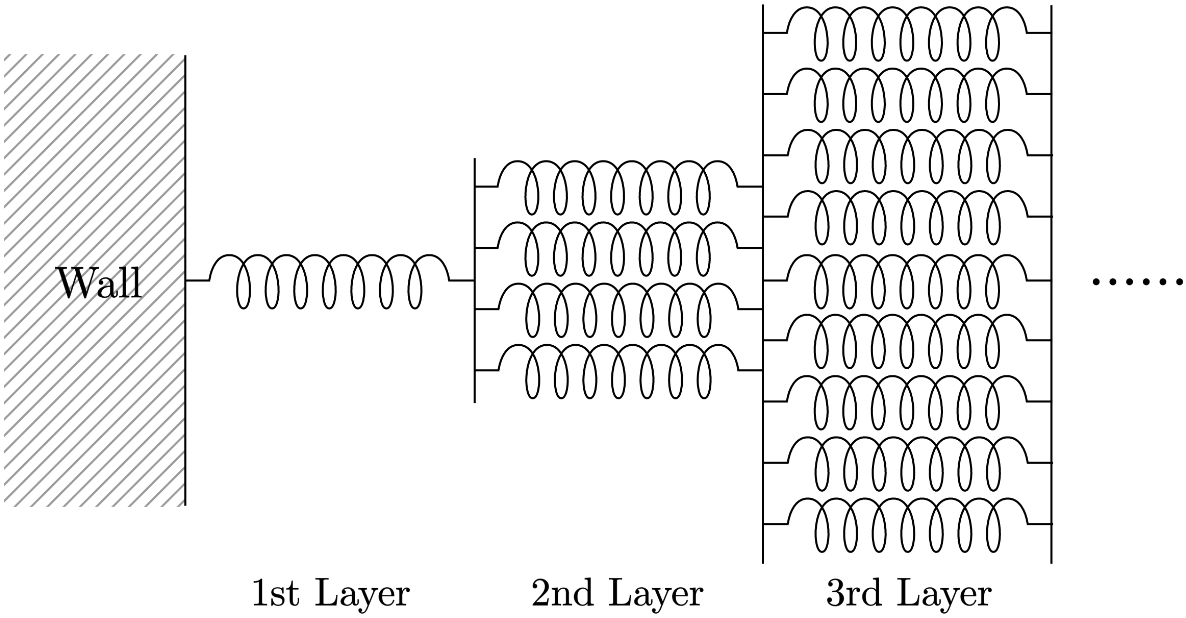
We can therefore isolate λ to get:

$$\lambda = \frac{1}{N}\left(\sqrt{d^2 + 4h_1^2} - \sqrt{d^2 + 4h_0^2}\right) \approx \boxed{490 \text{ nm}}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Problem 25: To Infinity and Beyond (4 points)

Benson has many light and ideal springs. These springs are identical, each having spring constant $k_0 = 1$ N m^{-1}. One day, as Benson was playing around, he accidentally arranged an *extremely large* number of them on a table in the following manner:



The arrangement comprises layers of springs separated by light plates. In the i^{th} layer, there are $n_i = i^2$ springs joining the $(i-1)^{\text{th}}$ plate to the i^{th} plate. The spring in the 1^{st} layer joins an immovable wall to the 1^{st} plate.

Benson now attaches a mass $m = 10$ kg to the last plate. He gives the mass a push so that it begins to oscillate. What would be the resultant period T of oscillation? Neglect damping.

Leave your answer to 2 significant figures in units of s.

Solution: Notice that in the i^{th} layer, the springs are arranged in parallel. Therefore, we can treat the n_i springs as one spring with equivalent spring constant k_i, where:

$$k_i = n_i k_0 = i^2 k_0$$

Consequently, we have transformed the set-up into that of a sequence of springs arranged in series, where the i^{th} spring has spring constant k_i.

We shall now define the effective spring constant, k_{eff}, of the whole system. For springs in series, we know that:

$$\frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \dots$$

Substituting the values for $k_1, k_2, k_3, \ldots$, we obtain:

$$\frac{1}{k_{\text{eff}}} = \frac{1}{k_0}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = \frac{\pi^2}{6k_0}$$

In the last equality, the sum of the reciprocals of squares is given by $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.[3]

We know that the period of a simple spring-mass system is given by $T = 2\pi\sqrt{m/k}$. Applying this formula to our system:

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} = \left(2\sqrt{\frac{5}{3}}\right)\pi^2 \approx \boxed{25 \text{ s}}$$

Setter: Shaun Quek, **shaun.quek@sgphysicsleague.org**

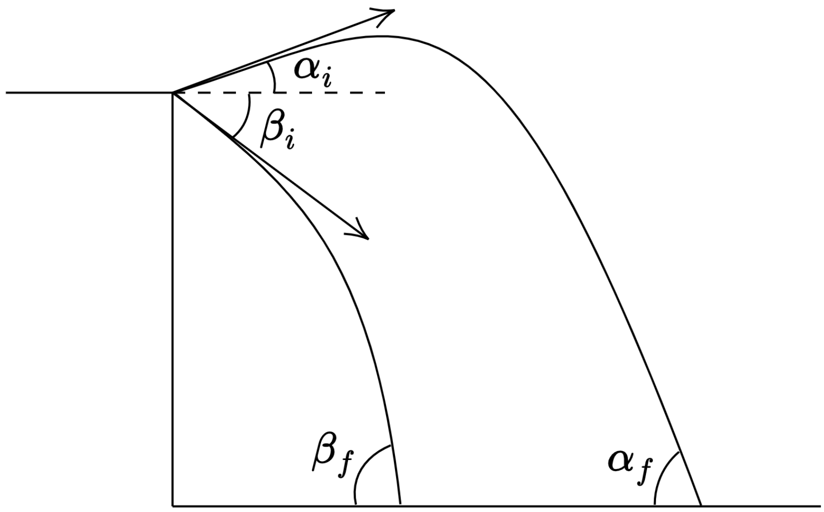
[3]This is in fact a famous historical problem (Basel Problem), solved by Leonhard Euler in 1734.

Problem 26: Landing Angles (4 points)

A projectile is launched from the edge of a cliff at angle $\alpha_i = 30°$ above the horizontal. At landing, it makes angle $\alpha_f = 60°$ with the flat horizontal ground below the cliff.

A second projectile is launched from the same point and at the same speed as the first projectile, but now at angle $\beta_i = 45°$ below the horizontal. At landing, what angle β_f does the second projectile make with the ground?

Leave your answer to 3 significant figures in units of degrees.



Solution: Both projectiles are launched at the same initial speeds, so their initial kinetic energies are equal. Additionally, since they were launched from the same point, they started out at the same height above the ground, so they also start out with equal gravitational potential energies. As such, both projectiles have the same total energy.

Since energy is conserved, this means that both projectiles land with the same final kinetic energies. In other words, both projectiles have exactly the same speed at landing.

Let the initial launch speeds of the projectiles be u, and their final landing speeds be v. The only force experienced by each projectile is its weight, so it does not have any acceleration in the horizontal direction. The horizontal component of its velocity is thus preserved throughout its motion. We may thus equate the horizontal components of the initial and final velocities for the first projectile:

$$u \cos \alpha_i = v \cos \alpha_f \implies v = u \frac{\cos \alpha_i}{\cos \alpha_f}$$

We can write a similar equation for the second projectile:

$$u \cos \beta_i = v \cos \beta_f$$

Substituting the previously found expression for v, we may express β_f purely in terms of α_i, α_f and β_i:

$$u \cos \beta_i = u \frac{\cos \alpha_i}{\cos \alpha_f} \cos \beta_f \implies \cos \beta_f = \cos \beta_i \frac{\cos \alpha_f}{\cos \alpha_i}$$

This gives $\cos \beta_f = \frac{1}{\sqrt{6}} \implies \beta_f \approx \boxed{65.9^\circ}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 27: Take a Chance on Me (3 points)

What is the probability that a sample of uranium-235 with mass $M = 15$ μg has at least 1 decay within time $T = 0.01$ s? You may take the half-life of uranium to be $t_{1/2} = 703.8$ million years.

Leave your answer to 2 significant figures.

Your answer should range between 0 and 1.

Solution: The probability of a single nucleus not decaying within time t is $e^{-\lambda t}$, where λ is the decay constant. For uranium-235, we can calculate the decay constant as follows:

$$\lambda = \frac{\ln 2}{t_{1/2}} \approx 3.121 \times 10^{-17}\ \mathrm{s}^{-1}$$

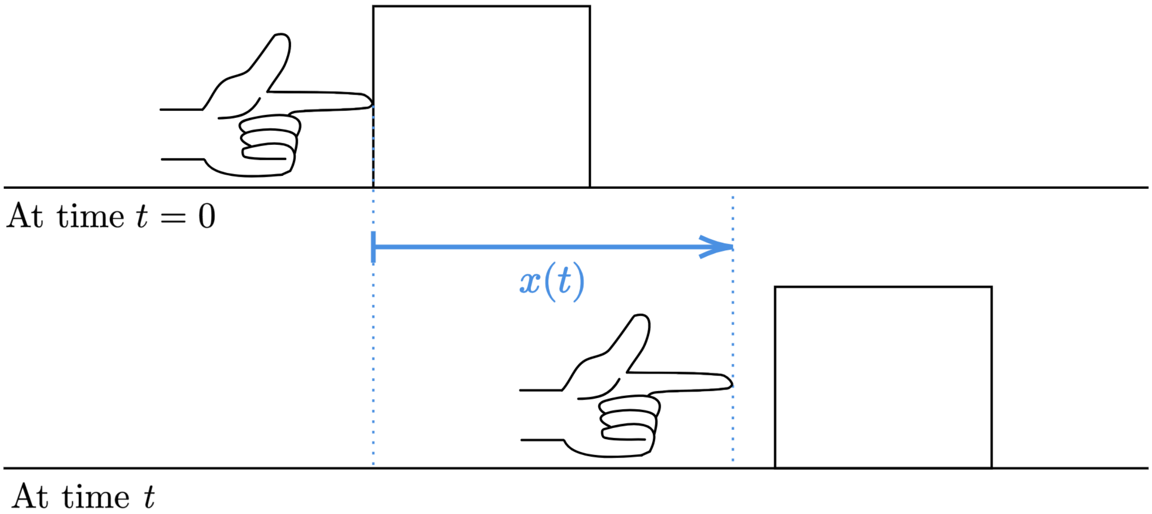
Since the decay of every nucleus is an independent event, the probability $P(0)$ that every nucleus does not decay within time t is given by $P(0) = \left(e^{-\lambda t}\right)^{N_0} = e^{-N_0 \lambda t}$, where N_0 refers to the initial total number of nuclei. Hence, the probability of at least one decay taking place within time t is $1 - P(0)$, which yields:

$$\begin{aligned}
1 - P(0) &= 1 - e^{N_0 \lambda T} \\
&= 1 - \exp\left(-\left(\frac{15 \times 10^{-6}}{235} \times 6.02 \times 10^{23}\right)(3.121 \times 10^{-17})(0.01)\right) \\
&\approx \boxed{0.012}
\end{aligned}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 28: Pushing a Block

A block is placed on a frictionless horizontal table. A finger begins to push the block from its left side. The displacement of the finger at time t (in s) is x (in cm), where x is given by $x = t^2 \sin t$. (The argument of the sin function is taken to be in units of radians.) We set the origin $x = 0$ at the initial position of the block, with x increasing rightwards.



(a) What is the position x_0 of the block when it first loses contact with the finger?

Leave your answer to 2 significant figures in units of cm. (3 points)

(b) What is the position x_1 of the block when it first regains contact with the finger?

Leave your answer to 2 significant figures in units of cm. (3 points)

If you think that the block does not regain contact with the finger, submit your answer as 0.

Solution:

(a) The only force acting on the block is supplied by the finger, thus, letting F be the force from the finger and m be the mass of the block, Newton's Second Law reads:

$$F = m\ddot{x}$$

Note further that $F > 0$ since the finger can push the block but cannot pull the block. Therefore $\ddot{x} > 0$ and contact is lost when this condition is violated. Differentiating the given expression for x yields:

$$\dot{x} = 2t \sin t + t^2 \cos t$$
$$\ddot{x} = (2 - t^2) \sin t + 4t \cos t > 0$$

By DESMOS or graphing calculator, the first value of t where $\ddot{x} > 0$ is violated is $t_0 \approx 1.52$ s, which yields $x_0 \approx \boxed{2.3 \text{ cm}}$.

Side-note: A classic blunder/rookie mistake would be to use $\dot{x} = 0$ as the condition for losing contact instead of $\ddot{x} = 0$.

(b) The velocity of the block immediately after losing contact with the finger is $v_0 = 2t_0 \sin t_0 + t_0^2 \cos t_0$.

By Newton's First Law, after losing contact with the finger, the block travels at constant velocity v_0 until regaining contact with the finger. As such, the graph of the block's position against time during this phase of its motion is linear, with gradient v_0. It passes through the coordinates (t_0, x_0).

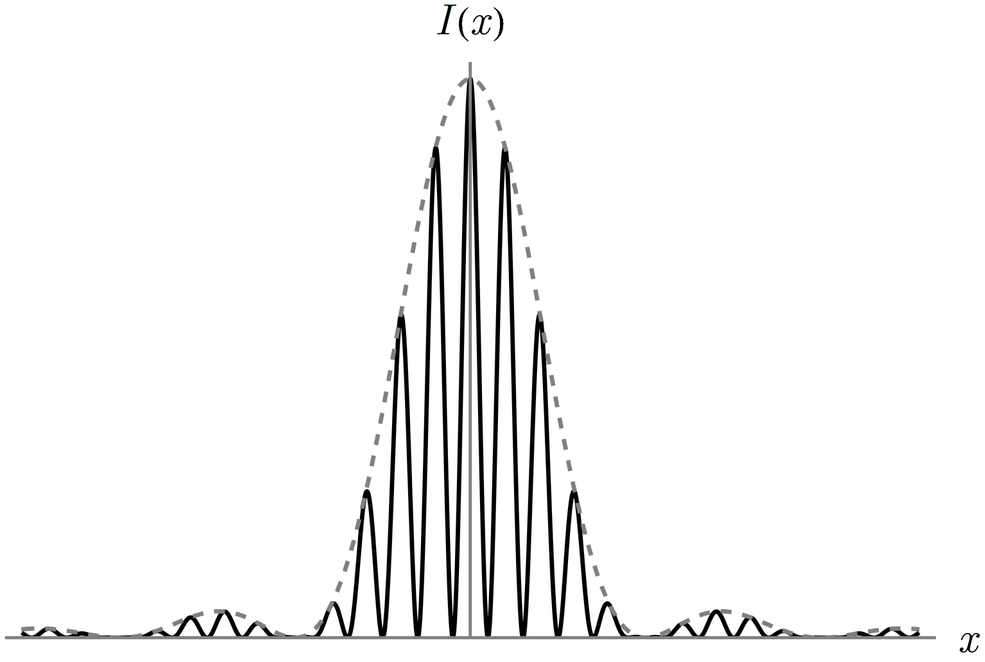
Hence, the function $x = x_0 + v_0(t - t_0)$ describes the displacement of the block between the first loss of contact and the first regaining of contact with the finger. Numerically, we have $x \approx 3.15t - 2.49$.

Graphing this using DESMOS or a suitable graphing calculator and comparing with the x-t graph of the finger, the next intersection is at $x_1 \approx \boxed{19 \text{ cm}}$.

Setter: Brian Siew, brian.siew@sgphysicsleague.org

Problem 29: Intensity Profiles (3 points)

In a far-field double-slit interference experiment, the two slits are separated by distance D between their centres, and each slit has finite width $D/5$. The resulting intensity profile across the screen $I(x)$ is plotted against position x along the screen, as shown below.



Another interference experiment is conducted under identical conditions, except that the separation between the centre of the slits is now $4D/5$. The width of each slit is unchanged. The new intensity profile obtained is given by $i(x)$.

Calculate the ratio

$$\frac{\int_{-\infty}^{\infty} i(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty} I(x)\,\mathrm{d}x}$$

Leave your answer to 2 significant figures.

Solution: It is useful to think about what the area under $I(x)$ represents. For convenience, let the x-direction be horizontal. Consider an infinitesimal rectangular slice of the screen with dimensions $\mathrm{d}x$ and L, where L is the vertical height of the screen. The area of this slice is $L\,\mathrm{d}x$. Since intensity is defined as power delivered per unit area, the power delivered to the slice is $I(x)L\,\mathrm{d}x$.

If the power reaching the screen is given by P, the area under $I(x)$ is given by the following relation:

$$\int_{-\infty}^{\infty} I(x)\,\mathrm{d}x = \frac{P}{L}$$

When the distance between slits was changed from D to $4D/5$, there is no change to the power from the slits reaching the screen, since the power of the light source was not changed. As such, with no changes to P or L, the area under $i(x)$ will still be equal to the area under $I(x)$:

$$\int_{-\infty}^{\infty} i(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} I(x)\,\mathrm{d}x$$

Hence, the required ratio $\frac{\int_{-\infty}^{\infty} i(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty} I(x)\,\mathrm{d}x} = \boxed{1.0}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 30: A Water Bubble (4 points)

Conventional wisdom suggests that bubbles float. However, under certain conditions, it turns out that bubbles can, in fact, sink!

A bubble contains an ideal gas with molar mass $\mu = 100$ g mol^{-1} and temperature $T = 300$ K. Find the minimum depth h of the bubble below sea level such that it sinks. Assume that Earth's gravitational field is uniform with magnitude $g = 9.81$ m s^{-2}, and that the density of water is uniform at $\rho_w = 1000$ kg m^{-3}. Neglect any change in state of the gas, and take the bubble's radius to be much smaller than h.

Leave your answer to 3 significant figures in units of m.

Solution: For the bubble to sink in water, the density of the bubble ρ_b has to be greater than the density of water. In other words:

$$\rho_b > \rho_w$$

We can express ρ_b in terms of μ, the number of moles n of gas in the bubble, and volume V of the bubble:

$$\rho_b = \frac{m}{V} = \mu\frac{n}{V}$$

Using the ideal gas law $n/V = p/RT$, we can rewrite ρ_b in terms of pressure p and temperature T:

$$\rho_b = \frac{p\mu}{RT}$$

Furthermore, we also know that the pressure p is given by $p = p_0 + \rho_w gh$. Thus, we obtain a final expression for ρ_b:

$$\rho_b = \frac{(p_0 + \rho_w gh)\mu}{RT}$$

Applying this expression to our initial inequality, we obtain:

$$\frac{(p_0 + \rho_w gh)\mu}{RT} > \rho_w \implies h > \frac{1}{g}\left(\frac{RT}{\mu} - \frac{p_0}{\rho_w}\right) \approx \boxed{2530 \text{ m}}$$

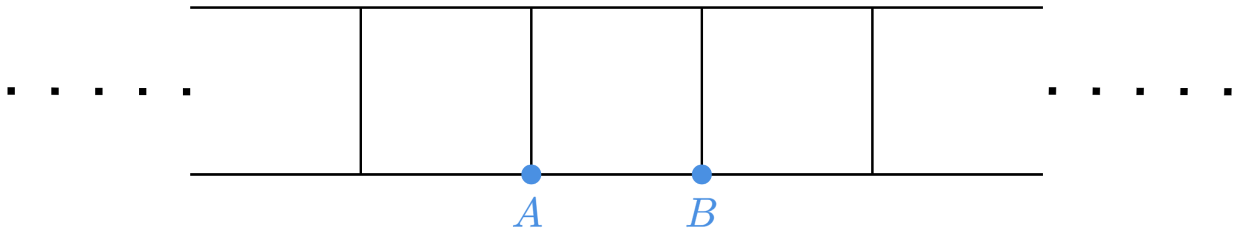
This corresponds to $p \approx 25$ MPa[4]. Consequently, it is unlikely that the gas obeys the ideal gas law, and in reality, it may actually liquefy or even solidify. Therefore, the simplistic assumptions above may not be valid.

Setter: Shaun Quek, shaun.quek@sgphysicsleague.org

[4]Quite a Brobdingnagian amount

Problem 31: Do You Prefer 2D or 3D?

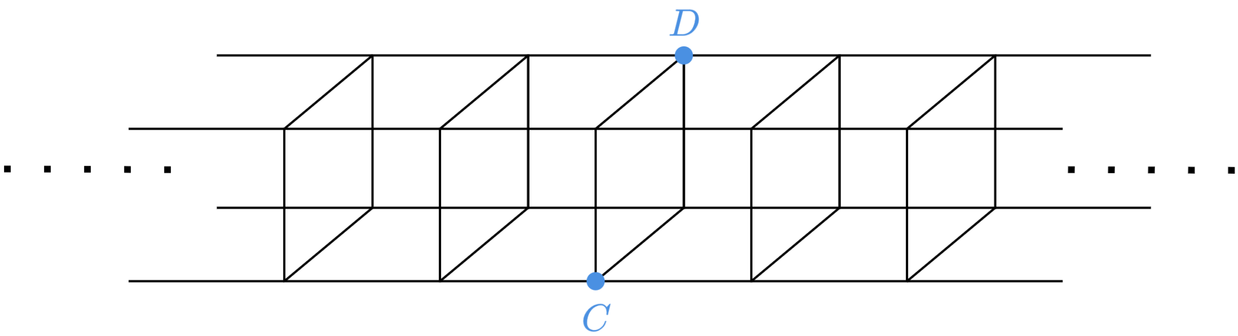
(a) Consider the planar infinitely-extending circuit as shown below, where each edge represents a resistor with resistance $R = 1\ \Omega$.



Let R_{AB} denote the effective resistance between the points A and B. Find R_{AB}.

Leave your answer to 3 significant figures in units of Ω. (4 points)

(b) Now, consider a similar infinitely-extending circuit that is cuboidal rather than planar as shown below, with each edge also having resistance $R = 1\ \Omega$.

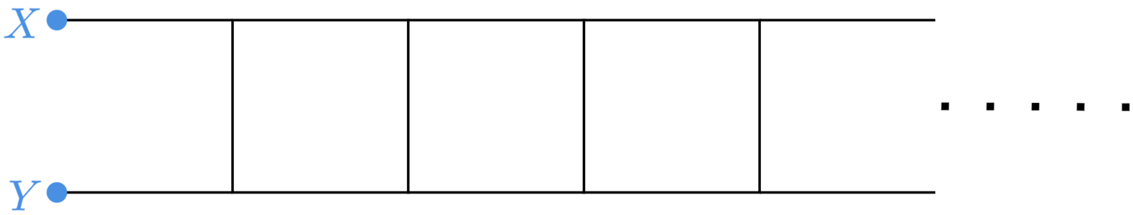


Let R_{CD} denote the effective resistance between the points C and D. Find R_{CD}.

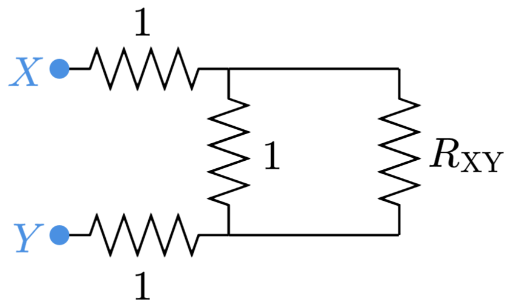
Leave your answer to 3 significant figures in units of Ω. (4 points)

Solution:

(a) To determine R_{AB}, rather than considering an infinite circuit, we consider a similar semi-infinite problem:



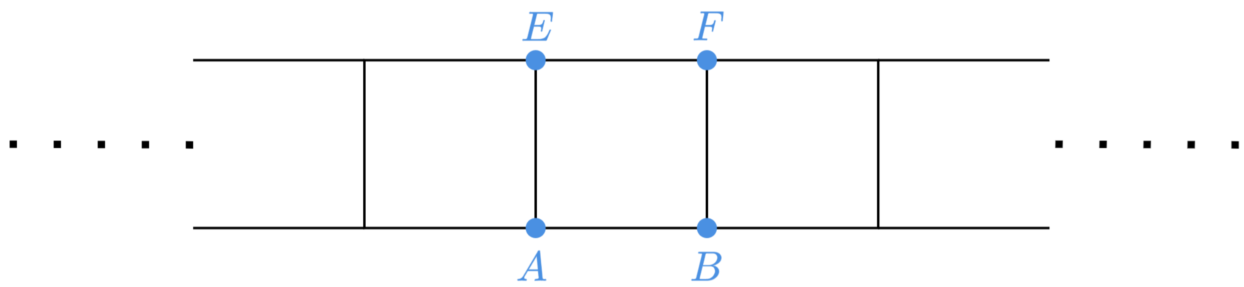
Exploiting the recursive nature of the circuit, we may re-draw the circuit as shown to solve for the effective resistance R_{XY} between X and Y:



This gives us the equation:

$$R_{XY} = 2 + \left(\frac{1}{1} + \frac{1}{R_{XY}}\right)^{-1} \implies R_{XY}^2 - 2R_{XY} - 2 = 0$$

Solving for R_{XY}, the only positive root is $R_{XY} = 1 + \sqrt{3}$. Now, we introduce two additional labels E and F:



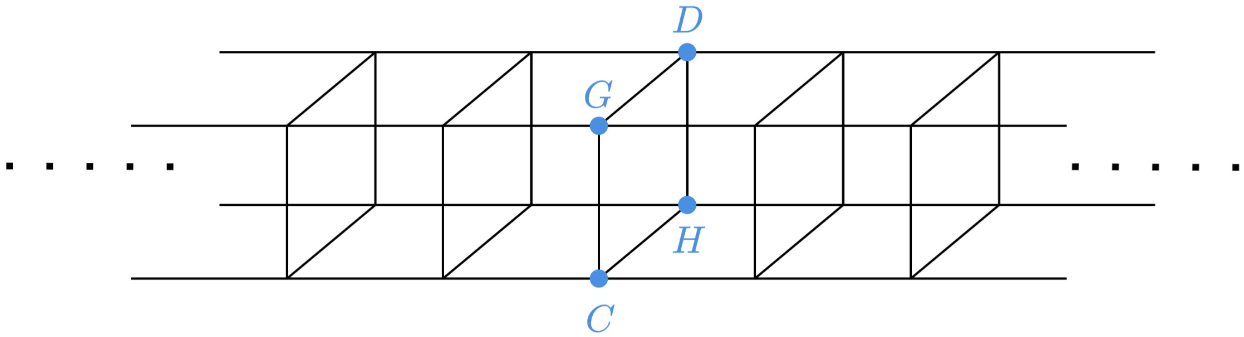
We then use our earlier result in parallel with a 1 Ω resistor to find the semi-infinite resistances R_{AE} and R_{BF}:

$$R_{AE} = R_{BF} = \left(\frac{1}{R_{XY}} + \frac{1}{1}\right)^{-1} = \frac{1+\sqrt{3}}{2+\sqrt{3}}$$

We may hence compute R_{AB} as follows:

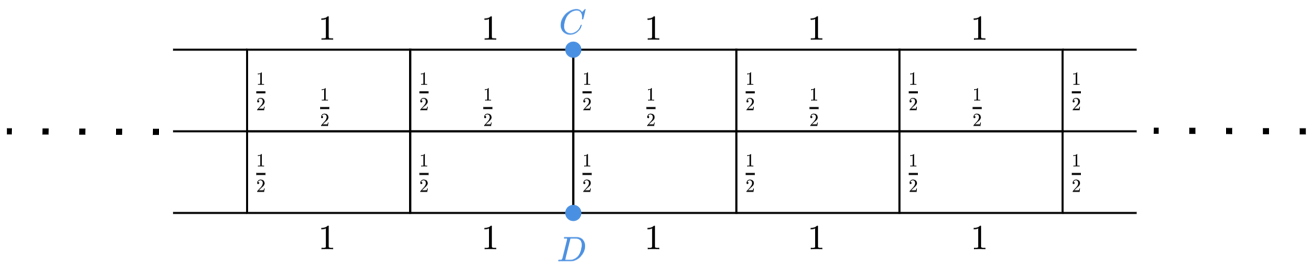
$$\begin{aligned} R_{AB} &= \left(\frac{1}{R_{AE} + R_{BF} + 1} + \frac{1}{1}\right)^{-1} \\ &= \left(\frac{1}{1} + \frac{2+\sqrt{3}}{4+3\sqrt{3}}\right)^{-1} = \frac{6-\sqrt{3}}{6} \approx \boxed{0.711\ \Omega} \end{aligned}$$

(b) Similarly to (a), we introduce the additional labels G and H:



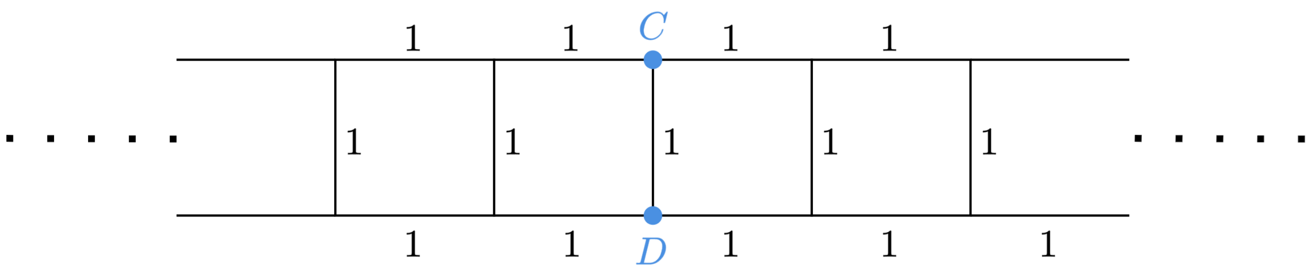
We first observe that by the reflective symmetry across the diagonal CD, the points G and H are of equal potential. We may thus treat G and H as the same point, with the effective resistance from C to this new joint point being that of two 1 Ω resistors in parallel, i.e. $\frac{1}{2}$ Ω. Similarly, the effective resistance from this joint point to D is $\frac{1}{2}$ Ω as well. (A similar argument applies for all other similarly directed diagonals.)

Our circuit can thus be re-drawn as such:



Once again, the symmetry of the circuit allows us to conclude no current flows through the central horizontal branch of the circuit.

This allows a final re-drawing of the circuit:



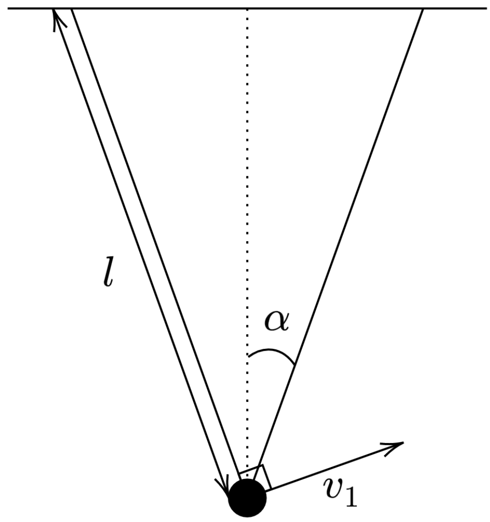
This is remarkably similar to our original problem! We can thereby calculate R_{CD}:

$$R_{CD} = \left(\frac{1}{1} + \frac{2}{1+\sqrt{3}}\right)^{-1} = \frac{\sqrt{3}}{3} \approx \boxed{0.577\ \Omega}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 32: Strange Pendulum

A mass is suspended from the ceiling by two light inextensible strings of equal lengths $l = 3$ m arranged in a "V" shape. The angle between either string and the vertical is $\alpha = \pi/18$ rad. The mass is given a push that causes it to swing rightwards with an initial velocity $v_1 = 0.5$ m s^{-1} perpendicular to the left string. Assume that the small-angle approximation is true, and that all motion takes place in the plane of the figure below.



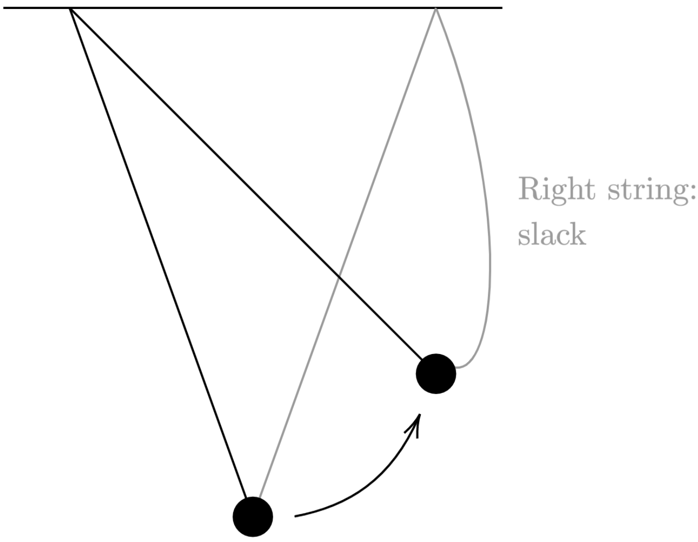
(a) Find the time taken for the mass to first return to its starting position, t_1.

Leave your answer to 2 significant figures in units of s. (3 points)

(b) Find the time taken for the mass to complete two oscillations, t_{total}.

Leave your answer to 2 significant figures in units of s. (4 points)

Solution: First, let's think about the situation qualitatively. After the mass is pushed, the left string will constrain it to move like an ordinary pendulum. The right string has no influence on the motion of the mass since it will be slack. This motion continues until the mass returns to its starting position. Then, the right string will constrain the motion of the mass, while the left string will be slack.



(a) For this period of time, under the small-angle approximation, the mass moves in simple harmonic motion. Thus, the angle $\theta(t)$ of the left string from the vertical can be described using a sinusoidal function:

$$\theta(t) = A\sin(\omega t + \phi)$$

where amplitude A denotes the maximum θ reached in the motion, angular frequency $\omega = \sqrt{g/l}$, and ϕ is the phase angle.

Conservation of energy gives us:

$$\frac{1}{2}mv_1^2 = mgl(\cos\alpha - \cos A) \implies A = \cos^{-1}\left(\cos\alpha - \frac{v_1^2}{2gl}\right)$$

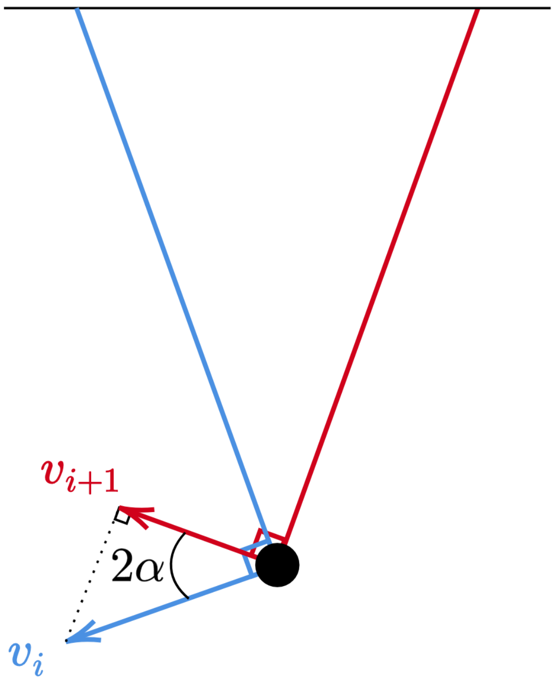
Since the string did not start out vertical, we have the initial condition $\theta(0) = \alpha$. Substituting this in, we obtain:

$$\phi = \sin^{-1}\left(\frac{\alpha}{A}\right)$$

To find t_1, we seek to solve for $\theta(t) = \alpha$ for the smallest non-zero t. Thus, t_1 satisfies $\omega t_1 + \phi = \pi - \phi$, which simplifies to give:

$$t_1 = \frac{\pi - 2\sin^{-1}(\alpha/A)}{\omega} \approx \boxed{0.54 \text{ s}}$$

(b) At the instant the constraining string changes, the new constraining string will deliver an instantaneous impulse to the mass parallel to the string. As such, relative to the new constraining string, the radial component of the mass' velocity is brought to zero, and only its tangential component is preserved. Therefore, the velocity of the mass will be reduced, and the period of each successive "half-oscillation" will also be reduced.



We define v_i as the velocity of the mass at the start of the ith half-oscillation (i.e. when the mass is right at the centre). By considering the geometry of the setup, we can write:

$$v_{i+1} = v_i \cos(2\alpha) \implies v_i = v_1 \cos^{i-1}(2\alpha)$$

Similar to part (a), we have

$$\frac{1}{2}mv_i^2 = mgl(\cos\alpha - \cos A_i) \implies A_i = \cos^{-1}\left(\cos\alpha - \frac{v_i^2}{2gl}\right)$$

where A_i denotes the amplitude of the ith half-oscillation, as well as

$$t_i = \frac{\pi - 2\sin^{-1}(\alpha/A_i)}{\omega}$$

where t_i is the time taken for the ith half-oscillation.

Computing the v_i, A_i, t_i values for $i = 1, 2, 3, 4$, we total up the values for t_i to give:

$$t_{\text{total}} = t_1 + t_2 + t_3 + t_4 \approx \boxed{2.0 \text{ s}}$$

Setter: Luo Zeyuan, zeyuan.luo@sgphysicsleague.org

Problem 33: Don't Change My Field! (4 points)

A circular loop of radius $r = 1.00$ m and resistance $R = 5.00 \times 10^{-4}\ \Omega$ is fixed and prevented from moving or rotating. It is placed in an external uniform magnetic field B_{ext} that is directed perpendicular to the plane of the loop.

This external field B_{ext} increases with time t as given by $B_{\text{ext}}(t) = B_0 e^{\alpha t}$. As such, the net magnetic field at the loop's centre $B_{\text{loop}}(t)$ also increases over time, but at a slower rate due to Lenz's law. In fact, the ratio $B_{\text{loop}}(t)/B_{\text{ext}}(t)$ is always constant for all time t. Find this ratio.

Take $\alpha = 1.00\ \text{s}^{-1}$; you may treat this to be sufficiently small for the quasistatic approximation to be valid.

Leave your answer to 3 significant figures.

Solution: The magnetic flux ϕ through the loop is given by $\phi = \pi r^2 B_{\text{ext}}$. (By the quasistatic approximation, the contribution to magnetic flux from any induced currents can be neglected.) Hence, by Faraday's Law, the magnitude of the emf ε induced in the loop due to changes in the magnetic flux is given by $\varepsilon = \dot{\phi} = \pi r^2 \dot{B}_{\text{ext}}$. So, letting I be the current through the loop:

$$I = \frac{\varepsilon}{R} = \frac{\pi r^2}{R}\dot{B}_{\text{ext}}$$

Since the loop is circular, the magnetic field B_{induced} generated at the loop's centre by the induced current is given by:

$$B_{\text{induced}} = \frac{\mu_0 I}{2r} = \frac{\mu_0 \pi r}{2R}\dot{B}_{\text{ext}}$$

Again, this relies on the quasistatic approximation; the field must be changing slowly enough for the magnetostatic expression to hold true.

The net magnetic field at the loop's center B_{loop} is made up of both the external field and the field produced by the loop. By Lenz's law, the induced current points in the direction that counteracts the change in magnetic flux. Hence, B_{loop} is given by:

$$B_{\text{loop}} = B_{\text{ext}} - B_{\text{induced}} = B_{\text{ext}} - \frac{\mu_0 \pi r}{2R}\dot{B}_{\text{ext}} = B_0 e^{\alpha t}\left(1 - \frac{\mu_0 \pi r \alpha}{2R}\right)$$

The required ratio is thus $1 - \frac{\mu_0 \pi r \alpha}{2R} \approx \boxed{0.996}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 34: I Thought Solutions were Unique

Suppose that we have an exotic gas that has the strange property of "reverse drag" – it increases the velocity of anything in it. A particle of mass $m = 1$ kg is placed in this gas. We denote its velocity to be $v(t)$ at time t after entering the gas. The force exerted by the gas on the particle is given by $k\sqrt{|v(t)|}$, where $k = 1$ N m$^{-1/2}$ s$^{1/2}$.

(a) Given $v(0) = 1$ m s^{-1} and $v(2) = 4$ m s^{-1}, find $v(12)$.

Leave your answer to 2 significant figures in units of m s^{-1}. (2 points)

(b) Given $v(0) = 0$ m s^{-1} and $v(2) = 1$ m s^{-1}, find $v(12)$.

Leave your answer to 2 significant figures in units of m s^{-1}. (2 points)

(c) Given $v(0) = 0$ m s^{-1} and $v(8) = 4$ m s^{-1}, find $v(12)$.

Leave your answer to 2 significant figures in units of m s^{-1}. (3 points)

Solution: First, noting that $v(0)$ for all 3 parts is non-negative and that the particle's velocity can only increase, $v(t)$ is always non-negative. Hence, $|v(t)| = v(t)$ is always true for the given conditions. By invoking Newton's Second Law, we have:

$$m\dot{v} = k\sqrt{v}$$

Noting that $k/m = 1$, this reduces to the following differential equation:

$$\dot{v} = \sqrt{v}$$

The usual way to solve this would be by separating variables:

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= \sqrt{v} \\ \int_{v_0}^{v(t)} \frac{\mathrm{d}v}{\sqrt{v}} &= \int_0^t \mathrm{d}t \\ 2\sqrt{v(t)} - 2\sqrt{v_0} &= t \\ v(t) &= \frac{1}{4}\left(t + 2\sqrt{v_0}\right)^2 \end{aligned}$$

where v_0 denotes the initial velocity of the particle.

(a) In the first case, this can be solved using the method above with $v_0 = 1$ to give

$$v(t) = \frac{1}{4}(t+2)^2$$

and we can verify $v(2) = 2$. Hence, $v(12) = \boxed{49 \text{ m s}^{-1}}$.

(b) Similarly, the second case this can be solved using the method above with $v_0 = 0$ to give

$$v(t) = \frac{1}{4}t^2$$

and we can verify $v(2) = 1$. Hence, $v(12) = \boxed{36 \text{ m s}^{-1}}$.

(c) The third case also involves $v_0 = 0$, but the solution from before gives $v(8) = 16$, which differs from the question's specification that $v(8) = 4$.

This is because the solution to the differential equation is not unique. We can actually write all possible solutions to $v_0 = 0$ (proof given in a later subsection):

$$v(t) = \frac{1}{4}\begin{cases} -(t-A)^2 & t \leq A \\ 0 & A \leq t \leq B \\ (t-B)^2 & t \geq B \end{cases}$$

for arbitrary $A \leq 0 \leq B$.

In this case, we are only concerned with $t > 0$. Using the question's requirement that $v(8) = 4$, we have $\frac{1}{4}(8 - B)^2 = 4$, hence $B = 4$ (we omit the other root as $B \leq 8$), giving us:

$$v(t) = \frac{1}{4}\begin{cases} 0 & t \leq 4 \\ (t-4)^2 & t \geq 4 \end{cases}$$

Hence, $v(12) = \frac{1}{4}(12 - 4)^2 = \boxed{16 \text{ m s}^{-1}}$.

From (b) and (c), we see that there are multiple possible forms of $v(t)$ that solve the differential equation. An intuitive reason for this is that the system is extremely unstable – any small perturbation would cause a significant change in the behaviour of $v(t)$. In our case, $v(t)$ can remain at 0 until supplied with an infinitesimal perturbation at any time, after which it takes on the form of the quadratic function.

To aid our intuition, it would therefore be useful to analyse the stability of the $v = 0$ equilibrium. We recall how we usually define stability: Given some force $F(x) = -V'(x)$, we say that the system is stable at an equilibrium point x_0 if $V''(x_0) = -F'(x_0) > 0$ and is unstable if $V''(x_0) = -F'(x_0) < 0$. The equation of motion in this case is given by $\ddot{x} = \frac{1}{m}F(x)$. We may thus rewrite the stability condition to be $\frac{\partial}{\partial x}\ddot{x} < 0$.

Substituting for $F(x)$ in our case, we have

$$\ddot{v} = \frac{\dot{v}\,\operatorname{sgn} v}{2\sqrt{|v|}}$$
$$\frac{\partial}{\partial v}\ddot{v} = -\frac{\dot{v}}{4\sqrt{v}^3} + \frac{\delta(v)}{2\sqrt{|v|}}$$

where $\delta(v)$ denotes the Dirac delta function. This tells us that at the equilibrium point $v = 0$, the system is extremely unstable as $\delta(v) = +\infty$. (The term $\frac{\dot{v}}{4\sqrt{v}^3}$ can be neglected since the Dirac delta term dominates.) Therefore, this allows for non-unique solutions.

At the exact point when the velocity of the particle starts increasing (at $t = 4$), even though the higher derivatives of v are discontinuous or diverge to infinity, the force is still finite (due to the fact that $\dot{v} = 0$ when $v = 0$). The solution provided is therefore still valid as Newtonian mechanics does not impose a smoothness condition on the higher derivatives of v.

Although we typically expect trajectories in reality to be "smooth", we often model them by functions that aren't smooth, for instance with the particle-in-box problem. Our problem thus gives an example where such models may fail due to the possibility of non-unique solutions to differential equations.

This lack of uniqueness appears in other places as well. For instance, consider a rod with 3 supports. We have 3 unknowns, which are the 3 forces from the supports, but only 2 equations from force and torque balance. As such, there are infinitely many solutions for the system at equilibrium – another example of an indeterminate situation in Newtonian mechanics.

Proof: (Readers, beware. Proceed at your own risk...)

This problem was inspired by Rudin Exr 5.27[5], which details the proof for the well known Picard-Lindelöf theorem. This theorem provides a simple criteria for the uniqueness of a solution to a first-order ordinary differential equation (ODE):

[5]Rudin, W. (1976). *Principles of Mathematical Analysis* (Vol. 3). New York: McGraw-Hill.

f is said to be Lipschitz continuous if

$$\sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|} < \infty$$

Suppose

$$\phi(t, x) : \{(t, x) : a \leq t \leq b, \alpha \leq x \leq \beta\} = \Omega \to \mathbb{R}$$

is a continuous function in t and Lipshitz continuous on x. Then the differential equation

$$\dot{x} = \phi(t, x), \quad x(t_0) = x_0, \quad (t_0, x_0) \in \Omega$$

has a unique solution.

Now we can go on to find all solutions to the differential equation $\dot{v} = \sqrt{|v|}, \ \ v(0) = 0$.

We have the trivial solution of $v(t) = 0$. Suppose otherwise that $v(t) \neq 0$, that there exists some T such that $v(T) = V \neq 0$. Note that since $\sqrt{|v|}$ is Lipshitz continuous on any closed set not containing 0, the solution to the differential equation is unique whenever v does not vanish in any point in the domain.

If $V > 0$, we have the solution

$$v(t) = \frac{\left(t - T + 2\sqrt{V}\right)^2}{4}, \quad T - 2\sqrt{V} \leq t$$

where the case for equality is valid due to continuity. Note that this solution is positive for $T - 2\sqrt{V} < t$, so $T - 2\sqrt{V}$ cannot be negative, otherwise $v(0) > 0$.

Similarly when $V < 0$, we have the solution

$$v(t) = -\frac{\left(t - T - 2\sqrt{-V}\right)^2}{4}, \quad t \leq T + 2\sqrt{-V}$$

Note that this solution is negative for $t < T + 2\sqrt{-V}$, so $T + 2\sqrt{V}$ cannot be positive, otherwise $v(0) < 0$.

This combined with the solution of $v = 0$ where the Lipshitz condition fails gives us all possible solutions to $\dot{v} = \sqrt{|v|}, \ \ v(0) = 0$:

$$v(t) = \frac{1}{4} \begin{cases} -(t - A)^2 & t \leq A \\ 0 & A \leq t \leq B \\ (t - B)^2 & B \leq t \end{cases} \quad A \leq 0 \leq B$$

Additional information for the mathematically inclined:

The Picard-Lindelöf theorem is sufficient, but not necessary to test for the uniqueness of a ODE solution. In other words, there exist ODEs with unique solutions that do not satisfy the Picard-Lindelöf theorem. In light of this, it is interesting to consider some other tests (of varying strength) for the uniqueness of ODE solutions. A brief survey of such tests is given below.

The most well-studied ODEs are first-order differential equations, where we have conditions that guarantee existence of solutions, such as Carathéodory, some that guarantee uniqueness, famously the Picard-Lindelöf theorem, and even necessary and sufficient conditions, such as Yosie's theorem.[6] With this, we also have some non-uniqueness theorems, the first of which is from Tamarkine, which states:

Let $\phi(t,x)$ is continuous for $|t| < T, |x| < X$ and suppose $\frac{d}{dt}s(t) = \phi(t, s(t))$, i.e. s is a solution to $\dot{x} = \phi(t,x)$. Suppose there exists some increasing continuous function g such that

(a) $|\phi(t,x) - \phi(t,s(t))| \geq g(|x - s(t)|)$

(b) $\lim_{\epsilon\to 0^+} \int_\epsilon^c \frac{dz}{g(z)} < \infty$ where c is an arbitrary number in the interior of the domain of g.

Then $\dot{x} = \phi(t,x)$ has at least two solutions.

We may apply Tamarkine's theorem to the differential equation in our problem. Let $\phi(t,x) = \sqrt{|x|}$, $s = 0$, $g(z) = \sqrt{z}$. Evaluating condition (b) with these parameters, we have:

$$\lim_{\epsilon\to 0^+} \int_\epsilon^c \frac{dz}{g(z)} = \lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{dz}{\sqrt{z}} = 2 < \infty$$

We see that both conditions of Tamarkine's theorem are satisfied, hence our differential equation has multiple solutions.

See Kalas' paper[7] for more information about the history and proofs of these results.

Setter: Ariana Goh, ariana.goh@sgphysicsleague.org

[6] Yosie, T. (1925). Über die Unität der Lösung der gewöhnlichen Differentialgleichungen erster Ordnung. *Japanese Journal of Mathematics* (Vol. 2, pp. 161-173). The Mathematical Society of Japan.

[7] Kalas, J. (2000). The use of Lyapunov functions in uniqueness and nonuniqueness theorems. *Archivum Mathematicum*, 36(5), 469-476.

Problem 35: Mirror Mirror on the Wall

The curator of the museum *Texallate* wishes to create a room with mirrors as walls for a museum installation. The room is a polygon with straight vertical walls of equal length. The curator stands in the centre of the room and shines his flashlight centered along a line of symmetry, whose beam can be modelled as a circular sector with beam angle α. You may take the system to be 2D – the flashlight is shone parallel to the floor.

(a) The first room the curator makes is a rhombus, with an interior angle of $\theta = 60°$. As the mirrored walls aren't polished yet, the light from the curator's flashlight can be reflected 1 time (totalled across all mirrors) before being fully dissipated. What is the minimum beam angle α of his flashlight so that light reaches every part of the room?

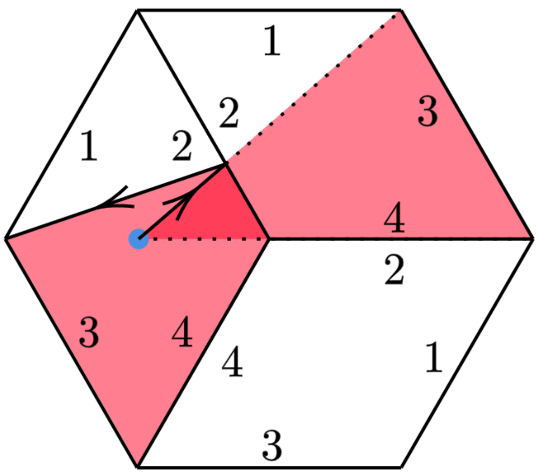
Leave your answer to 3 significant figures in units of degrees. (3 points)

(b) To save costs, the museum's director suggests that the curator build a rectangular room instead. The room has length $l = 2$ m and width $w = 1$ m. With the walls now polished, each light ray can be reflected $n = 100$ times (totalled across all mirrors) before being fully dissipated. What is the minimum beam angle α of his flashlight so that light reaches every part of the room?

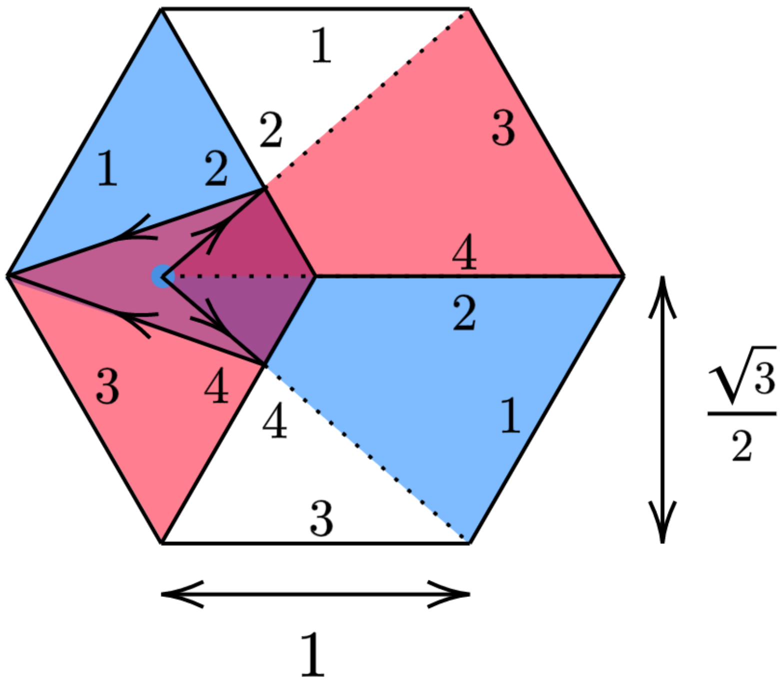
Leave your answer to 3 significant figures in units of degrees. (3 points)

Solution:

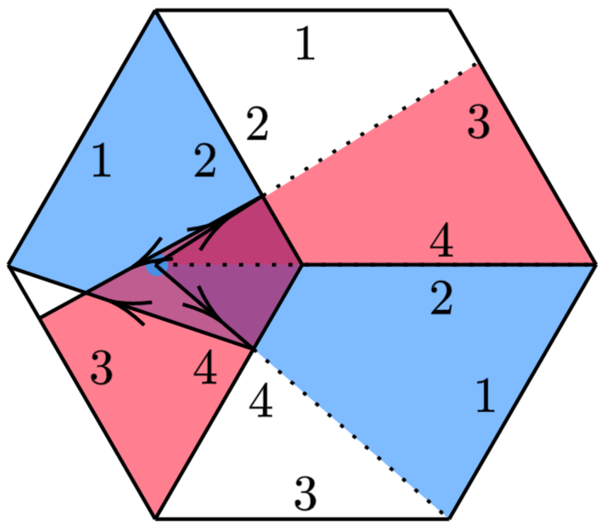
(a) Consider the diagram below. The left rhombus is the original room, while the top and bottom right rhombi are the images of the original room as seen in mirrors 2 and 4 respectively. The path of a light beam is illustrated below:



To light up the entire room, we just need its corresponding areas to be lit up in one of its images. The minimum case where no regions are left unlit will be when the "corners" are furthest from the centre:



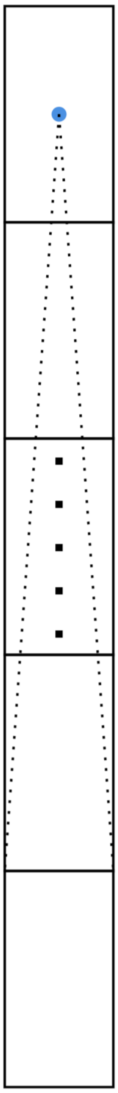
If the beam were any narrower, we see that there will be a triangular region that remains unlit:

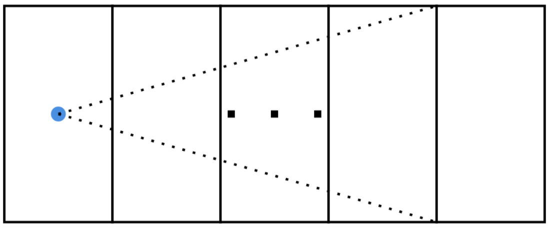


The beam angle required can then be calculated to be $\alpha = 2\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \approx \boxed{81.8^\circ}$.

If we shine the light at the other corner instead, it can be proven that the minimum α required is 180°. Therefore, shining the light at this corner is less optimal.

(b) Let us take a look at the images of the rooms. They are in fact tessellations of rectangles. By considering reflection through the length and width of the rectangle respectively, there are two extremal angles we can investigate:





We require the last rectangle to provide the last reflection for the beam to reach the side walls.

By observation, the left diagram yields our minimal beam angle, which can be calculated to be:

$$\alpha = 2\tan^{-1}\left(\frac{w/2}{(n-1)l + l/2}\right) = 2\tan^{-1}\left(\frac{w}{l}\frac{1}{2n-1}\right) \approx \boxed{0.288^\circ}.$$

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Problem 36: Oscillating E (4 points)

A few students are given an electrical source that supplies AC voltage at a fixed amplitude and frequency $f = 50.0$ Hz. They are tasked to set up an oscillating electric field whose amplitude is as large as possible. The electric field is to be set up between the plates of a capacitor with capacitance $C = 100\ \mu$F.

Shuan says, "The best way to do this would be to directly connect the plates of the capacitor to the AC source." This way, he achieves amplitude E for the electric field.

But Chris objects, "No, an increased amplitude E' for the electric field could be achieved by connecting an inductor in series with the capacitor!"

Surprisingly, Chris is right. If he uses an inductor with self-inductance $L = 80.0$ mH, what is the ratio E'/E? You may neglect any resistance present in the circuit.

Leave your answer to 3 significant figures.

Solution: The electric field inside the capacitor at any instant is directly proportional to the voltage across the capacitor plates. As such, to maximise the amplitude of the electric field, we seek to maximise the amplitude of voltage across the capacitor. We thus focus our attention on solving for the voltage across the capacitor employing Chris' method of using an LC series circuit.

Let us write the complex voltage of the source as $\tilde{V}_{\text{in}} = V_0 e^{i\omega t}$, where angular frequency $\omega = 2\pi f$. The complex voltage across the capacitor is thus $\tilde{V}_C = \frac{Z_C}{Z_C + Z_L}\tilde{V}_{\text{in}}$, where $Z_C = \frac{1}{i\omega C}$ is the impedance of the capacitor and $Z_L = i\omega L$ is the impedance of the inductor. Simplifying for $\tilde{V}_C$, we can obtain:

$$\tilde{V}_C = \frac{V_0}{1 - \omega^2 LC} e^{i\omega t}$$

Hence, the real voltage across the capacitor $V_C(t)$ can be found by taking the real part of $\tilde{V}_C$:

$$V_C(t) = \frac{V_0}{1 - \omega^2 LC} \cos \omega t$$

As such, using Chris' method, the amplitude of the voltage across the capacitor is $\frac{V_0}{1-\omega^2 LC} = 4.75 V_0$. In contrast, Shuan's method does not involve an inductor ($L = 0$), thus the amplitude is simply V_0. Chris' approach thus yields an amplitude that is increased by a factor of $\boxed{4.75}$.

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Problem 37: Weird Flux But Ok

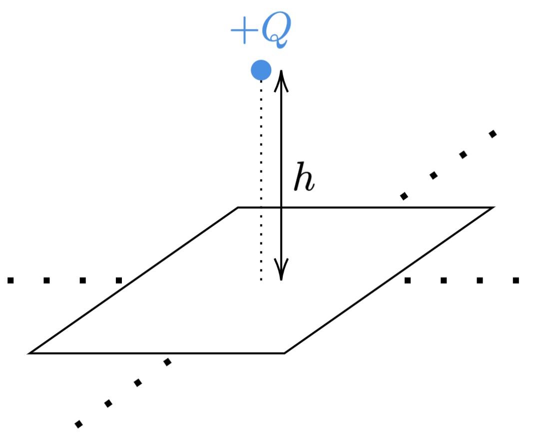
A point charge $Q = +2.5$ nC is placed at vertical distance $h = 0.15$ m above the centre of a horizontal rectangular insulating plate. Determine the total electric flux through the plate if...

(a) the plate has infinite length and breadth.

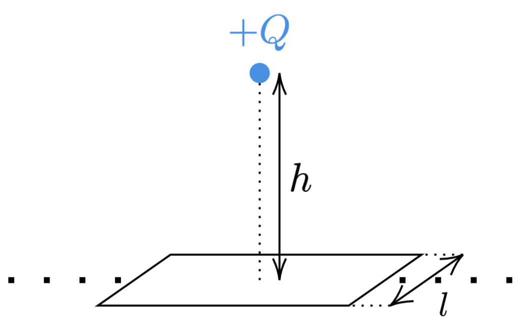
Leave your answer to 2 significant figures in units of V m. (2 points)

(b) the plate has infinite length and finite breadth $l = 0.10$ m.

Leave your answer to 2 significant figures in units of V m. (4 points)



(a) spans infinitely in all directions



(b) spans infinitely in length but not in breadth

Solution:

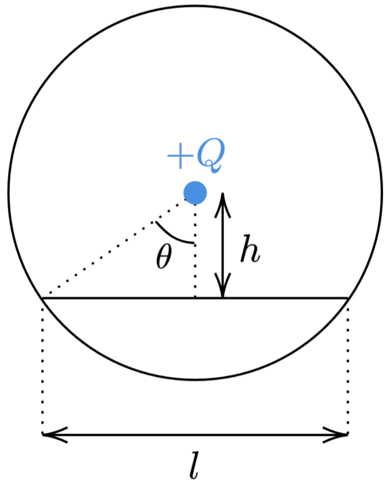
(a) Consider the Gaussian surface to be a cuboid whose top and bottom faces are infinitely wide and separated by vertical height $2h$, with the charge positioned at the center of the cuboid. The total electric flux through the cuboid is given by Gauss' Law to be Q/ε_0.

Among the six faces of the cuboid, only the top and bottom faces have nonzero electric flux, since all other faces are infinitely distant from the charge. By symmetry, the top and bottom faces of the cuboid have equal electric flux $Q/2\varepsilon_0$.

The plate is analogous to the bottom face of this cuboid. Hence, the electric flux through the plate is also $Q/2\varepsilon_0 \approx \boxed{140 \text{ V m}}$.

(b) One could attempt a similar approach to (a), but with the cuboid's top and bottom faces having finite breadth l and infinite length. However, not all of the side faces would be infinitely far from the charge, and thus have some electric flux through them which would have to be accounted for. Such an approach would thus be unviable in determining the electric flux through the plate.

Instead, we now consider our Gaussian surface to be an infinitely long cylinder whose axis is parallel to the plate's length. The cylinder is centered on the charge. The radius of the cylinder is picked such that the plate exactly fits within the curved surface of the cylinder, with perpendicular distance h between the plate and the cylinder's center.



By Gauss' Law, the electric flux through the curved surface of the cylinder is Q/ε_0. All of this electric flux goes through the curved surface of the cylinder, as the flat ends of the cylinder are infinitely far away from the charge and thus have no flux through them.

Define θ as the angle that the line between the center of the cylinder and the end of the plate makes with the vertical, as drawn above. From trigonometry, we may determine $\theta = \tan^{-1} \frac{l}{2h}$.

Let us consider the closed surface comprising the plate and the bottom portion of the curved surface that lie below the plate, i.e. the parts that subtend angle 2θ. There is no enclosed charge within this closed surface, so there can be no net electric flux through the surface. As such, the electric flux through the plate must be equal to the electric flux through the bottom portion of the curved surface below the plate.

To find the electric flux through this bottom portion, notice that the rotational symmetry of the cylinder implies that:

$$\text{Flux through bottom portion } = \frac{\theta}{\pi} \times \text{ Flux through whole curved surface}$$

Further invoking the fact that the flux through the plate is equal to the flux through the bottom portion of the curved surface, we may calculate the electric flux through the plate:

$$\text{Flux through plate } = \frac{Q}{\pi\varepsilon_0} \tan^{-1} \frac{l}{2h} \approx \boxed{29 \text{ V m}}$$

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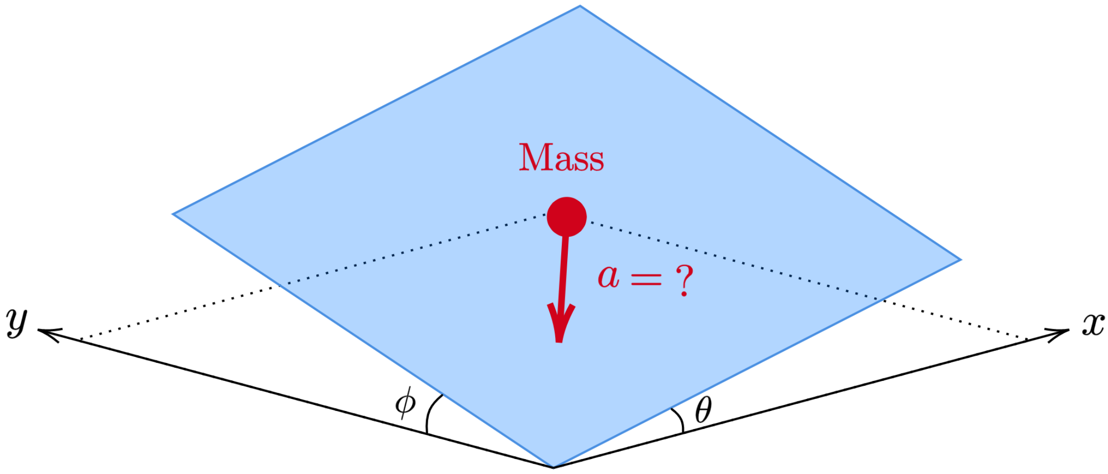
Problem 38: Inclined Plane (5 points)

A fixed frictionless plane is inclined at angle $\theta = 30^\circ$ above the x-axis and angle $\phi = 60^\circ$ above the y-axis, where the xy-plane is horizontal. Find the acceleration a of a mass that slides down the inclined plane.

Leave your answer to 2 significant figures in units of m s^{-2}.



Solution: Define the z-axis to be the vertical axis, with upwards defined as positive. The acceleration of the mass is simply the component of gravity along the plane. That is, if $\hat{n}$ is a unit vector normal to the plane, a would be given by:

$$a = \left| \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \times \hat{n} \right|$$

To find $\hat{n}$, we may express it as the normalised cross product of any two vectors $\hat{p}$ and $\hat{q}$ that lie along the surface of the plane:

$$\hat{n} = \frac{\hat{p} \times \hat{q}}{|\hat{p} \times \hat{q}|}$$

We may consider $\hat{p}$ to be the unit vector along the plane above the x-axis, and $\hat{q}$ to be the unit vector along the plane above the y-axis. This enables us to write $\hat{p}$ and $\hat{q}$

in terms of x, y, z components:

$$\hat{p} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} 0 \\ \cos\phi \\ \sin\phi \end{pmatrix}$$

Correspondingly, we can compute $\hat{n}$:

$$\hat{n} = \frac{\hat{p} \times \hat{q}}{|\hat{p} \times \hat{q}|} = \frac{1}{\sqrt{\cos^2\theta + \cos^2\phi - \cos^2\theta\cos^2\phi}} \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\phi\cos\theta \\ \cos\theta\cos\phi \end{pmatrix}$$

With $\hat{n}$, we can thus determine a:

$$a = \left| \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \times \hat{n} \right| = \frac{1}{\sqrt{\cos^2\theta + \cos^2\phi - \cos^2\theta\cos^2\phi}} \left| \begin{pmatrix} -g\sin\phi\cos\theta \\ -g\sin\theta\cos\phi \\ 0 \end{pmatrix} \right|$$

$$\implies a = g\sqrt{1 - \frac{1}{\tan^2\theta + \tan^2\phi + 1}}$$

As a check, if $\phi = 0°$, a reduces to $g\sin\theta$; whereas if $\theta = 0°$, a reduces to $g\sin\phi$. This agrees with the basic result of the most classic inclined plane problem where the plane is only inclined by one angle. Additionally, if either $\theta \to 90°$, or $\phi \to 90°$, or both, then $a \to g$. This makes sense, because these limits would imply that the plane is vertical.

Plugging in the values in the question, $a \approx \boxed{8.6 \text{ m s}^{-2}}$.

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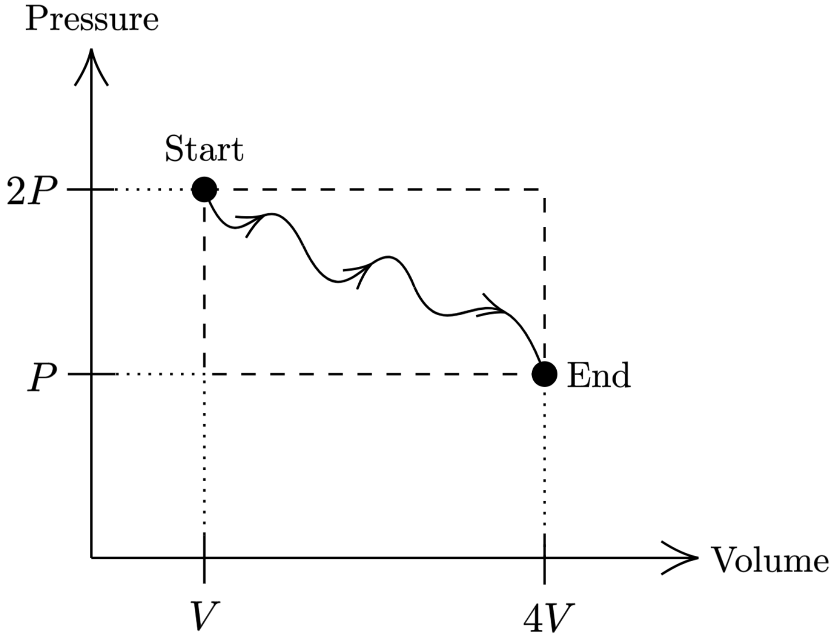
Problem 39: Strategic Syringe (5 points)

Ideal monatomic gas is contained within a sealed syringe. The piston in the syringe can be moved such that the volume can be freely varied between V and $4V$. Simultaneously, valves in the syringe allow the pressure to be freely controlled between P and $2P$.

The gas starts out at pressure $2P$ and volume V. It is to be brought to final pressure P and volume $4V$. Depending on how the piston and the valves are controlled over time, the heat absorbed by the gas in the process can range between Q_{min} and Q_{max}. Find the ratio $Q_{\text{max}}/Q_{\text{min}}$.

Assume that the piston and the valves are controlled such that the process undergone by the gas is reversible, and that the path traced by the process on the pressure-volume graph does not self-intersect.

Leave your answer to 2 significant figures.



Solution: It is helpful to visualise the process undergone by the gas in a pressure-volume graph. There are infinitely many possible paths for this process. The path of the process must have its start and end points fixed at $(V, 2P)$ and $(4V, P)$ respectively. This path must also be entirely contained within the rectangle with vertices $(V, P), (V, 2P), (4V, P), (4V, 2P)$, as shown in the diagram in the question.

Internal energy U is a state function. Thus, regardless of the path chosen, the change in internal energy of the gas ΔU from start to end is given by:

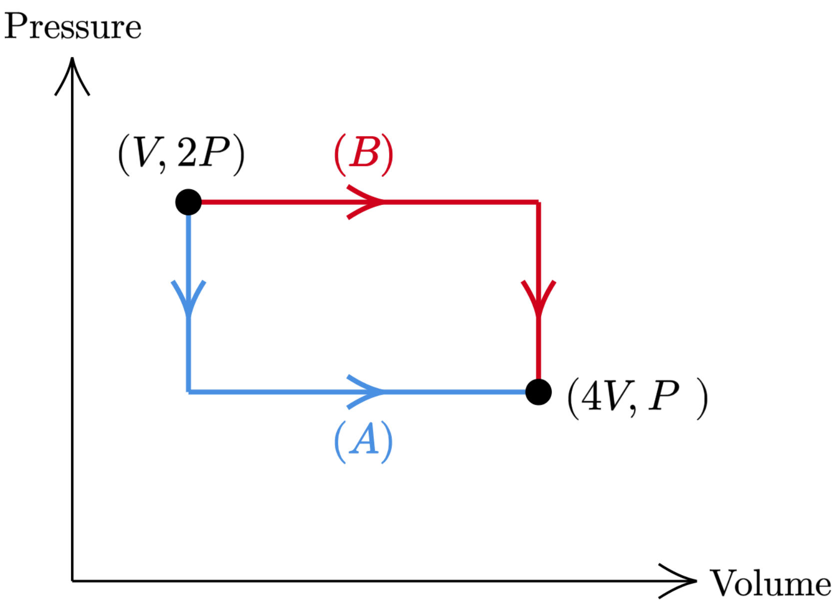
$$\Delta U = \frac{3}{2}\left[(P)(4V) - (2P)(V)\right] = 3PV$$

Using the First Law of Thermodynamics, we can write:

$$Q = \Delta U + W = 3PV + W$$

where Q is the heat absorbed by the gas and W is the work done by the gas.

As such, Q is minimised when W is minimised, and Q is maximised when W is maximised. Since W is given by the area under the path traced by the process on the pressure-volume graph, we can identify the paths needed to minimise and maximise W as A and B respectively (shown below).



The area under path A is given by $P(4V - V) = 3PV$, so $W = 3PV$. Hence $Q_{\text{min}} = 6PV$.

The area under path B is given by $2P(4V - V) = 6PV$, so $W = 6PV$. Hence $Q_{\text{max}} = 9PV$.

Therefore, $Q_{\text{max}}/Q_{\text{min}} = \frac{9PV}{6PV} = \boxed{1.5}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

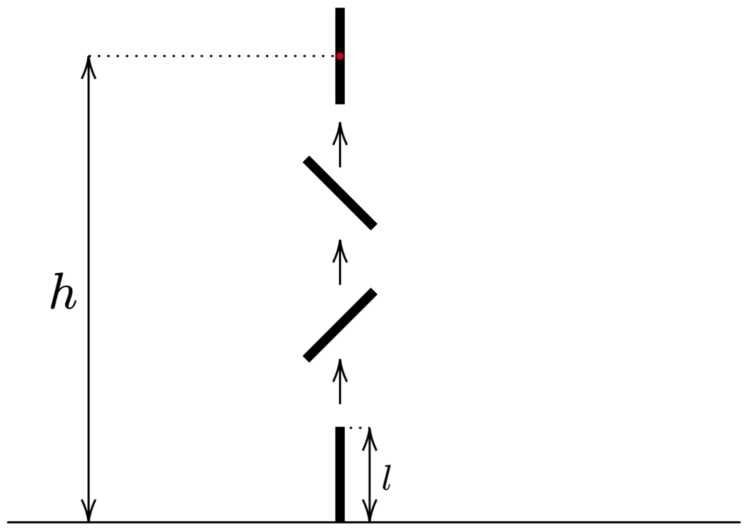
Problem 40: Optimised Backflip (5 points)

Let's investigate the most efficient backflip physically possible. In a perfect backflip, the person starts out standing straight on his feet. He jumps, and his body rotates mid-air such that by the time he returns to the ground, he lands vertically on his feet.

As a toy model of a backflip, consider the person's body to be a uniform rigid stick of length l that initially stands on the ground with its axis vertical. Suppose that the stick is imparted the theoretical minimum energy required to perform a perfect backflip. As a result, the stick's centre reaches maximum height h above the ground. Find the ratio h/l.

Neglect air resistance.

Leave your answer to 3 significant figures.



Solution: When energy is imparted to the stick, it is given both a translational velocity in the vertical direction, and a rotational velocity. Let its initial vertical velocity be u and angular velocity be ω.

The time duration t for which the stick is in the air can be determined using kinematics, by considering the time when the stick has zero vertical displacement:

$$ut - \frac{1}{2}gt^2 = 0 \implies t = \frac{2u}{g}$$

For a perfect backflip to be performed, the stick must complete exactly one revolution in the air. There is no torque acting on the stick when it is in the air, so we may take its angular velocity to be constant at ω. This allows us to obtain the following relation between u and ω:

$$\omega t = 2\pi \implies \omega = \frac{g\pi}{u}$$

Let E be the energy initially imparted to the stick. E is given by the sum of translational kinetic energy (due to vertical motion), and rotational kinetic energy where moment of inertia $I = \frac{1}{12}ml^2$:

$$E = \frac{1}{2}mu^2 + \frac{1}{24}ml^2\omega^2 = \frac{1}{2}mu^2 + \frac{ml^2g^2\pi^2}{24u^2}$$

Our task now is to find the value of u that minimises E. This can be determined by setting $\frac{\mathrm{d}E}{\mathrm{d}u} = 0$:

$$u - \frac{l^2g^2\pi^2}{12u^3} = 0 \implies u = \sqrt{\frac{lg\pi}{2\sqrt{3}}}$$

This gives us the value of u required to perform the most efficient possible backflip. (Correspondingly, the required value of ω may also be found, using the previous relation between u and ω.)

Using this value of u, we can find the maximum vertical displacement y of the stick, invoking the fact that the stick has no vertical velocity when it is at its peak:

$$0 = u^2 - 2gy \implies y = \frac{u^2}{2g} = \frac{\pi}{4\sqrt{3}}l$$

We can thus find the maximum height h reached by the stick's centre above the ground:

$$h = y + \frac{l}{2} = l\left(\frac{\pi}{4\sqrt{3}} + \frac{1}{2}\right) \approx 0.953l$$

Hence, the ratio $h/l \approx \boxed{0.953}$.

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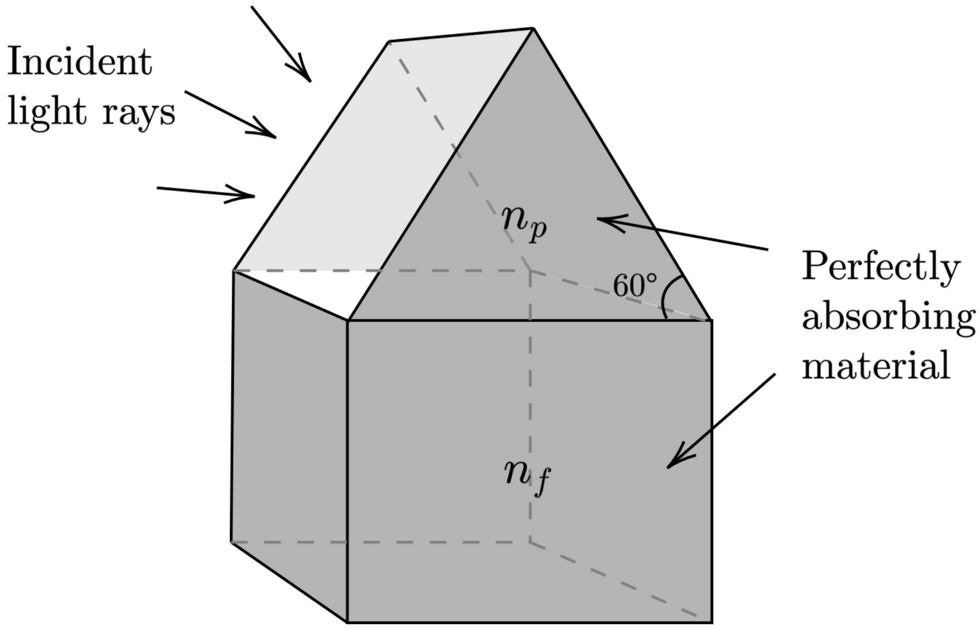
Problem 41: Stop Blocking My View (5 points)

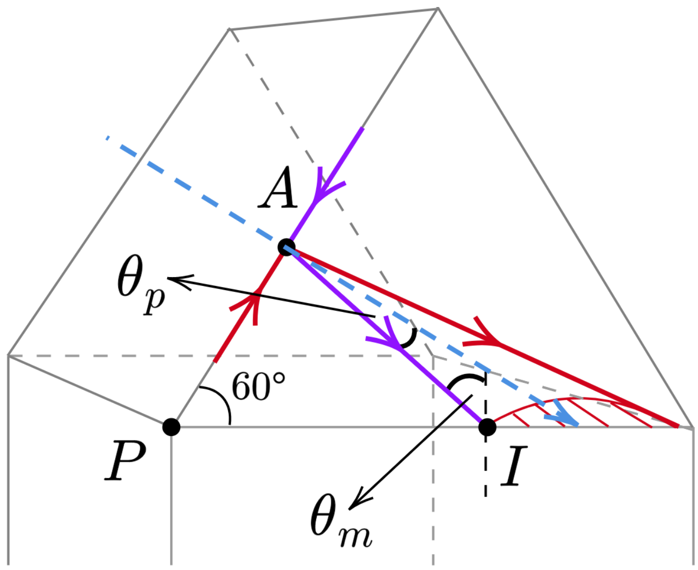
A silicon carbide prism of refractive index $n_p = 2.63$ is placed above a tank. The tank is in the shape of a cuboid, open at the top, and is completely filled with fluid. The cross section of the prism is in the shape of an equilateral triangle. The outer surface of the prism-and-tank setup is covered entirely by perfectly absorbing material, except for one of the slanted faces of the prism. What is the minimum refractive index n_f that the fluid must have such that any observer outside the set-up is unable to see light from inside the fluid?

Leave your answer to 3 significant figures.



Solution: Notice that, due to the principle of reversibility of light, the problem is equivalent to finding the minimum n_f such that no light from outside the set-up can reach the fluid.

The critical angle θ_p at the prism-air interface is the angle of refraction of a light ray that strikes the prism at an angle of incidence of 90°, nearly parallel to the interface. This angle is given by Snell's law, $n_p \sin\theta_p = \sin 90° = 1$. Since the angle of incidence cannot go higher than 90°, the angle of refraction cannot go higher than θ_p within the prism. Thus, all incoming light rays are confined to cones of half-apex angle θ_p whose axes are normal to the interface, as shown below.



Now let us consider what happens at the prism-tank interface. Light from one of these allowed cones strikes the prism-tank interface at a restricted range of incidence angles, and the smallest possible incidence angle θ_m occurs at the ray nearest to the prism-air interface, as illustrated above.

This angle θ_m can be calculated by angle chasing. Let the apex of the cone be A, the nearest point at the bottom edge of the prism be P, and the intersection of the nearest ray with the bottom of the prism be I, as shown in the diagram. Then $\angle IAP = 90^\circ - \theta_p$, $\angle API = 60^\circ$, and since the sum of angles in a triangle is 180°, we find that $\angle AIP = 180^\circ - 60^\circ - (90^\circ - \theta_p) = 30^\circ + \theta_p$, and therefore $\theta_m = 90^\circ - (30^\circ + \theta_p) = 60^\circ - \theta_p$.

In order for a light ray to pass from the prism to the tank, it must strike at an incidence angle below the critical angle θ_t for the prism-tank interface. This critical angle corresponds once again to a parallel outgoing ray, where $n_p \sin\theta_t = n_f$, with rays at higher incidence angles undergoing total internal reflection instead. Thus, for light to reach the fluid in the tank, we must have $\theta_m \leq \theta_t$. From here, we can deduce that:

$$\theta_m \leq \theta_t \implies 60^\circ - \theta_p = 60^\circ - \sin^{-1}\frac{1}{n_p} \leq \sin^{-1}\frac{n_f}{n_p}$$

Since both sides are smaller than 90° and $\sin(x)$ is increasing for $0^\circ \leq x \leq 90^\circ$, we can take the sine of both sides and multiply both sides by n_p to obtain:

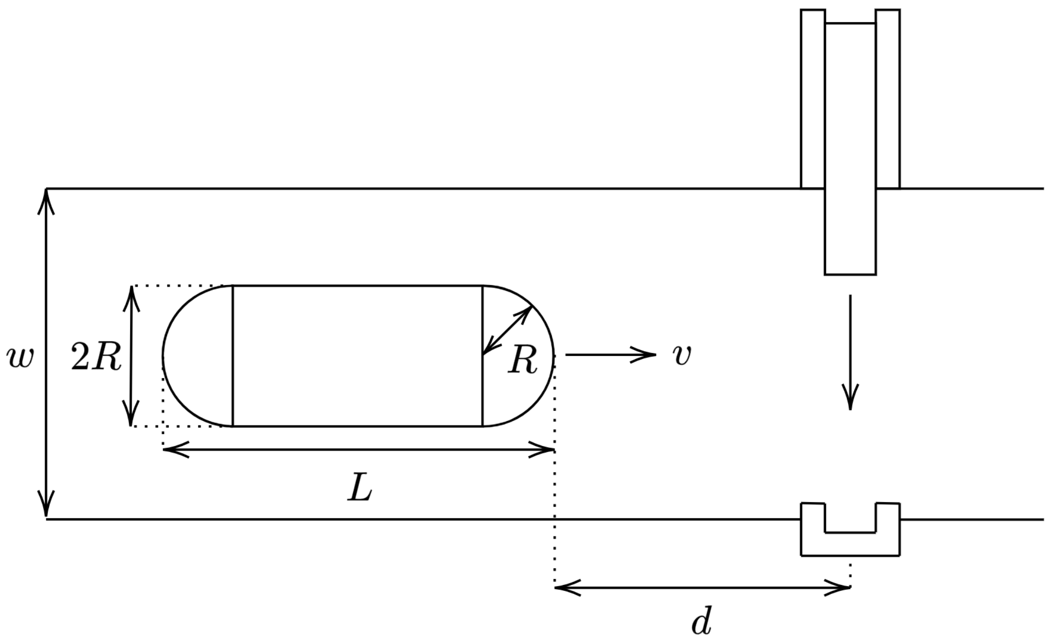
$$n_f \geq n_p \sin\left(60^\circ - \sin^{-1}\frac{1}{n_p}\right) \approx \boxed{1.61}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

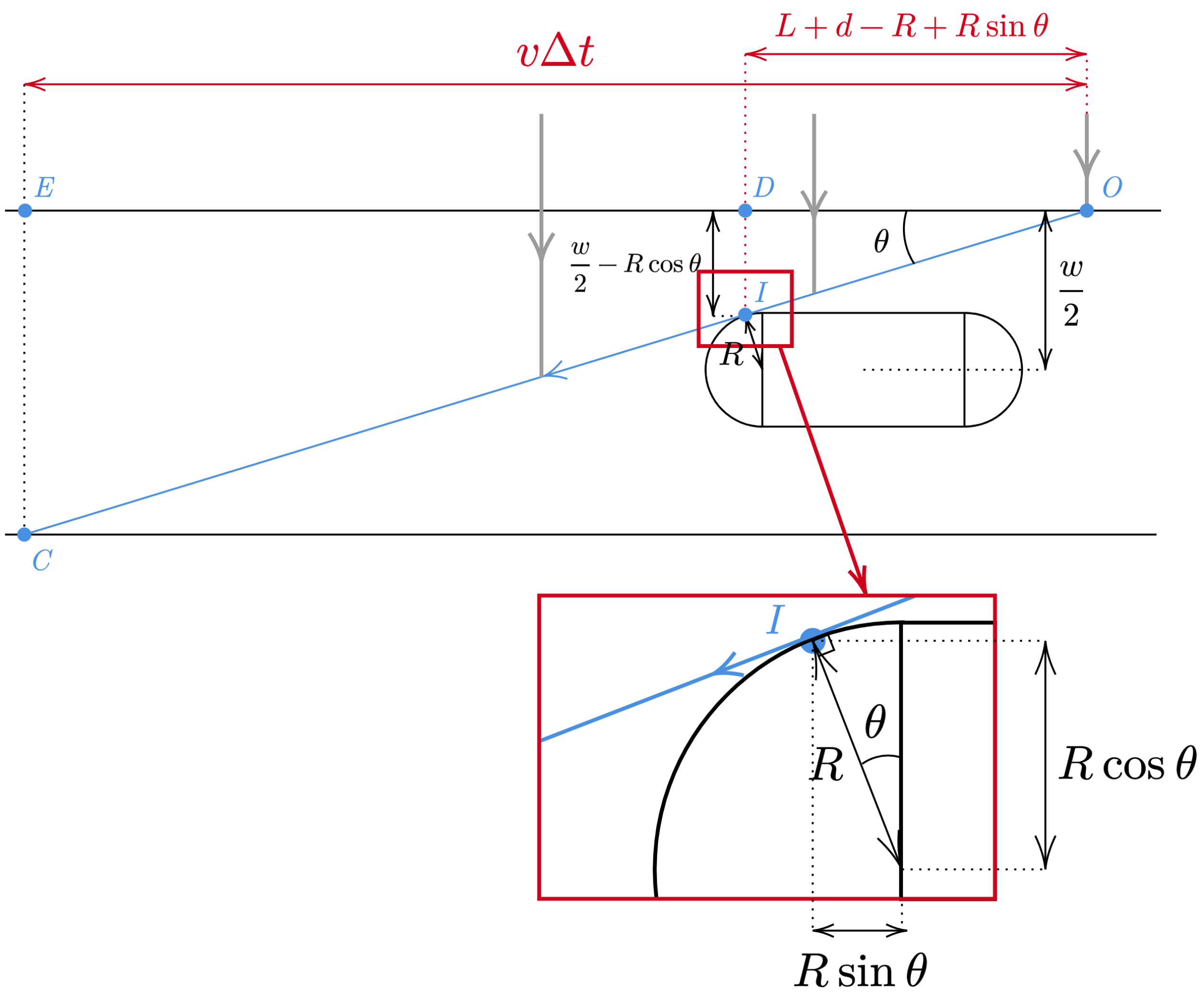
Problem 42: Closing Gate (5 points)

A car has semicircular bumpers on the front and back ends of an otherwise rectangular body. Each bumper has radius $R = 1.0$ m. Overall, the car has width $2R$ and length $L = 4.0$ m, as shown below. It drives at a steady speed along the center of a road of width $w = 10$ m towards a sliding gate at distance $d = 50$ m ahead of the car, which has just started closing at a constant rate. Given that the gate takes time $\Delta t = 30$ s to close fully, what minimum speed v must the car travel at in order to make it through the gate without contacting the gate?

Leave your answer to 3 significant figures in units of m s^{-1}.



Solution: Let us consider this problem from a frame of reference moving with the car at speed v. In this frame, the front of the gate moves both towards the car and perpendicular to the road simultaneously, and this traces a diagonal line along the road. In order for the car to make it through, it must remain in front of and cannot intersect this diagonal line. In the limiting case of the minimum velocity, the car's outline would naturally be tangent to this diagonal line. The problem then reduces to one of geometry.



Let the diagonal line make an angle θ with the road. With reference to the above diagram, the intersection point I of the line with the rear bumper of the car will thus be a distance $OD = L + d - R + R\sin\theta$ from the open gate and a distance $DI = w/2 - R\cos\theta$ from the nearer side of the road. Since these two quantities form a right triangle ODI with angle θ, we must have:

$$\tan\theta = \frac{w/2 - R\cos\theta}{L + d - R + R\sin\theta}$$

We can solve this equation numerically to give $\theta \approx 4.3°$. Alternatively, for an analytical solution, we can re-arrange the equation to obtain

$$\frac{w}{2}\cos\theta - (L + d - R)\sin\theta = R$$

and use the identity $a\cos\theta - b\sin\theta = \sqrt{a^2 + b^2}\cos\left(\theta + \tan^{-1}\frac{b}{a}\right)$ to arrive at

$$\sqrt{\frac{w^2}{4} + (L + d - R)^2}\cos\left(\theta + \tan^{-1}\frac{2(L + d - R)}{w}\right) = R$$

which gives us

$$\theta = \cos^{-1}\frac{R}{\sqrt{\frac{w^2}{4}+(L+d-R)^2}} - \tan^{-1}\frac{w}{2(L+d-R)} \approx 4.3^\circ$$

From triangle CEO, we can deduce that:

$$\tan\theta = \frac{w}{v\Delta t}$$

As such, the required value of v is:

$$v = \frac{w}{\tan\theta \cdot \Delta t} = \boxed{4.42 \text{ m s}^{-1}}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Problem 43: Solar Eclipse (5 points)

Consider a parallel universe where, just like our own universe, an Earth month is $T_M \approx 30$ days long, an Earth year is $T_S \approx 365$ days long, the Moon has radius $r_M \approx 1,700$ km, and the Sun has radius $r_S \approx 700,000$ km. However, in this universe, the ratio m_E/m_S (where the Earth's mass is m_E and the Sun's mass is m_S) is different from ours. We do not know the precise value of this ratio.

But we do know one thing: While the Earth of this universe can sometimes observe *partial* solar eclipses, *total* solar eclipses can **never** happen. Using this fact, find the minimum possible value of $\ln(m_E/m_S)$.

You may assume the following properties of this parallel universe:

- The same laws of physics apply
- The mass of Moon is very small compared to the mass of Earth, and the mass of Earth is very small compared to the mass of Sun
- The radius of Earth is very small compared to the Sun-Earth distance and the Moon-Earth distance
- All orbits are circular
- The centres of the Sun, Earth and Moon always lie on the same plane

Leave your answer to 2 significant figures.

Solution: The orbital period of the Moon around the Earth is T_M. Let the mass of the Moon be m_M and the radius of the Moon's orbit around the Earth be d_M. The gravitational force between the Earth and the Moon supplies the centripetal force for its circular orbit:

$$\frac{Gm_E m_M}{d_M^2} = m_M d_M \left(\frac{2\pi}{T_M}\right)^2 \implies d_M^3 = \frac{Gm_E T_M^2}{4\pi^2}$$

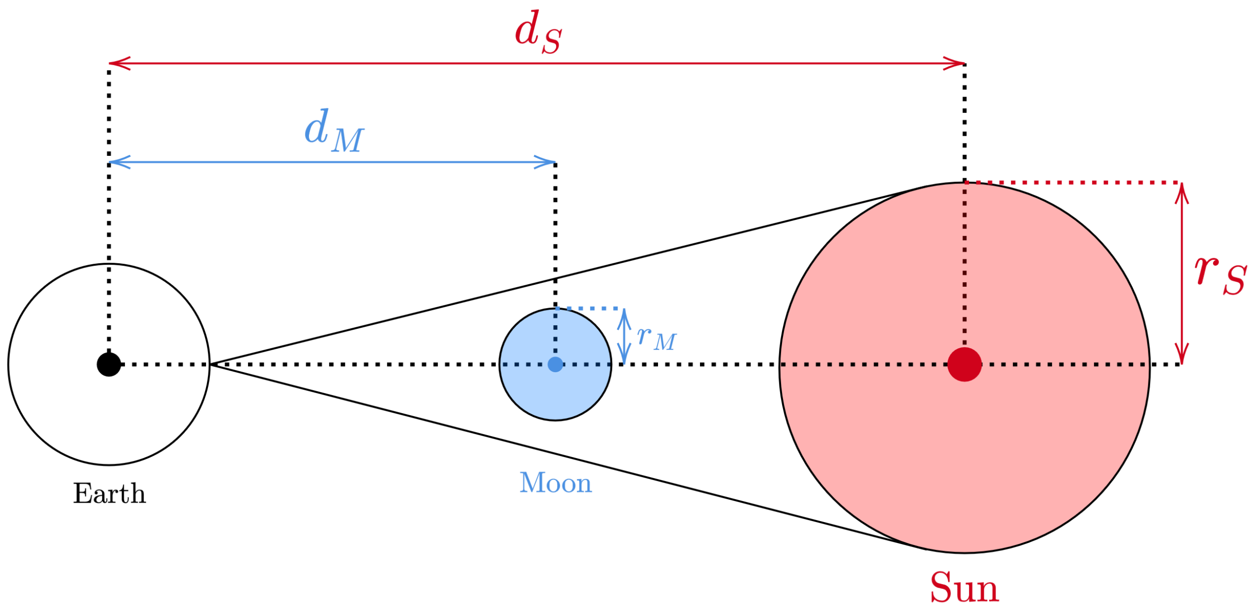
Likewise, we can find an expression for the radius of the Earth's orbit around the Sun, d_S:

$$d_S^3 = \frac{Gm_S T_S^2}{4\pi^2}$$

With reference to the diagram below, consider the sunlight incident on the point on Earth closest to the Sun.[8] If the ratio d_M/d_S is large enough, the Moon will be too far

[8] Technically, the poles of Earth are the most likely points where a total solar eclipse can be observed. However, since we assume that $r_E \ll d_S$ and $r_E \ll d_M$, all points on Earth can be effectively treated to be equivalent, in the context of this question.

from the Earth to be able to block out all of the sunlight, resulting only in a partial solar eclipse, rather than a total solar eclipse.



Using similar triangles, a total solar eclipse will not occur if

$$r_M < \frac{d_M - r_E}{d_S - r_E} r_S$$

where r_E is the Earth's radius. Since $r_E \ll d_S$ and $r_E \ll d_M$, the right-hand side reduces to $r_S(d_M/d_S)$. Using the previously derived expressions for d_M and d_S, we have:

$$\frac{m_E}{m_S} > \frac{r_M^3 T_S^2}{r_S^3 T_M^2} \approx 2 \times 10^{-6}$$

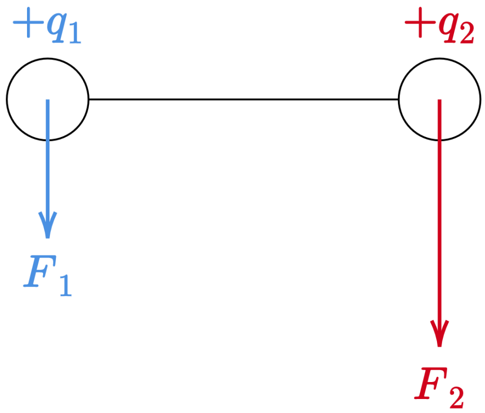
Hence, $\ln(m_E/m_S) > \boxed{-13}$. As a further note, the value of $\ln(m_E/m_S)$ in our universe is approximately -12.7, which is above the minimum value found in this question. But, we do observe total solar eclipses, contrary to the premise of this question! This just means the assumptions used are not exactly true – particularly, the last two assumptions of circular orbits and planets lying on the same plane are inaccurate enough for the calculation to fail in reality.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 44: Unbalanced Fall (6 points)

Two point charges $+q_1$ and $+q_2$, each of equal mass $m = 1.0$ kg, are joined to opposite ends of a massless rigid rod. There is a uniform downward electric field that exerts constant downward forces $F_1 = 1.0$ N and $F_2 = 5.0$ N on charges q_1 and q_2 respectively. The charges are simultaneously released from rest with the rod initially horizontal. In its subsequent motion, find the maximum downward acceleration of charge q_1. Neglect gravity.

Leave your answer to 2 significant figures in units of m s^{-2}.



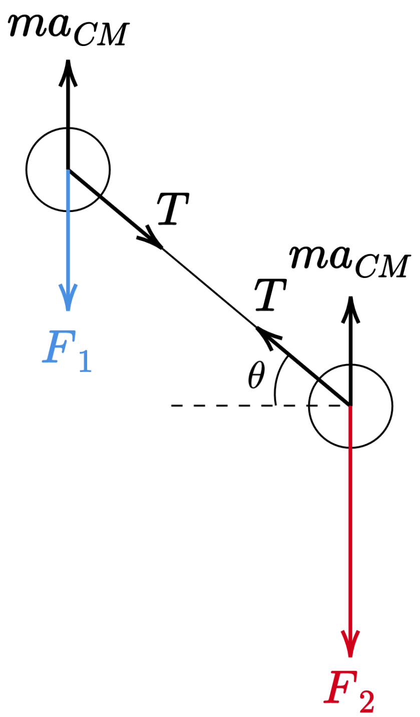
Solution: Upon release, the system of charges will undergo a combination of translational and rotational motion.

Considering the combined system of both charges, the only external forces are F_1 and F_2. Using Newton's Second Law, the acceleration of the centre of mass of the system a_{CM} can be deduced:

$$F_1 + F_2 = 2ma_{\text{CM}} \implies a_{\text{CM}} = \frac{F_1 + F_2}{2m}$$

This is a constant acceleration in the downward direction. Let us consider the motion of the charges viewed from the reference frame of their centre of mass. In this frame, both charges are rotating around their centre of mass with equal angular velocities. Due to the reference frame having a downward acceleration, each charge experiences a constant upward fictitious force of $ma_{\text{CM}} = (F_1 + F_2)/2$.

At any point in the motion, let the tension in the rod be T and the angle of the rod from the horizontal be θ. (There are also forces arising from the electrostatic repulsion between the charges – let's subsume them all under T.) The free-body diagram of the setup is drawn below.



We can write the radial and tangential components of the net forces exerted on each individual charge:

$$F_{\text{tan}} = \frac{F_2 - F_1}{2}\cos\theta$$
$$F_{\text{rad}} = T - \frac{F_2 - F_1}{2}\sin\theta$$

Let the length of the rod be l and the angular velocity of the charges be ω. Newton's Second Law in the tangential direction gives $F_{\text{tan}} = m\left(\frac{l}{2}\right)\frac{\mathrm{d}\omega}{\mathrm{d}t}$, so we can obtain an expression for $\frac{\mathrm{d}\omega}{\mathrm{d}t}$:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{F_2 - F_1}{ml}\cos\theta$$

Rewriting $\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$, we may form a differential equation in ω and θ that can be integrated to give a relation between ω and θ:

$$\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \frac{F_2 - F_1}{ml}\cos\theta$$
$$\int_0^{\omega}\omega\,\mathrm{d}\omega = \frac{F_2 - F_1}{ml}\int_0^{\theta}\cos\theta\,\mathrm{d}\theta$$
$$\implies \omega = \sqrt{\frac{2(F_2 - F_1)}{ml}\sin\theta}$$

In the reference frame of the centre of mass, the charges are undergoing pure circular motion, which is possible with F_{rad} supplying the necessary centripetal acceleration where $F_{\text{rad}} = m\left(\frac{l}{2}\right)\omega^2$. Using the previous expressions for ω and F_{rad}:

$$T - \frac{F_2 - F_1}{2}\sin\theta = (F_2 - F_1)\sin\theta \implies T = \frac{3}{2}(F_2 - F_1)\sin\theta$$

Returning to the lab frame, the downward component F_y of the net force on charge q_1 can be written as:

$$F_y = F_1 + T\sin\theta = F_1 + \frac{3}{2}(F_2 - F_1)\sin^2\theta$$

F_y is maximised when $\theta = 90°$, giving a maximum value of $(3F_2 - F_1)/2$. Hence, the maximum acceleration experienced by charge q_1 is given by $(3F_2 - F_1)/2m = \boxed{7.0 \text{ m s}^{-2}}$. Surprisingly, this acceleration is greater than a single charge could have achieved if both F_1 and F_2 were acting on it!

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Problem 45: Leonard the Oiler (5 points)

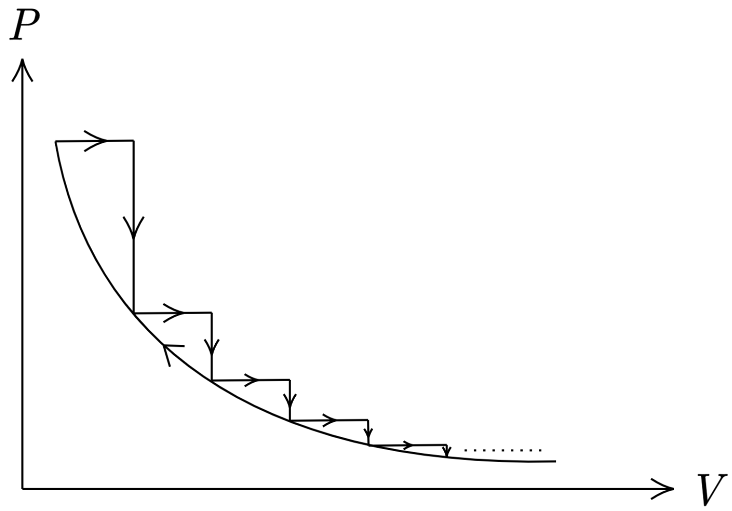
Leonard proposes a machine with a frictionless piston in an infinitely extending cylinder, contained in which is an ideal gas.

The piston starts off at infinity, with work W_1 done on the gas to compress it isothermally until the gas reaches $P_0 = 120$ kPa and a finite volume $V_0 = 1\ \mathrm{m}^3$. The gas then undergoes an alternating series of reversible processes: first, an isobaric expansion to increase the volume by $\Delta V = +V_0$, followed by an isochoric reduction in pressure to restore it to its initial temperature. These iterations are performed until the piston returns to infinity, and the total work done by the gas during these iterations is W_2.

Determine the net work done **by** the gas throughout the entire procedure, $W = W_2 - W_1$.

Leave your answer to 2 significant figures in units of kJ.

Solution: Let the volume V of the gas at any point in the process be given by $V = aV_0$. During the isothermal compression, the pressure of the gas P_1 is given by $P_1(V) = \frac{P_0}{a}$. During the iterative process in the second step, we may express the pressure of the gas P_2 as $P_2(V) = \frac{P_0}{\lfloor a \rfloor}$. Both steps are plotted in the diagram below.



We may then obtain the net work W done by the gas via integration:

$$W = \int (P_2 - P_1)\,\mathrm{d}V = P_0 V_0 \int_1^\infty \left(\frac{1}{\lfloor a \rfloor} - \frac{1}{a} \right) \mathrm{d}a$$

Note that while neither $\int P_1\,\mathrm{d}V$ nor $\int P_2\,\mathrm{d}V$ converge individually, their difference is finite and converges to a constant.

One may proceed to approximate the integral computationally with a sufficient number of terms for it to converge, such as using a finite upper bound for the integral, or splitting it into a harmonic sum and standard $\frac{1}{x}$ integral and using known results (both of which are solvable online using WolframAlpha). However, some participants may realise that the integral is simply the celebrated Euler-Mascheroni constant, which has a numerical value of $\gamma = 0.5772156649...$

To complete the problem, the work done on the gas is simply $W = P_0 V_0 \gamma \approx \boxed{69\ \text{kJ}}$.[9]

Setter: Paul Seow, paul.seow@sgphysicsleague.org

[9]The insightful may realise that the names in this problem have hinted towards this constant's relevance in this problem, and that the author thought of this question at 2a.m. in the morning.

Problem 46: Rebounding Particles

A neutral atom of unknown rest mass m_n is travelling rightwards, towards a positive ion of unknown rest mass m_p and charge $+e$ (where e is the elementary charge) that is travelling leftwards. There is also an electric field $E = 1.0$ kV m^{-1} pointing rightwards. Both particles collide with each other at the origin, with equal and opposite velocities of magnitude $v_0 = 0.50c$ at the instant of collision.

After the collision, they stick together. The resulting composite particle moves leftwards for a while, before reversing and returning to the origin at a time $\Delta t = 15.0$ ms after the collision. If this composite particle is determined to have total (relativistic) energy $E_T = 30.0$ GeV upon returning to the origin, find

(a) the original rest mass of the neutral atom m_n.

Leave your answer to 2 significant figures in units of u (atomic mass unit), where 1 $u = 1.66 \times 10^{-27}$ *kg.* (3 points)

(b) the original rest mass of the positive ion m_p.

Leave your answer to 2 significant figures in units of u (atomic mass unit), where 1 $u = 1.66 \times 10^{-27}$ *kg.* (3 points)

Solution: We will work exclusively in the lab frame. Let the x-axis be the horizontal axis, with the rightward direction taken to be positive. The four-momenta of the atom and ion respectively are:

$$\vec{p}_n = \begin{bmatrix} \frac{m_n c^2}{\sqrt{1-v_0^2/c^2}} \\ \frac{m_n v_0}{\sqrt{1-v_0^2/c^2}} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{p}_p = \begin{bmatrix} \frac{m_p c^2}{\sqrt{1-v_0^2/c^2}} \\ -\frac{m_p v_0}{\sqrt{1-v_0^2/c^2}} \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the composite particle has four momentum

$$\vec{p}_c = \begin{bmatrix} \frac{(m_n+m_p)c^2}{\sqrt{1-v_0^2/c^2}} \\ \frac{(m_n-m_p)v_0}{\sqrt{1-v_0^2/c^2}} \\ 0 \\ 0 \end{bmatrix}$$

and we shall denote its components, the relativistic energy and momenta, as $p_c^0 = \frac{(m_n+m_p)c^2}{\sqrt{1-v_0^2/c^2}} = E_T$ and $p_c^1 = \frac{(m_n-m_p)v_0}{\sqrt{1-v_0^2/c^2}}$ respectively. The composite particle has charge $+e$ and thus experiences a constant force $F = +eE$ in the positive x-direction. When it passes the origin after making a turn, it must do so at the same speed as when it first left the origin. This is because the particle is at the same electric potential (i.e the electric field cannot have done any net work on the particle), which implies that the speed of the particle remains unchanged, with only its direction being reversed.

Since the force it experiences is constant and we know the initial and final momenta are simply $\pm p_c^1$, we can infer that:

$$(-p_c^1) - (p_c^1) = F\Delta t \implies -2p_c^1 = eE\Delta t \implies p_c^1 = \frac{(m_n - m_p)v_0}{\sqrt{1 - v_0^2/c^2}} = -\frac{eE\Delta t}{2}$$

Since we also know the relativistic energy $p_c^0 = \frac{(m_n+m_p)c^2}{\sqrt{1-v_0^2/c^2}} = E_T$, we can use these two equations to solve for m_n and m_p to obtain:

$$\text{(a)} \quad m_n = \frac{1}{2}\sqrt{1 - v_0^2/c^2}\left(\frac{E_T}{c^2} - \frac{eE\Delta t}{2v_0}\right) \approx \boxed{12\ u}$$

$$\text{(b)} \quad m_p = \frac{1}{2}\sqrt{1 - v_0^2/c^2}\left(\frac{E_T}{c^2} + \frac{eE\Delta t}{2v_0}\right) \approx \boxed{16\ u}$$

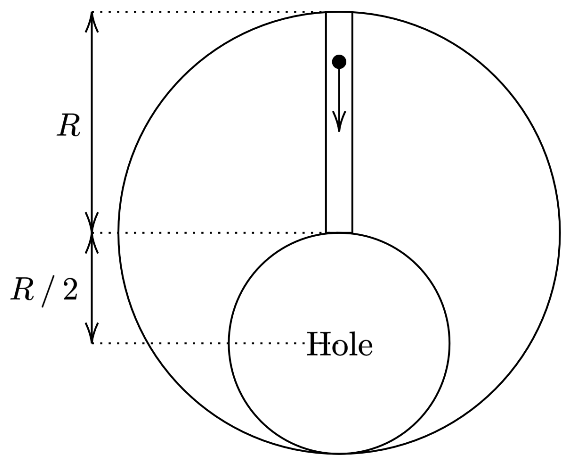
Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Problem 47: Journey to the Centre of the Earth (8 points)

A large spherical hole with radius $R/2 = 3200$ km is drilled in the Earth such that it is tangent to the surface of the Earth. A tunnel is then drilled along the common axis of the Earth and the hole. A stone is dropped from rest into the tunnel from the end that is further from the hole. Find the maximum distance the stone will fall from its starting position.

Assume that the Earth is a perfect sphere of uniform density with radius $R = 6400$ km, and neglect effects due to friction and the rotation of the Earth.

Leave your answer to 3 significant figures in units of km.



Solution: We consider the system to be a superposition of a uniform solid sphere of mass M and radius R, and a uniform solid sphere of negative mass $-M'$ (corresponding to the hole) and radius $R/2$. The force acting on the stone is given by the sum of the gravitational forces from both objects. Let the force due to the solid sphere be F_E, and the force due to the sphere with negative mass be F_H, and let their corresponding gravitational potential energies be U_E and U_H respectively. Take r to denote the distance of the stone from its initial position.

We can write expressions for F_E and F_H:

$$F_E = GMm\left(\frac{R-r}{R^3}\right)$$

$$F_H = -GM'm\begin{cases} \frac{1}{(\frac{3}{2}R-r)^2} & 0 < r < R \\ \frac{\frac{3}{2}R-r}{(\frac{R}{2})^3} & R < r < 2R \end{cases}$$

Since M and M' have the same density, M' is given by:

$$\frac{M'}{M} = \left(\frac{R/2}{R}\right)^3 = \frac{1}{8} \implies M' = \frac{M}{8}$$

Let $U = 0$ at $r = 0$. Then, we can integrate the above expressions from 0 to r to write U_E and U_H:

$$U_E = \frac{1}{2}GMm\left(\frac{(R-r)^2}{R^3} - \frac{1}{R}\right)$$
$$U_H = GM'm\begin{cases}\frac{1}{\frac{3}{2}R-r} - \frac{1}{\frac{3}{2}R} & 0 < r < R\\ \frac{1}{\frac{3}{2}R-R} - \frac{1}{\frac{3}{2}R} + \frac{1}{2}\frac{(\frac{1}{2}R)^2}{(\frac{R}{2})^3} - \frac{1}{2}\frac{(\frac{3}{2}R-r)^2}{(\frac{R}{2})^3} & R < r < 2R\end{cases}$$

The stone starts out at rest, hence its initial kinetic energy is zero. Since its initial potential energy is defined as zero at $r = 0$, its total energy must be zero. When the stone comes to a halt at the maximum distance from its entry point, its kinetic energy is also zero, so by conservation of energy, $U_E + U_H = 0$ at this point. Since this can happen anywhere between $r = 0$ and $r = 2R$, we have to solve for the ranges $0 < r < R$ and $R < r < 2R$ separately.

Assuming $0 < r < R$, we have

$$\frac{1}{2}\left(\frac{(R-r)^2}{R^3} - \frac{1}{R}\right) + \frac{1}{8}\left(\frac{1}{\frac{3}{2}R-r} - \frac{1}{\frac{3}{2}R}\right) = 0$$
$$\implies \frac{1}{2}(1-x)^2\left(\frac{3}{2}-x\right) - \frac{7}{12}\left(\frac{3}{2}-x\right) + \frac{1}{8} = 0$$

where $x = r/R$. Numerically, $x \approx 1.27$. This implies $r > R$, which contradicts our assumption that $0 < r < R$. Hence, the assumption is wrong, and we conclude that $R < r < 2R$.

Given that $R < r < 2R$, we have:

$$\frac{1}{2}\left(\frac{(R-r)^2}{R^3} - \frac{1}{R}\right) + \frac{1}{8}\left(\frac{1}{\frac{3}{2}R-R} - \frac{1}{\frac{3}{2}R} + \frac{1}{R} - \frac{1}{2}\frac{(\frac{3}{2}R-r)^2}{\left(\frac{R}{2}\right)^3}\right) = 0$$
$$\implies \frac{1}{2}(1-x)^2 - \frac{1}{2}\left(\frac{3}{2}-x\right)^2 - \frac{5}{24} = 0$$

We can solve the quadratic equation to get $x = 5/3$. This is within the range $R < r < 2R$, so it is consistent with our assumption. Hence:

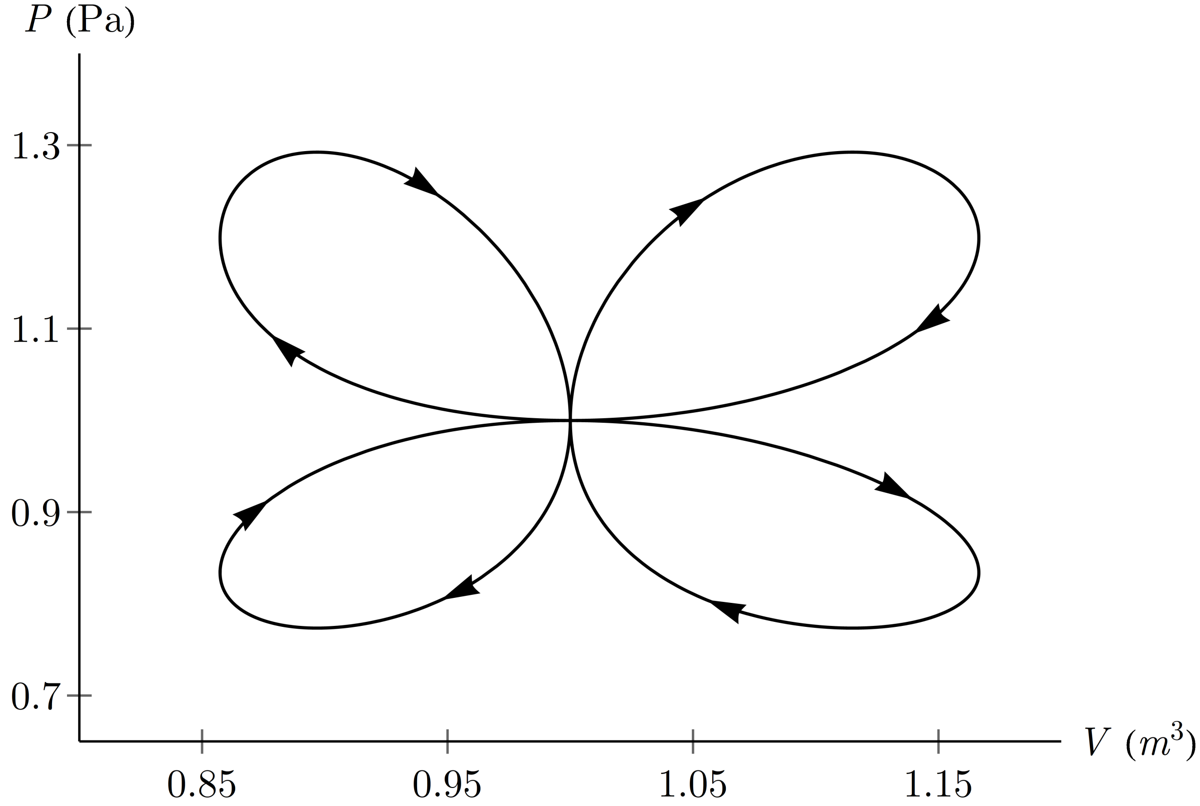
$$r = \frac{5}{3}R = \frac{5}{3} \times 6400\text{ km} = \boxed{10700\text{ km}}$$

Setter: Luo Zeyuan, zeyuan.luo@sgphysicsleague.org

Problem 48: Entropic Star (7 points)

Consider n mol of ideal monatomic gas that undergoes a reversible thermodynamic cycle. Let P and V respectively denote the pressure and volume of the gas at any point in the cycle. The $P-V$ graph is given by the following parametric equation for $0 \leq t \leq 2\pi$:

$$(P, V) = \left(\left(e^{\frac{1}{3}\sin 2t \cos t}\right) \text{Pa}, \left(e^{\frac{1}{5}\sin 2t \sin t}\right) \text{m}^3\right)$$



Suppose that the maximum entropy reached by the gas throughout the cycle is S_{max}, while the minimum entropy it reaches is S_{min}. Letting R denote the molar gas constant, compute the dimensionless quantity:

$$\left(\frac{S_{\text{max}} - S_{\text{min}}}{nR}\right)^2$$

Leave your answer to 2 significant figures.

Solution: We recall that for an infinitesimal step of a thermodynamic process, the change in entropy $\mathrm{d}S = \delta q / T$ where δq is the heat transferred to the gas during that step and T is the gas temperature.

To find δq, we invoke the First Law of Thermodynamics

$$\delta q = \mathrm{d}U + \delta w$$

where δw is the work done by the gas and $\mathrm{d}U$ is the change in internal energy of the gas.

For a monatomic ideal gas, internal energy U is given as follows. We differentiate to obtain $\mathrm{d}U$:

$$U = \frac{3}{2}P(t)V(t) \implies \mathrm{d}U = \frac{3}{2}P'(t)V(t)\,\mathrm{d}t + \frac{3}{2}P(t)V'(t)\,\mathrm{d}t$$

δw is given by:

$$\delta w = P\,\mathrm{d}V = P(t)V'(t)\,\mathrm{d}t$$

Substituting the expressions for $\mathrm{d}U$ and δw back into the First Law, we have:

$$\delta q = \mathrm{d}U + \delta w = \frac{3}{2}P'(t)V(t)\,\mathrm{d}t + \frac{5}{2}P(t)V'(t)\,\mathrm{d}t$$

To express T in terms of the quantities given in the question, we invoke the ideal gas law:

$$T = \frac{P(t)V(t)}{nR}$$

We are thus finally able to compute $\mathrm{d}S$

$$\begin{aligned}
\mathrm{d}S = \frac{\delta q}{T} &= nR\left(\frac{3}{2}\frac{P'(t)}{P(t)}\,\mathrm{d}t + \frac{5}{2}\frac{V'(t)}{V(t)}\,\mathrm{d}t\right)\\
\int_{S(0)}^{S(t)} \mathrm{d}S &= \int_0^t nR\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}\ln P(t) + \frac{5}{2}\ln V(t)\right)\mathrm{d}t\\
\Delta S &= nR\left(\frac{3}{2}\ln\frac{P(t)}{P(0)} + \frac{5}{2}\ln\frac{V(t)}{V(0)}\right)
\end{aligned}$$

where ΔS is the change in entropy from the state $(P(0), V(0))$ to $(P(t), V(t))$.

Applying the parametric equations for $P(t), V(t)$ from the question to this expression for entropy, we obtain:

$$\begin{aligned}
\Delta S &= nR\left(\frac{3}{2}\ln\frac{e^{\frac{1}{3}\sin 2t\cos t}\ \mathrm{Pa}}{1\ \mathrm{Pa}} + \frac{5}{2}\ln\frac{e^{\frac{1}{5}\sin 2t\sin t}\ \mathrm{m}^3}{1\ \mathrm{m}^3}\right)\\
&= \frac{1}{2}nR\sin 2t\,(\cos t + \sin t)\\
&= -\frac{\sqrt{2}}{2}nR\cos\left(2\left(t+\frac{\pi}{4}\right)\right)\sin\left(t+\frac{\pi}{4}\right)\\
&= \frac{\sqrt{2}}{2}nR\left[2\sin^3\left(t+\frac{\pi}{4}\right) - \sin\left(t+\frac{\pi}{4}\right)\right]
\end{aligned}$$

Note that the maximum and minimum values of $2u^3 - u$ are 1 and -1 respectively for $-1 \leq u \leq 1$. Hence the maximum and minimum values of

$$2\sin^3\left(t+\frac{\pi}{4}\right) - \sin\left(t+\frac{\pi}{4}\right)$$

are 1 and -1 respectively. Therefore:

$$\begin{aligned} S_{\text{max}} - S_{\text{min}} &= \sqrt{2}nR \\ \left(\frac{S_{\text{max}} - S_{\text{min}}}{nR}\right)^2 &= \boxed{2.0} \end{aligned}$$

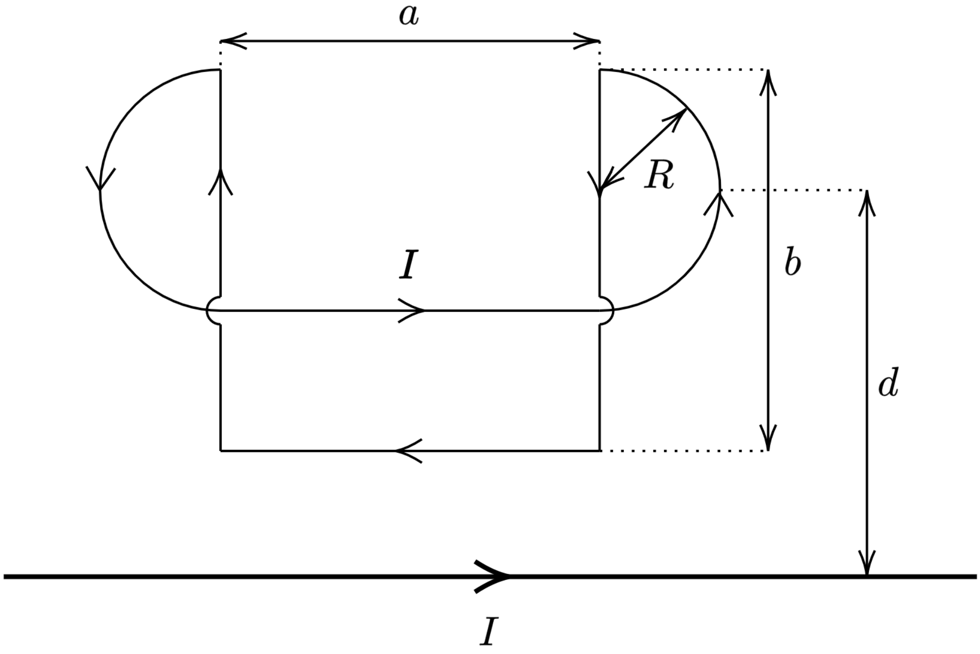
Setter: Ariana Goh, ariana.goh@sgphysicsleague.org

Problem 49: Strangely Shaped Current Loop (7 points)

A rigid wire loop is made of two semicircular arcs of radius $R = 0.50$ m, and these arcs are connected by two horizontal segments of length $a = 1.00$ m and two vertical segments of unknown length b, as drawn below. (The loop appears to overlap itself in the diagram but small bends in the wire outside the plane of the loop ensure it does not self-intersect. The effect of these bends are negligible beyond preventing self-intersection, and the loop is otherwise planar.)

A fixed long straight wire coplanar with the loop is located parallel to and a distance d from a line segment joining the centers of the semicircular arcs. The same current $I = 200$ A is passed through both the wire and the loop, as shown below. If the loop settles into stable equilibrium when $d = 1.70$ m, find b. Neglect gravity.

Leave your answer to 3 significant figures in units of m.



Hint: The following integral might be useful, where $|\alpha| < 1$:

$$\int_0^{\pi} \frac{1}{1 + \alpha \cos\theta}\, \mathrm{d}\theta = \frac{\pi}{\sqrt{1 - \alpha^2}}$$

Solution: Based on the diagram, the magnetic field from the long wire points out of the page in the region of the loop, and has magnitude $B = \frac{\mu_0 I}{2\pi r}$ at a distance r from the wire. We can compute the force on different parts of the loop separately. For the sake of consistency, we will denote a repulsive force away from the long wire as positive.

The vertical parts of the loop perpendicular to the long wire experience a purely horizontal magnetic force, as we can verify using the right-hand rule. Since the two parts are identical except for the direction of their current flow, the corresponding forces on them cancel exactly, leaving no net force.

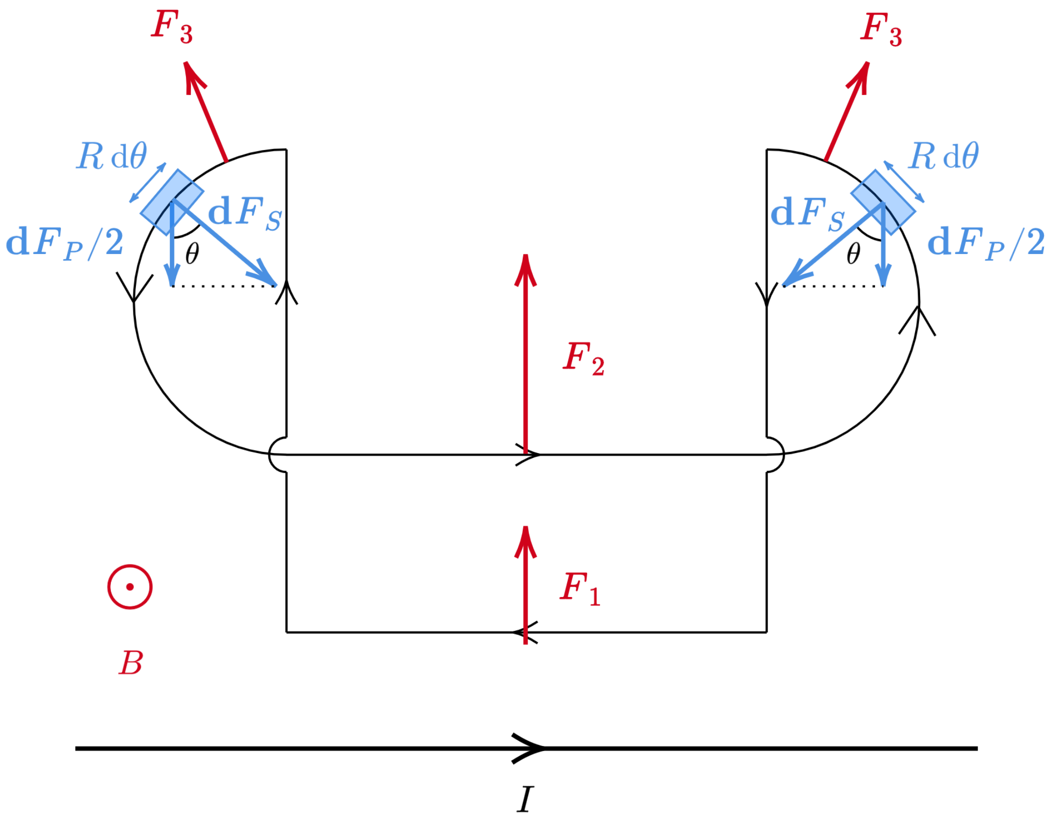
The horizontal parts of the loop parallel to the long wire experience forces entirely in the vertical direction. The closer segment lies a distance $d+R-b$ from the long wire and harbours a current antiparallel to it. Therefore, a repulsive force of

$$F_1 = I \cdot a \cdot \frac{\mu_0 I}{2\pi(d+R-b)} = \frac{\mu_0 I^2 a}{2\pi(d+R-b)}$$

acts on it, pulling it in the upwards direction. The other horizontal segment lies at a distance of $d-R$ and carries an opposite current, and therefore experiences an attractive downwards force of

$$F_2 = -\frac{\mu_0 I^2 a}{2\pi(d-R)}$$

Finally, let us consider the semicircular segments. Since the parts with parallel current flow relative to the long wire is closer to it than parts with antiparallel current flow, we expect the segments to experience a net attractive force. We can find the force by considering pairs of small segments of the arcs as shown in the diagram.



For a pair of small segments subtending an angle of $d\theta$ at an angle of θ from the vertical, the magnitude of the force $\mathrm{d}F_s$ experienced by a single segment is

$$\mathrm{d}F_s = IB\,\mathrm{d}l = \frac{\mu_0 I^2 R}{2\pi(d+R\cos\theta)}\,\mathrm{d}\theta$$

where the force is directed radially outward. Since the horizontal components of this force cancel for each pair of these segments, and the vertical components add together, we can find the net force on each pair of segments $\mathrm{d}F_p$ as

$$\mathrm{d}F_p = 2\,\mathrm{d}F_s \cos\theta = \frac{\mu_0 I^2 R \cos\theta}{\pi(d + R\cos\theta)}\,\mathrm{d}\theta$$

To find the total force on the circular segments, F_3, we integrate this from $\theta = 0$ to $\theta = \pi$:

$$F_3 = \int_{\theta=0}^{\theta=\pi} \mathrm{d}F_p = \int_0^{\pi} \frac{\mu_0 I^2 R\cos\theta}{\pi(d + R\cos\theta)}\,\mathrm{d}\theta = \frac{\mu_0 I^2}{\pi}\left(\pi - \int_0^{\pi} \frac{1}{1 + \frac{R}{d}\cos\theta}\,\mathrm{d}\theta\right)$$

The last integral can be expressed as $\int_0^{\pi} \frac{1}{1+\alpha\cos\theta}\,\mathrm{d}\theta = \frac{\pi}{\sqrt{1-\alpha^2}}$, which gives us the final expression for F_3:

$$F_3 = \mu_0 I^2 \left(1 - \frac{1}{\sqrt{1 - R^2/d^2}}\right)$$

This quantity is negative, and thus the circular segments experience a net attractive force, as expected. Since the system is in stable equilibrium, the forces must add to zero:

$$F_1 + F_2 + F_3 = \frac{\mu_0 I^2 a}{2\pi(d + R - b)} - \frac{\mu_0 I^2 a}{2\pi(d - R)} + \mu_0 I^2\left(1 - \frac{1}{\sqrt{1 - R^2/d^2}}\right) = 0$$

Solving for b yields:

$$b = d + R - \left(\frac{1}{d - R} + \frac{2\pi}{a}\left(\frac{d}{\sqrt{d^2 - R^2}} - 1\right)\right)^{-1} \approx \boxed{1.31\ \mathrm{m}}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Problem 50: Falling Hinges

A uniform rod of length $L = 4.20$ m is fixed to a pivot, A, around which it can freely rotate. Its other end is attached to the end of an identical rod at H, forming a freely rotating hinge. The remaining end of the second rod, B, carries a ball with the same mass as a single rod. The ball is constrained to only slide frictionlessly along the vertical line passing through A. The rods are initially held horizontal and are then released from rest. Let the angle formed by either rod with the horizontal be $\theta(t)$ where t is the time after release.

(a) At any time, the magnitude v of the velocity of the ball can be expressed as

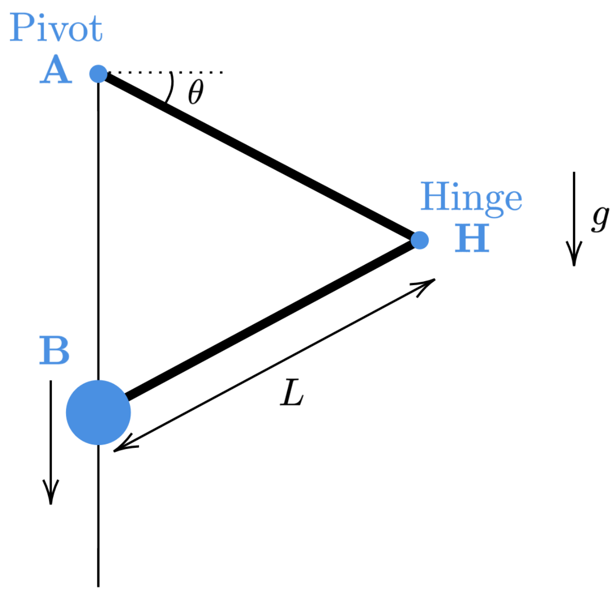
$$v = \alpha L \frac{\mathrm{d}\theta}{\mathrm{d}t} \cos\theta$$

where α is a positive numerical coefficient. Find α.

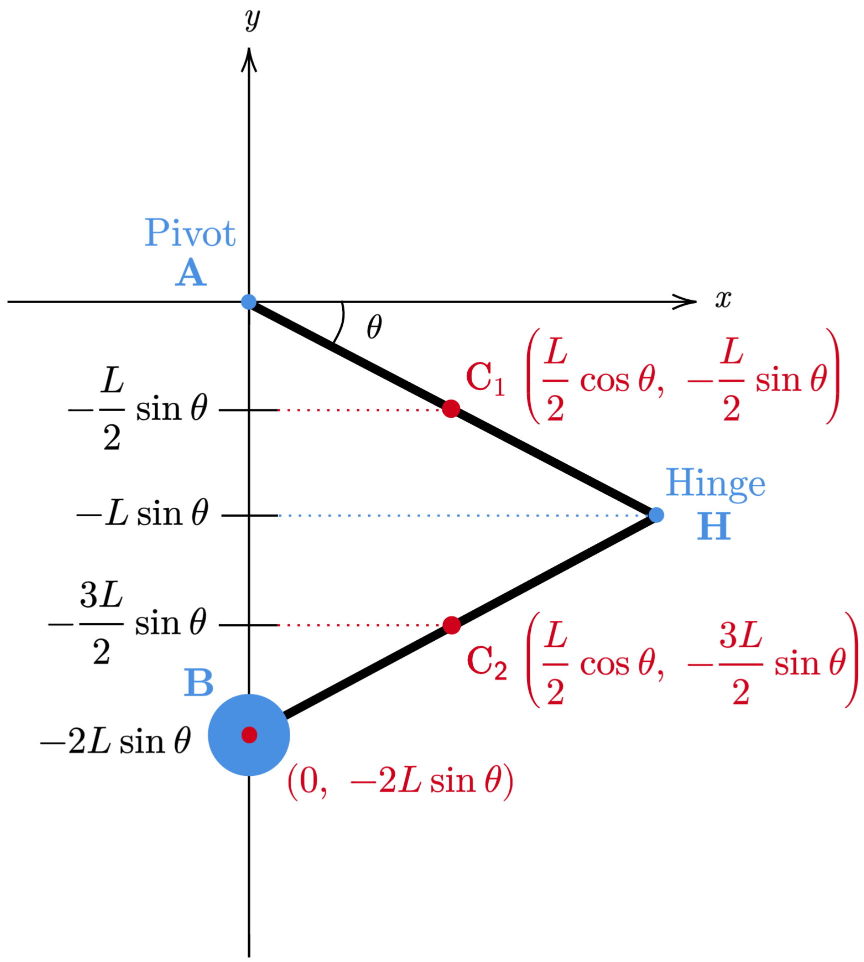
Leave your answers to 3 significant figures. (2 points)

(b) At what angle θ_* would v be maximal?

Leave your answers to 3 significant figures in units of degrees. (8 points)



Solution: In the solution, we will take $\dot{x} := \mathrm{d}x/\mathrm{d}t$ to denote a derivative of some quantity x with respect to time t for brevity.



(a) As illustrated above, we may determine the ball to be a distance $x = 2L\sin\theta$ from A at any time. v can thus be obtained via differentiation:

$$v = \dot{x} = 2L\dot{\theta}\cos\theta \implies \alpha = \boxed{2}$$

(b) Let the mass of a single rod be M.

We will deduce an equation of motion for $\theta(t)$ using conservation of energy. The center of masses of the two rods and ball are

$$\vec{C}_1 = \begin{pmatrix} \frac{L}{2}\cos\theta \\ -\frac{L}{2}\sin\theta \end{pmatrix}, \vec{C}_2 = \begin{pmatrix} \frac{L}{2}\cos\theta \\ -\frac{3L}{2}\sin\theta \end{pmatrix}, \vec{B} = \begin{pmatrix} 0 \\ -2L\sin\theta \end{pmatrix}$$

respectively. We can differentiate these position vectors with respect to time to find the velocities of the three points:

$$\dot{\vec{C}}_1 = \begin{pmatrix} -\frac{L}{2}\dot{\theta}\sin\theta \\ -\frac{L}{2}\dot{\theta}\cos\theta \end{pmatrix}, \dot{\vec{C}}_2 = \begin{pmatrix} -\frac{L}{2}\dot{\theta}\sin\theta \\ -\frac{3L}{2}\dot{\theta}\cos\theta \end{pmatrix}, \dot{\vec{B}} = \begin{pmatrix} 0 \\ -2L\dot{\theta}\cos\theta \end{pmatrix}$$

We can now find the translational kinetic energy T_T of the system just by multiplying the squared magnitudes of these three vectors by $M/2$.

$$\begin{aligned}
T_T &= \frac{1}{2}M\left|\dot{\vec{C_1}}\right|^2 + \frac{1}{2}M\left|\dot{\vec{C_2}}\right|^2 + \frac{1}{2}M\left|\dot{\vec{B}}\right|^2 \\
&= \frac{1}{2}M\left[\left(-\frac{L}{2}\dot{\theta}\sin\theta\right)^2 + \left(-\frac{L}{2}\dot{\theta}\cos\theta\right)^2\right. \\
&\left. + \left(-\frac{L}{2}\dot{\theta}\sin\theta\right)^2 + \left(-\frac{3L}{2}\dot{\theta}\cos\theta\right)^2 + \left(-2L\dot{\theta}\cos\theta\right)^2\right] \\
&= ML^2\dot{\theta}^2\left(\frac{1}{4} + 3\cos^2\theta\right)
\end{aligned}$$

The rotational kinetic energy T_R is more straightforward. Both rods share angular velocity $\dot{\theta}$ and moment of inertia $I = \frac{1}{12}ML^2$ about their centers of mass, and thus:

$$T_R = 2 \cdot \frac{1}{2}I\dot{\theta}^2 = \frac{1}{12}ML^2\dot{\theta}^2$$

Finally, the gravitational potential energy of the rods and ball relative to point A is:

$$U = Mg\left(-\frac{L}{2}\sin\theta\right) + Mg\left(-\frac{3L}{2}\sin\theta\right) + Mg\left(-2L\sin\theta\right) = -4MgL\sin\theta$$

The initial system has zero kinetic energy and no gravitational potential energy relative to point A. Thus, conservation of energy gives:

$$0 = T_T + T_R + U = ML^2\dot{\theta}^2\left(\frac{1}{4} + 3\cos^2\theta\right) + \frac{1}{12}ML^2\dot{\theta}^2 - 4MgL\sin\theta$$

which can be rearranged to give a first-order differential equation for θ:

$$\dot{\theta}^2 = \frac{12\sin\theta}{1 + 9\cos^2\theta} \cdot \frac{g}{L}$$

There is no need to solve for $\theta(t)$ explicitly. Instead, we just need to use this equation to express the downward velocity v of the ball in terms of θ:

$$v = 2L\dot{\theta}\cos\theta = 2L \cdot \sqrt{\frac{12\sin\theta}{1 + 9\cos^2\theta} \cdot \frac{g}{L}} \cdot \cos\theta = 4\sqrt{3gL} \cdot \sqrt{\frac{\sin\theta\cos^2\theta}{1 + 9\cos^2\theta}}$$

The angle θ_* which maximizes this velocity is simply the angle that maximizes the function:

$$f(\theta) = \sqrt{\frac{\sin\theta\cos^2\theta}{1 + 9\cos^2\theta}}$$

Thus, we can find θ_* by differentiating this function and setting its derivative to zero:

$$f'(\theta) = \frac{1}{2}\left(\frac{\sin\theta\cos^2\theta}{1+9\cos^2\theta}\right)^{-1/2}\frac{\cos\theta(9\cos^4\theta+3\cos^2\theta-2)}{(1+9\cos^2\theta)^2} = 0$$

This reduces to solving the equation:

$$0 = 9\cos^4\theta + 3\cos^2\theta - 2 = (3\cos^2\theta - 1)(3\cos^2\theta + 2)$$

Since $\cos^2\theta \geq 0$, we must have $3\cos^2\theta - 1 = 0$, or:

$$\theta_* = \cos^{-1}\frac{1}{\sqrt{3}} = \boxed{54.7^\circ}$$

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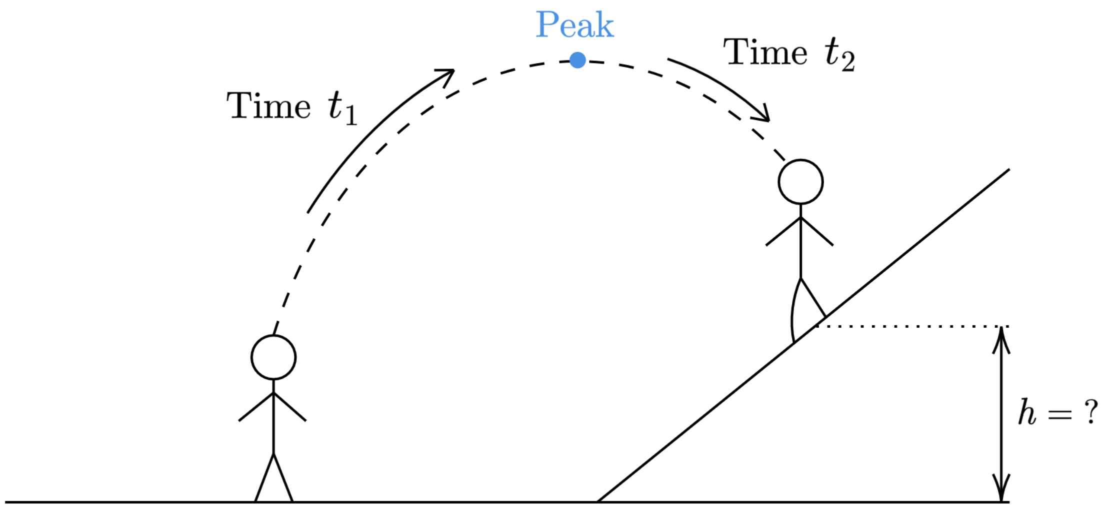
Half Hour Rush M1: I Have The High Ground (3 points)

In *Star Wars: Revenge of the Sith*, Obi-Wan and Anakin engage in a fierce duel on the planet Mustafar. When Obi-Wan reaches a position higher than Anakin, he exclaims, "It's over Anakin! I have the high ground!" In response, Anakin jumps towards Obi-Wan.

Based on the movie timestamps, he takes approximately $t_1 = 1.6$ s to reach the peak of his trajectory, and another $t_2 = 1.2$ s thereafter to reach Obi-Wan's position.

Use this information to determine the height difference h between Obi-Wan and Anakin's initial position. You may assume that the gravitational field on the surface of Mustafar is the same as that on Earth.

Leave your answer to 2 significant figures in units of m.



Solution: Throughout this problem, we're only concerned with Anakin's motion in the vertical direction.

Let the vertical component of Anakin's initial velocity be u. At the peak of his trajectory, which took time t_1, the vertical component of his velocity must have reached zero:

$$u - gt_1 = 0 \implies u = gt_1$$

Hence, the height difference y_1 between his trajectory's peak and his initial position is given by:

$$y_1 = ut_1 - \frac{1}{2}gt_1^2 = \frac{1}{2}gt_1^2$$

After reaching the peak, he falls for another time t_2 before reaching Obi-Wan's position. So, the height difference y_2 between his trajectory's peak and Obi-Wan's position is:

$$y_2 = \frac{1}{2}gt_2^2$$

With y_1 and y_2, we can easily find h:

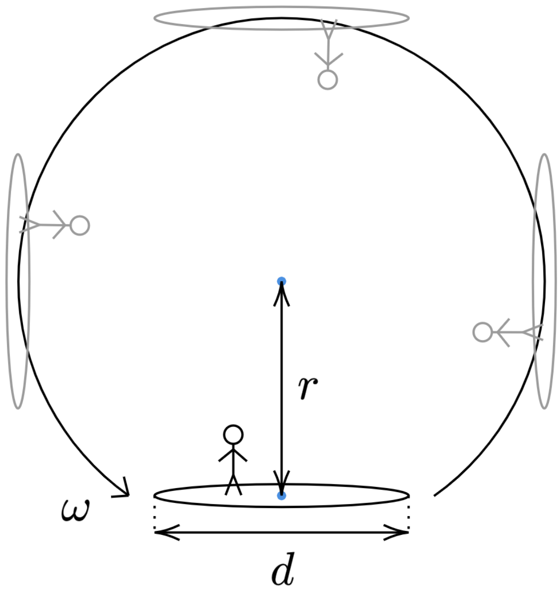
$$h = y_1 - y_2 = \frac{1}{2}g(t_1^2 - t_2^2) \approx \boxed{5.5 \text{ m}}$$

Setter: Christopher Ong, **chris.ong@sgphysicsleague.org**

Half Hour Rush M2: Millennium Falcon (4 points)

In the *Star Wars* franchise, passengers in the Millennium Falcon appear to experience gravitational forces similar to those on Earth. While this is explained with fictional devices, artificial gravity is, in fact, possible in space, through rotation.

Consider the Falcon as a disc with diameter $d = 25.6$ m, which rotates with constant angular velocity about an axis parallel to the disc's flat surface at perpendicular distance $r = 50.0$ m away from the disc's centre, as illustrated below.



Luke stands on the flat surface of the disk and releases a ball from his position. Depending on where he stands, the initial magnitude of acceleration of the ball from his perspective differs. Denoting the maximum and minimum initial accelerations of the ball as a_{max} and a_{min} respectively, find the ratio $a_{\text{max}}/a_{\text{min}}$.

Assume there are no celestial bodies nearby, and that the ball's initial height above the floor is much smaller than r.

Leave your answer to 3 significant figures.

Solution: Let us consider the ball of mass m a distance R away from the centre of rotation, in the reference frame that rotates at an angular velocity of ω. It is clear that the acceleration of this ball is a result of the fictitious centrifugal force, which is given by $F_c = m\omega^2 R$. Dividing through by m, we obtain $a = \omega^2 R$ as an expression for the acceleration of the ball.

Since ω is constant, to find a_{min} and a_{max}, it suffices to consider the points on the disc closest to and furthest from the centre of rotation respectively (when R is minimum and maximum).

By applying the Pythagorean Theorem, we find that, for a point on the disc a distance

x away from the disc's centre, its distance from the centre of rotation is given by $R = \sqrt{r^2 + x^2}$.

The point on the disc closest to the centre of rotation is given by $x = 0$, i.e. at the centre of the disc. Hence:

$$a_{\text{min}} = m\omega^2\sqrt{r^2 + 0^2} = m\omega^2 r$$

On the other hand, the point on the disc furthest from the centre of rotation is given by $x = d/2$, i.e. at the edge of the disc. Hence:

$$a_{\text{max}} = m\omega^2\sqrt{r^2 + \left(\frac{d}{2}\right)^2} = m\omega^2\sqrt{r^2 + \frac{d^2}{4}}$$

Therefore, the required ratio $a_{\text{max}}/a_{\text{min}}$ is:

$$\frac{a_{\text{max}}}{a_{\text{min}}} = \frac{m\omega^2\sqrt{r^2 + \frac{d^2}{4}}}{m\omega^2 r} = \frac{\sqrt{r^2 + \frac{d^2}{4}}}{r} \approx \boxed{1.03}$$

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Half Hour Rush M3: Death Star (4 points)

In the original 1977 *Star Wars* film, the Death Star, a space station of the Galactic Empire, annihilated a rebellious planet, Alderaan, in a matter of seconds using a superlaser, reducing it to a cloud of small asteroids scattered very far apart. Suppose that Alderaan was a uniform spherical planet of radius $R = 6000$ km and mass $M = 6.00 \times 10^{24}$ kg, similar to our Earth. If the Death Star had completed its deed within a time period of $\Delta t = 10.0$ s, what was the minimum average power P that the Death Star had to supply over this period to achieve this feat of destruction?

Leave your answer to 2 significant figures in units of $L_\odot$ *(solar luminosity), where* $1\ L_\odot$ *is the power output of a single Sun, or* $1\ L_\odot = 3.828 \times 10^{26}$ W.

Solution: After being annihilated, the gravitational potential energy of the cloud of asteroids is essentially zero since they are spaced far apart. Thus, by conservation of energy, the total work done by the Death Star must be greater than or equal to the negative of the gravitational potential energy of the original planet.

Hence, we seek to determine the gravitational potential energy U of the original planet, which is a uniform sphere with radius R and mass M. Let $\rho \equiv \frac{M}{\frac{4}{3}\pi R^3}$ denote the density of the sphere. We can consider assembling the sphere by successively pulling spherical shells of infinitesimal thickness $\mathrm{d}r$ from infinity. At an intermediate stage of this process, the sphere has radius r. The energy $\mathrm{d}U$ required to bring the next shell from infinity is thus:

$$\mathrm{d}U = -\frac{G(\frac{4}{3}\pi r^3 \rho)(4\pi r^2 \rho\,\mathrm{d}r)}{r} = -\frac{3GM^2 r^4}{R^6}\,\mathrm{d}r$$

The total gravitational potential energy of the sphere can then be obtained via integration:

$$U = \int_{r=0}^{r=R} \mathrm{d}U = -\frac{3GM^2}{R^6}\int_0^R r^4 \mathrm{d}r = -\frac{3GM^2}{5R}$$

Alternatively, this well-known result can be looked up online.

Since the Death Star had to supply at least an energy $-U$ to annihilate Alderaan, and it was done over a time interval Δt, the minimum power P required works out to be:

$$P = \frac{-U}{\Delta t} = \frac{3GM^2}{5R\Delta t} = 2.40 \times 10^{31}\ \mathrm{W} \approx \boxed{63000\ L_\odot}$$

That's quite a lot of power for a space station just a couple hundred kilometers wide!

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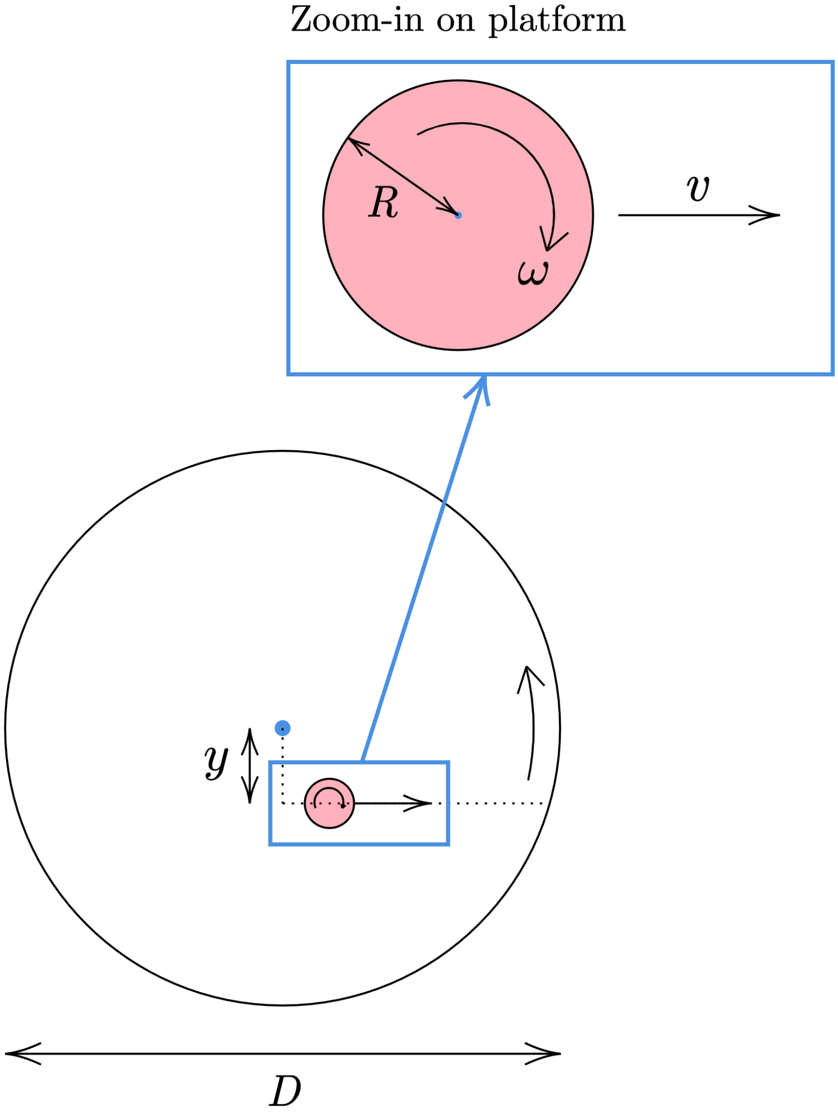
Half Hour Rush M4: I Am The Senate (5 points)

In *Star Wars: Revenge of the Sith*, Yoda fights with Palpatine in the Senate chamber. Assume that the Senate is fixed in place and is circular with diameter $D = 100$ m.

Yoda stands $y = 20$ m south of the centre and hurls a spinning platform towards Palpatine in the x-direction as shown below. Assume the platform is a uniform disc of mass $M = 100$ kg and radius $R = 1$ m, with clockwise spin $\omega_0 = 1$ rad s^{-1} and speed $v_0 = 4$ m s^{-1}. However, he narrowly misses, and the platform collides inelastically with the walls of the Senate. After collision with the walls, the platform rolls along the walls of the Senate without slipping.



You may assume that all motion takes place in the xy plane, which is horizontal, and that there is no gravity. Find the time taken t for the platform to complete 1 full revolution around the Senate. Do not include the time before the collision.

Leave your answer to 3 significant figures in units of s.

Solution: When considering the rotational motion (more specifically, the angular momentum) of an object, two special points are typically of particular interest: the centre of mass, and instantaneous centre of rotation. Specifically, the absence of a net torque taken about either centre is necessary and sufficient to conclude that angular momentum is conserved about that axis. In the case of a circle rolling without slipping along a surface, this instantaneous centre is the point of contact. Incidentally, this also implies that any forces during the collision (normal and frictional forces necessary for a change in linear momentum) will act at this same contact point, leaving no net torque about the point and hence leaving angular momentum unchanged.

We must first determine the angular displacement θ of the point of collision from Yoda's position, which is done by taking the Senate chamber to be a circle of radius $D/2 - R$ to account for the platform's radius. Hence, $\theta = \cos^{-1} \frac{y}{D/2-R}$.

To model the inelastic collision, we take the collision point as our pivot. By conserving angular momentum of the platform about the collision point, we can find the final value of its angular velocity ω_f:

$$\left(\frac{1}{2}MR^2 + MR^2\right)\omega_f = Mv_0R\cos\theta + \frac{1}{2}MR^2\omega_0$$

This gives an angular velocity of $\omega_f \approx 1.422$ rad s^{-1}.

We then notice that to complete a full round, the platform rotates only 49 times (not $D/2R = 50$, as it rotates oppositely by 1 revolution to traverse 50 circumferences)[10], hence $t = 49 \times \frac{2\pi}{\omega_f} \approx \boxed{217 \text{ s}}$.

An alternative method is to consider the impulse by the normal force and friction at the contact point for an instantaneous collision and to solve for it simultaneously. This method yields the same result as above.

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[10]Refer to *SPhL 2021 Solutions*, Problem 16: Rolling Pennies for a more detailed explanation.

Half Hour Rush E1: Cheap Christmas (3 points)

It's the most wonderful time of the year, and Paul needs to power all $n = 30$ of his identical Christmas lights via a constant voltage source. But Paul is broke, and electricity is expensive, so he seeks the cheapest way to do this!

Depending on how he wires the lights, the total light output may vary, and the total power they consume may range between P_{min} and P_{max}. Calculate $P_{\text{max}}/P_{\text{min}}$.

Leave your answer to 2 significant figures.

Solution: Let the voltage provided by the source be V, and the equivalent resistance of the circuit of lights be R_{eq}. The power P consumed by all the lights may be written as:

$$P = \frac{V^2}{R_{\text{eq}}}$$

Let the resistance of each light be R. R_{eq} is maximised when all of the lights are connected in series, such that $R_{\text{eq}} = nR$. This yields the smallest possible power P_{min}:

$$P_{\text{min}} = \frac{V^2}{nR}$$

On the other hand, R_{eq} is minimised when all of the lights are connected in parallel, such that $R_{\text{eq}} = R/n$. This results in maximum power consumed P_{max}:

$$P_{\text{max}} = \frac{nV^2}{R}$$

Hence, $P_{\text{max}}/P_{\text{min}} = n^2 = \boxed{900}$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Half Hour Rush E2: Cutting Costs (3 points)

Paul needs to build a circuit that ordinarily uses a silver wire of length $L = 5.00$ m and resistance $R = 0.10\ \Omega$, but Paul switched the wire to copper instead to save on costs, as he is a very broke man. If the new wire still has the same length and resistance, how much money did Paul save with this move? Use the properties of silver and copper as listed below.

Metal	Cost c	Electrical Resistivity r	Density ρ
Silver	\$722.43 per kg	$1.63 \times 10^{-8}\ \Omega$ m	10490 kg m^{-3}
Copper	\$9.68 per kg	$1.68 \times 10^{-8}\ \Omega$ m	8960 kg m^{-3}

Leave your answer to 2 decimal places in units of dollars (\$).

Solution: We shall let the total cost of the wire be C, the cost per unit mass be c, the cross-sectional area be a, the density be ρ and the resistivity be r. The total cost for a wire of either material is given by:

$$C = c \cdot \text{mass of wire} = c\rho \cdot \text{volume of wire} = c\rho a L$$

Since the resistance must be fixed, we have for both materials:

$$R = \frac{rL}{a} \implies a = \frac{rL}{R}$$

Thus, we can re-express the total cost C:

$$C = c\rho r L^2 / R$$

The difference in cost would be the difference in this quantity between the two materials:

$$\Delta C = C_s - C_c = \frac{L^2}{R}\left(c_s\rho_s r_s - c_c\rho_c r_c\right) \approx \boxed{\$30.52}$$

where the subscripts "c" and "s" refer to copper and silver respectively.

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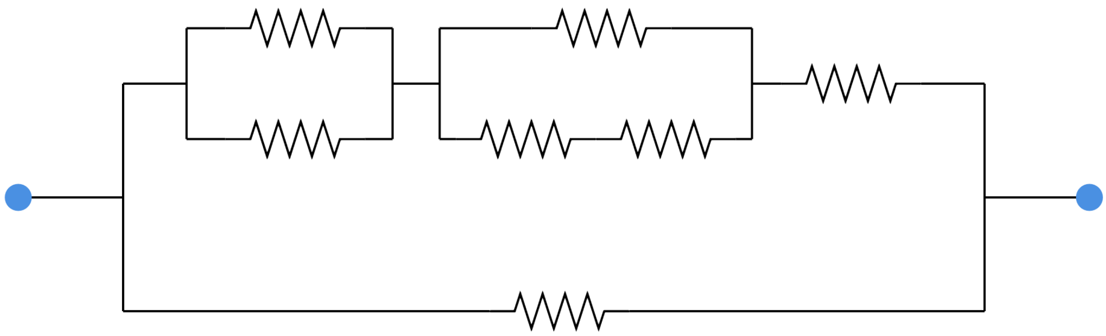
Half Hour Rush E3: Careful Construction (5 points)

Paul has a large number of $R = 1\ \Omega$ resistors, which he wants to arrange (using only series and parallel connections) into a network with an equivalent resistance of $R_{\text{eff}} = \frac{13}{19}\ \Omega$. To save money, he wants to use as few resistors as possible. What is the minimum number of $1\ \Omega$ resistors he needs? You may assume all wires are ideal.

Leave your answer as an integer.

Solution: $\boxed{7}$. By construction, we see that there is a possible solution with 7 resistors:



To conclude this is the optimal solution, you can test everything with 6 and fewer resistors.

Alternative solution: To obtain an upper bound on the number of resistors used, we can take the following intuitive strategies to help us "reduce" a value of R_{eff} to a more "fundamental" value:

- If the value is less than $1\ \Omega$, remove a parallel $1\ \Omega$ resistor.
- Look for fractions of the form $\frac{ab}{a+b}, \frac{a+b}{ab}$ or with whole number additions, and try to minimise the sum $a + b$.

If we apply this to the value $\frac{13}{19}\ \Omega$, notice that we can first remove a parallel resistor to get a top branch of resistance $\frac{13}{6}\ \Omega$. We can then decompose this into $\frac{13}{6} = 1 + \frac{7}{6}$, thus we can remove a $1\ \Omega$ resistor in series to get $\frac{7}{6}\ \Omega$. Lastly, we realise that $\frac{7}{6} = \frac{1}{2} + \frac{2}{3}$, both of which are irreducible fractions of the form $\frac{ab}{a+b}$. $\frac{1}{2}\ \Omega$ can be obtained with 2 resistors in parallel, and $\frac{2}{3}\ \Omega$ can be obtained with 1 resistor in parallel with 2 resistors. Therefore, this gives us a network equivalent to the diagram above, with 7 resistors in total.

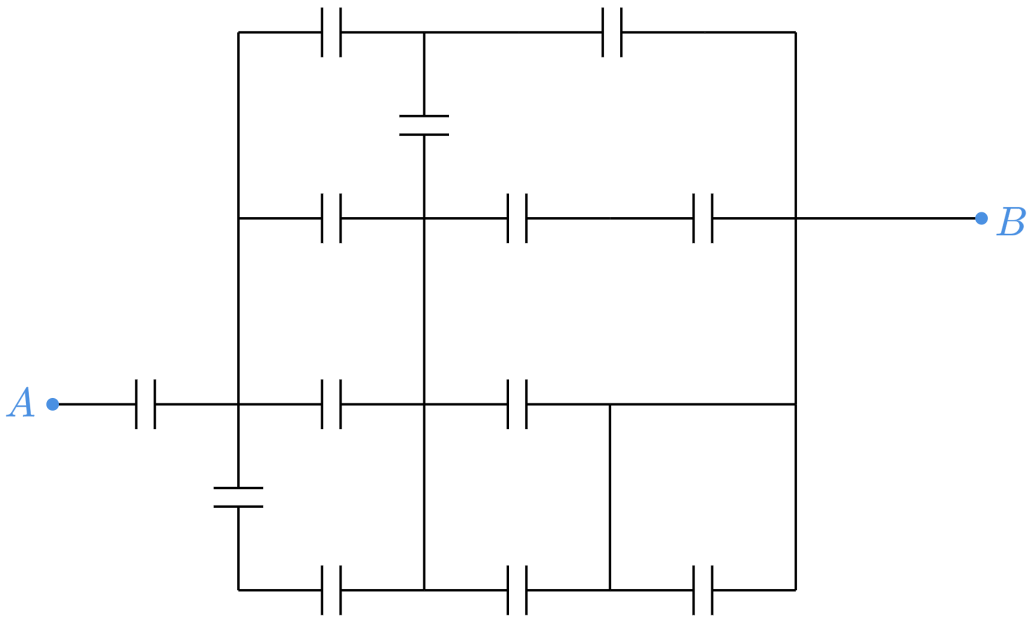
For a more rigorous approach, one may realise that among all possible series and parallel arrangements of a fixed number of resistors, the denominator is upper bounded by the Fibonacci sequence; for a circuit with 6 identical resistors, the largest possible denominator is in fact 13. Hence, to achieve a denominator of 19, there must be at least 7 resistors. Achieving such a result is then possible by construction.[11]

Setter: Paul Seow, paul.seow@sgphysicsleague.org

[11]This approach is covered in greater detail in: Khan, S.A (2012). Farey sequences and resistor networks. Proc Math Sci 122, 153–162.

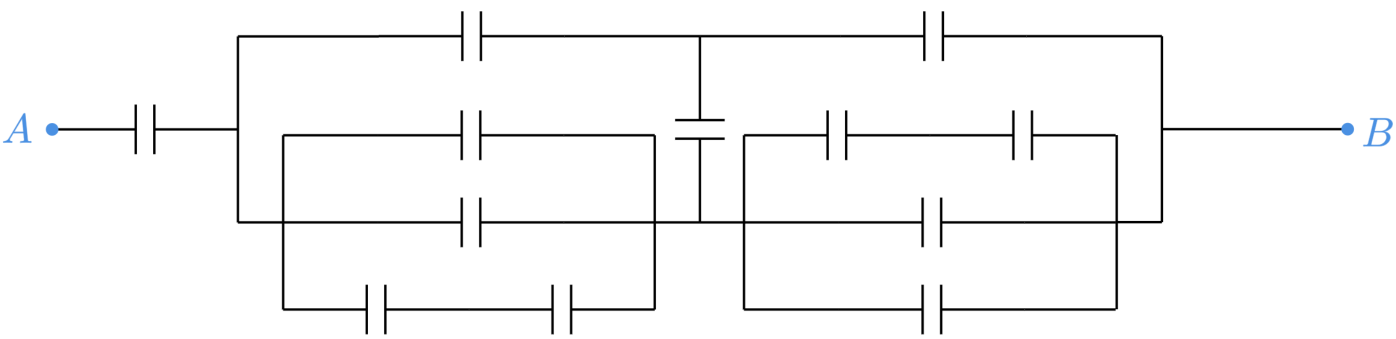
Half Hour Rush E4: Capacitor Chaos (4 points)

The following circuit consists of identical capacitors, each of capacitance $C_0 = 2200$ F. It's extremely wasteful, and Paul wishes to be prudent. Hence, he replaces this with a single capacitor of capacitance C_{eq} between A and B. Find C_{eq}.

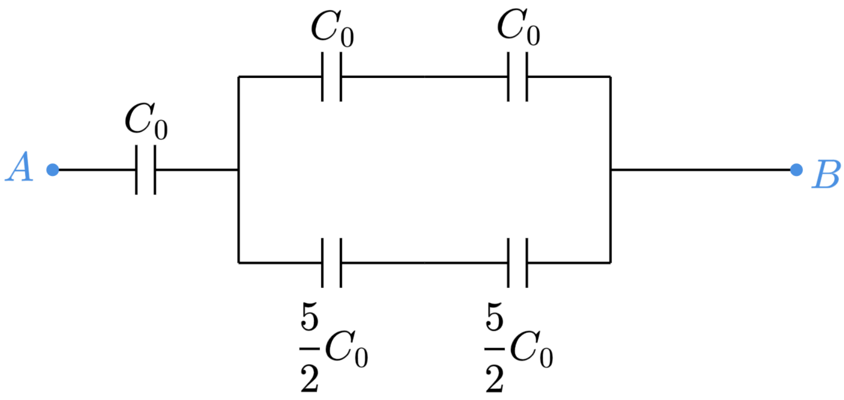


Leave your answer to 3 significant figures in units of F.

Solution: The circuit can be redrawn as such (note that the bottom right capacitor can be removed as both plates are equipotential and hence no charge is stored):



It can then be simplified to the following (again, note that the central capacitor can be removed as both plates are equipotential by symmetry):



The effective capacitance can then be easily found as follows:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_0} + \frac{1}{\frac{C_0}{2} + \frac{5C_0}{4}} \implies C_{\text{eq}} = \frac{7C_0}{11} \approx \boxed{1400 \text{ F}}$$

I just added capacitors to the grid randomly, never thought it would end up being so symmetrical.

Setter: Galen Lee, galen.lee@sgphysicsleague.org

Half Hour Rush X1: Shrinking a Grape (3 points)

A spherical grape of radius $R = 2$ cm is dried in an oven. After a time $T = 24$ h in the oven, the grape has turned into a spherical raisin of radius $r = 1$ cm. Assuming that the missing volume is due entirely to evaporated water, and that all the energy from the oven is used to evaporate the water, what is the average power supplied by the oven?

Leave your answer to 2 significant figures in units of W.

Solution: The volume of water evaporated is $V = \frac{4}{3}\pi(R^3 - r^3)$. Correspondingly, the mass of water evaporated is $m = \rho V = \frac{4}{3}\rho\pi(R^3 - r^3)$.

Letting the specific latent heat of vaporisation of water be l_v, the total energy E supplied by the oven to evaporate this water is given by $E = ml_v = \frac{4}{3}\rho\pi(R^3 - r^3)l_v$. Therefore, the average power from the oven is $E/T \approx \boxed{0.77 \text{ W}}$.

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Half Hour Rush X2: Throwing a Fish (3 points)

During lunch, Chris was mildly irritated to find that a piece of fish in his food had already become cold. He grabs the fish and throws it repeatedly against the table in an attempt to heat it up. If the piece of fish has mass $M = 300$ g and specific heat capacity $c = 1.67$ kJ kg^{-1} K^{-1} and it is thrown against the table uniformly at a speed of $v = 3.0$ m s^{-1} each time, what is the minimum number of throws N that Chris needs to heat the fish from the room temperature of $T_0 = 25°$C to a comfortably warm temperature of $T_1 = 60°$C? Assume that the fish remains intact throughout the throwing process.

Leave your answer to 2 significant figures.

Solution: The maximum thermal energy U that the fish can gain during each throw is the entirety of its kinetic energy, or:

$$U = \frac{1}{2}Mv^2$$

The total energy U_T needed to raise the fish from $T_0 = 25°$C to $T_1 = 60°$C is:

$$U_T = Mc(T_1 - T_0)$$

As such, the number of throws Chris needs is minimally:

$$N = \frac{U_T}{U} = \frac{2c(T_1 - T_0)}{v^2} \approx \boxed{13000}$$

Even in such grossly idealised conditions, that's still quite a lot of effort just to heat up a piece of fish.

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Half Hour Rush X3: Cloudy With A Chance of Meatballs (3 points)

A meatball of mass $m = 60$ g and specific heat capacity $c = 1670$ J kg^{-1} K^{-1} is cooked by dropping it from height $h = 830$ m onto a plate. The drag force, F, is given by $F = kv^2$ where $k = 1.18 \times 10^{-3}$ kg m^{-1} and v is the velocity of the meatball.

In order to heat the meatball to $T = 175$°C from an initial temperature $T_0 = 25$°C, what initial velocity, v_0, does the meatball have to be dropped with? You may assume that the meatball reaches terminal velocity quickly, and neglect heat loss to the surroundings. Assume that Earth's gravitational field is uniform with magnitude $g = 9.81$ m s^{-2}.

Leave your answer to 2 significant figures in units of m s^{-1}.

Solution: Let the terminal velocity be denoted v_t, which is obtained when the meatball is in equilibrium, i.e. $mg = F = kv_t^2 \implies v_t^2 = \frac{mg}{k}$.

Letting $\Delta T = T - T_0$, we may invoke conservation of energy to obtain the following equation:

$$mgh + \frac{1}{2}mv_0^2 = \frac{1}{2}mv_t^2 + mc\Delta T \implies v_0 = \sqrt{\frac{mg}{k} + 2c\Delta T - 2gh} \approx \boxed{700 \text{ m s}^{-1}}$$

Obviously, cooking is not done by heating food for such short durations, but it's still a fun thought experiment to ponder while enjoying those Swedish meatballs at IKEA.

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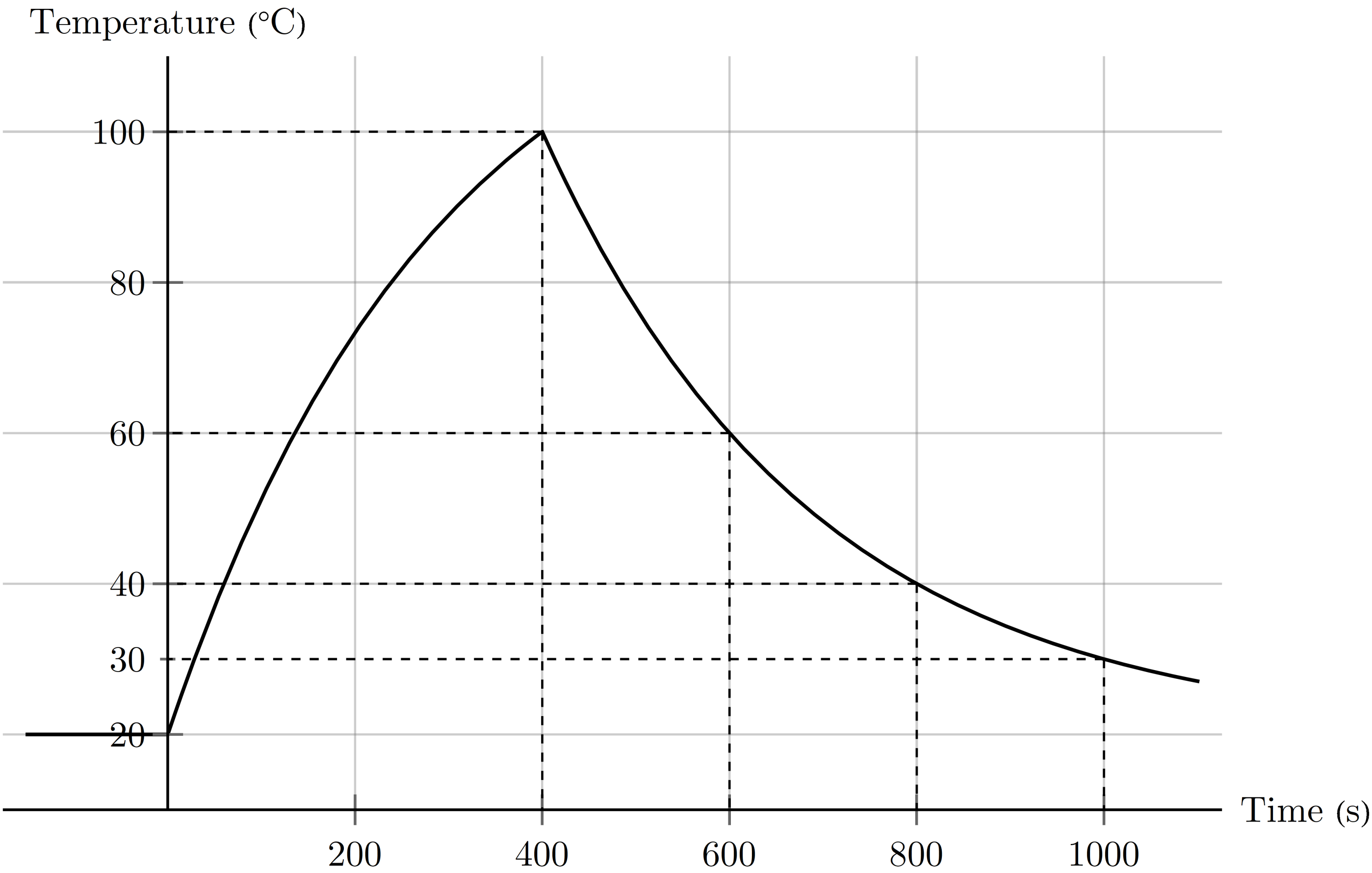
Half Hour Rush X4: Mala Hotpot (5 points)

Shuan is craving for some Mala soup, but he hasn't cooked in years, so he first checks that his stove is working. He fills a pot with mass $m = 1.0$ kg of water initially at room temperature $T_s = 20°\text{C}$. At time $t = 0$ s, he turns on the stove.

Later on, the moment he notices water coming to a boil, he switches off the stove. The graph of the water's temperature against time t is plotted below. Use this information to find the power P supplied by the stove to the water.

The stove's power and the mass of water in the pot may be assumed to be constant over time.



Leave your answer to 3 significant figures in units of W.

Solution: Let the water's temperature over time be $T(t)$. Firstly, consider the cooling portion of the graph. We notice that at the start of the cooling process (at $t = 400$ s), $T - T_s = 80°\text{C}$. After time interval $\Delta t = 200$ s, $T - T_s = 40°\text{C}$. After another time interval $\Delta t = 200$ s, $T - T_s = 20°\text{C}$. After yet another time interval $\Delta t = 200$, $T - T_s = 10°\text{C}$. In summary, the value of $T - T_s$ halves after every time interval $\Delta t = 200$ s.

This means the cooling curve follows an exponential decay with "half-life" given by

$\Delta t = 200$ s. The exponential nature of the curve indicates that the cooling of the water is governed by Newton's law of cooling, which states that the rate of heat loss to the surroundings at any time is given by $hA(T - T_s)$ where h is the heat transfer coefficient (in W m^{-2} K^{-1}) and A is the surface area of the system (in m^2).

Hence, the differential equation governing the cooling portion of the graph is

$$C\frac{dT}{dt} = -hA(T - T_s)$$

where C denotes the heat capacity of water. We may solve for T to give:

$$T(t) - T_s = [T(0) - T_s]\, e^{-\frac{hA}{C}t}$$

The "half-life" Δt can be related to h, A and C, enabling us to find an expression for the product hA in terms of known quantities:

$$\frac{hA}{C}\Delta t = \ln 2 \implies hA = \frac{C}{\Delta t}\ln 2$$

We now consider the heating portion of the graph. The differential equation is similar to the one governing the cooling portion, but with an additional term to account for the stove's power P:

$$C\frac{dT}{dt} = P - hA(T - T_s)$$

Under the initial condition that $T(0) = T_s$, the solution for $T(t)$ is:

$$T(t) = T_s + \frac{P}{hA}\left(1 - e^{-\frac{hA}{C}t}\right)$$

This may be rewritten by substituting the product hA as derived above:

$$T(t) = T_s + \frac{P\Delta t}{C\ln 2}\left(1 - 2^{-\frac{t}{\Delta t}}\right) \implies P = [T(t) - T_s]\frac{C\ln 2}{\Delta t\left(1 - 2^{-\frac{t}{\Delta t}}\right)}$$

We know that at $t = 400$ s, $T(t) = 100$°C. We also know half-life $\Delta t = 200$ s from the graph, and heat capacity $C = mc = 4.19 \times 10^3$ J K^{-1}. Hence, we may substitute these values to arrive at $P \approx \boxed{1550 \text{ W}}$.

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