# 4$^{th}$ Singapore Physics League

**Date:** 6 July 2024
**Website:** sgphysicsleague.org
**Email:** contact@sgphysicsleague.org

---

Singapore Physics League (SPhL) is strongly supported by the Institute of Physics Singapore (IPS) and the Singapore Ministry of Education (MOE), and is sponsored by Micron.

**SPhL 2024 Organising Team**

**Chen Guangyuan**
Chief Organiser

**Roger Zhang**
Deputy Chief Organiser

**Gerrard Tai**
Lead Developer, Problem Setter

**Huang Ziwen**
Lead Developer, Problem Setter

**Huang Zehan**
Team Leader, Problem Setter

**Paul Seow**
Team Leader, Problem Setter

**James He**
Problem Setter

**Li Xinrui**
Problem Setter

**Li Yichen**
Problem Setter

**Liu Yueyang**
Problem Setter

**Shanay Jindal**
Problem Setter

**Shen Xing Yang**
Problem Setter

**Sun Yu Chieh**
Problem Setter

**Tran Duc Khang**
Problem Setter

**Prannaya Gupta**
Web Developer

**Yap Yuan Xi**
Web Developer

**Tan Chien Hao**
Systems Engineer

**Theodore Lee**
Systems Engineer

**Keith Chan**
Graphic Designer

**Jed Lim**
Data Analyst

SPhL 2024 was made possible with the assistance of Christopher Ong, Tan Jun Wei, Wong Yee Hern, Brian Siew, Luo Zeyuan, Shaun Quek, Tian Shuhao and Man Juncheng.



Document version: 2.1 (Last modified: July 17, 2024)

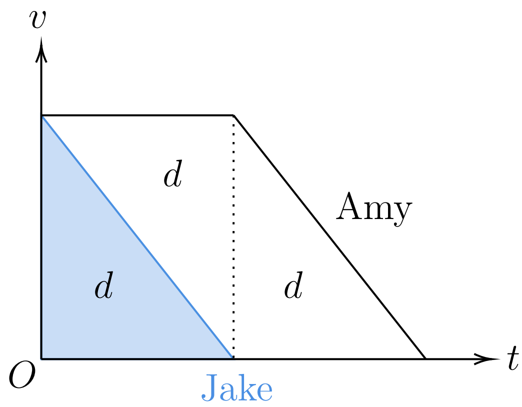
**Problem 1: Simp** (3 points)

Amy has a crush on Jake. Amy is hot on Jake's heels, and they are both cycling at the same constant velocity some distance apart. Jake engages the brakes and coasts to a stop at constant deceleration over a distance $d = 20$ m. At the exact moment Jake comes to a stop, Amy engages her brakes and coasts to a stop at the same constant deceleration. Coincidentally, Amy stops exactly where Jake is. How far behind Jake was Amy at the start?

*Leave your answer to 2 significant figures in units of* m.

---

*Solution:* We consider the velocity-time graph. Recall that the area under said graph corresponds to the distance travelled.




Clearly, Amy travels a distance $2d = 40$ m more than Jake. Thus, she was $\boxed{40 \text{ m}}$ behind Jake at the start.

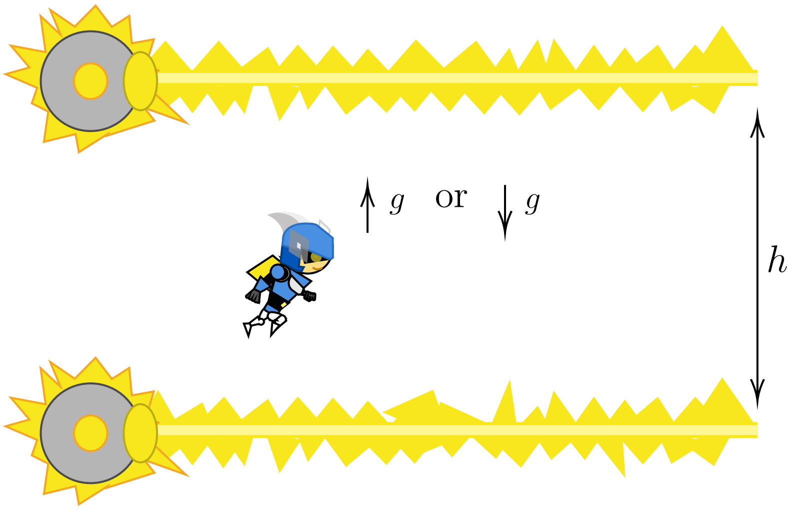
**Setter: Tan Jun Wei, junwei.tan@sgphysicsleague.org**

**Problem A: Jetpack Joyride** (3 points)

*Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.*

Gravity Guy is a character in Jetpack Joyride who can reverse the direction of gravity. As Gravity Guy approaches the middle of a pair of long horizontal zappers spaced a distance $h = 0.5$ m apart vertically, he presses his gravity-switching button at a regular time interval $t$. What is the maximum value of $t$ such that he can stay between the zappers indefinitely? Assume that the magnitude of gravitational acceleration remains constant at $g$, and that he is small enough to be considered a point mass.

*Leave your answer to 2 significant figures in units of* s.




---

*Solution:* Gravity Guy will trace a parabolic trajectory during each portion of motion where the direction of gravity does not change, with the peak of each arc at the top and bottom zappers. As he switches the direction of gravity at regular intervals, both the upper and bottom arcs must be identical, hence he must switch the direction of gravity when he is at the middle of the two zappers.

The time taken to travel from one zapper to the middle is $\frac{t}{2}$, hence:

$$\frac{h}{2} = \frac{1}{2} g \left(\frac{t}{2}\right)^2$$

Solving for $t$, we obtain:

$$t = 2\sqrt{\frac{h}{g}} \approx \boxed{0.45 \text{ s}}$$

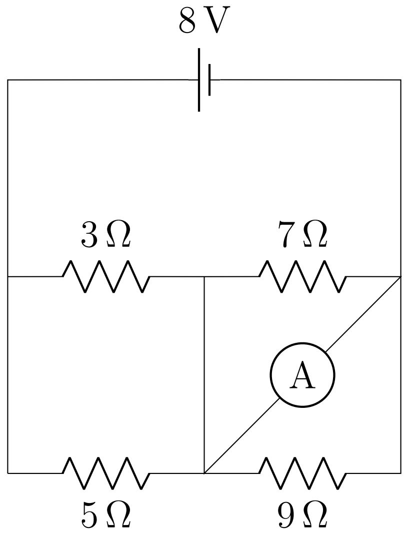
**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Problem B: Innocent Circuit** (3 points)

*Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.*

A circuit is set up with an ideal ammeter as shown below. What is the ammeter reading $I$?

*Leave your answer to 3 significant figures in units of* A.




---

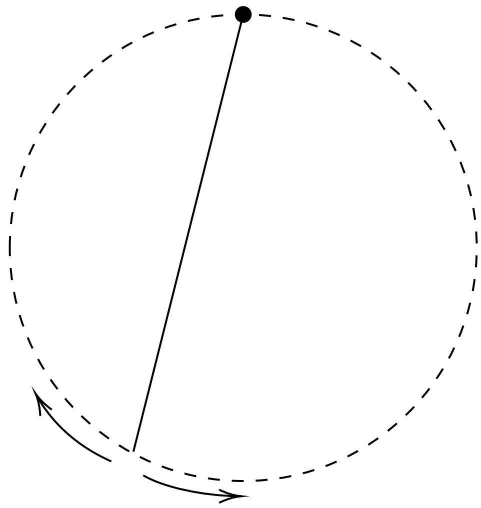
*Solution:* Let the effective resistance of the circuit be $R_{\text{eff}}$. Since an ideal ammeter has zero resistance, all the current will bypass the $7\,\Omega$ and $9\,\Omega$ resistors and flow through the ammeter instead. Hence, we only need to consider the $3\,\Omega$ and $5\,\Omega$ resistors to find $I$:

$$I = \frac{V}{R_{\text{eff}}} = \frac{8}{\left(\frac{1}{3} + \frac{1}{5}\right)^{-1}} \approx \boxed{4.27\text{ A}}$$

**Setter: Roger Zhang, roger.zhang@sgphysicsleague.org**

**Problem 2: Crossing a Circle** (3 points)

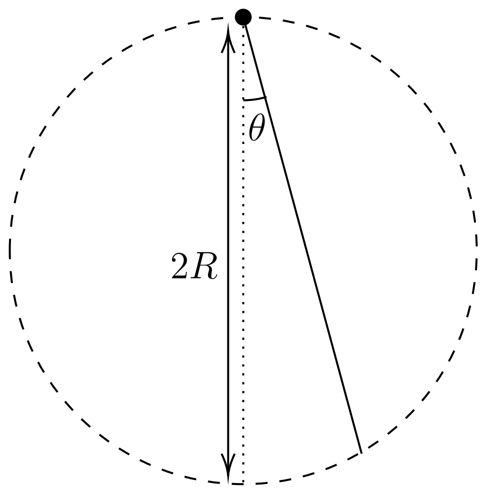
Consider a vertical ring of radius $R = 20$ cm. A slope is constructed between two points: the top of the ring, and another point chosen anywhere on the ring. A point mass is then placed at the top of the slope and released from rest. Find the minimum possible time taken $t$ for the mass to travel the length of the slope.



*Leave your answer to 2 significant figures in units of* s.

---

*Solution:* Let $\theta$ be the angle the slope makes with the vertical.




By considering the geometry of the circle, we have:

$$L = 2R\cos\theta$$

Along the slope's path, the acceleration is provided by the component of gravity in the direction of the path, which is $a = g\cos\theta$.

Hence, the time taken to travel the full length of the path is:

$$\begin{aligned} t &= \sqrt{\frac{2L}{a}} \\ &= \sqrt{\frac{4R\cos\theta}{g\cos\theta}} \\ &= \sqrt{\frac{4R}{g}} \end{aligned}$$

Notice that as all terms dependent on $\theta$ cancel out, our answer is independent of the point chosen! Hence, the time taken is always $t \approx \boxed{0.29\text{ s}}$.

**Setter: Huang Zehan, zehan.huang@sgphysicsleague.org**

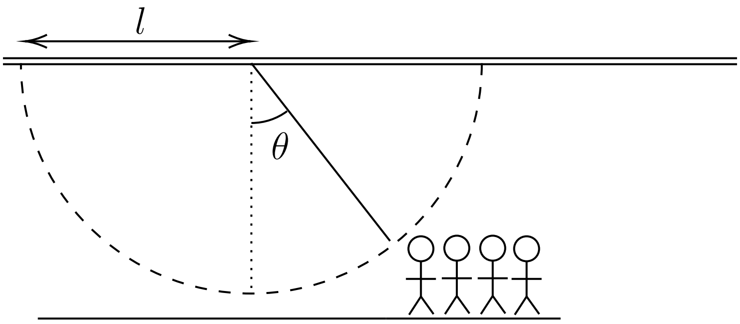
**Problem C: Me And The Boys** (3 points)

*Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.*

Paul and his friends found a massless rope of length $l = 11$ m with one end tied to a tree branch and decided to have some fun. First, Paul holds the lower end of the rope and swings from rest and back with an amplitude $\theta_0 = 20°$, completing the $1^{st}$ oscillation. When Paul reaches his maximum height, his friend grabs onto him from rest, and the two friends swing for another full oscillation, completing the $2^{nd}$ oscillation. This process repeats itself for the infinitely many friends Paul has.

Let $v_n$ be the maximum velocity of Paul and his friends during the $n^{th}$ oscillation. Determine $v_4$. Assume Paul and his friends are all point masses with mass $m = 67$ kg, and neglect resistive forces.

*Leave your answer to 2 significant figures in units of* m s$^{-1}$.




---

*Solution:* Let $v_1$ be the maximum velocity Paul reaches during the 1st oscillation. This occurs at the lowest point of the oscillation i.e. when $\theta = 0°$. Using conservation of energy, we have:

$$v_1 = \sqrt{2gl(1 - \cos\theta_0)}$$

Since Paul's friends all board the swing from rest and when the swing is at rest, we can apply the same equation for conservation of energy for each oscillation. The result for the maximum velocity is independent of mass, hence:

$$v_4 = \sqrt{2gl(1 - \cos\theta_0)} \approx \boxed{3.6 \text{ m s}^{-1}}$$

**Setter: Roger Zhang,** roger.zhang@sgphysicsleague.org

**Problem D: Icy Friendship** (3 points)

*Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.*

Adam stands on a frozen lake next to Bane, who is standing on wet ground. Adam's coefficient of static friction on ice is $\mu_{\text{ice}} = 0.05$ and Bane's coefficient of static friction on the wet ground is $\mu_{\text{wet}} = 0.15$. Adam has mass $m_a$, while Bane has mass $m_b = 30$ kg. When Adam pushes Bane, Adam is able to remain stationary on the ice, while causing Bane to slide on the wet ground.

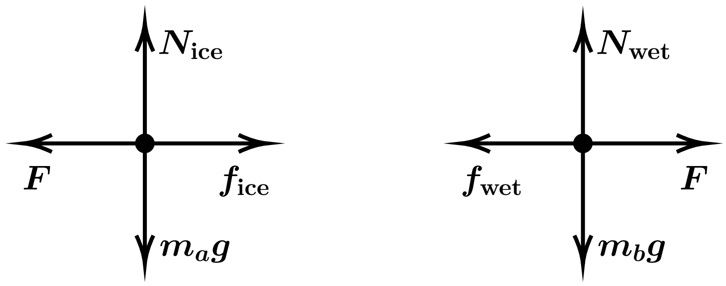
What is the minimum value of $m_a$ for this situation to occur?

*Hint*: The maximum static friction force between surfaces $f \leq \mu N$, where $N$ is the normal force between the surfaces.

*Leave your answer to 2 significant figures in units of* kg.

---

*Solution:* Suppose that Adam pushes on Bane with a force $F$. We can draw the free body diagram of Adam and Bane respectively, accounting for the normal forces $N$ and frictional forces $f$.




For Bane to slip, we have the condition:

$$F \geq \mu_{\text{wet}} N_{\text{wet}} = \mu_{\text{wet}} m_b g$$

For Adam to not slip, we have the condition:

$$F \leq \mu_{\text{ice}} N_{\text{ice}} = \mu_{\text{ice}} m_a g$$

Combining these 2 inequalities to eliminate $F$, we have:

$$m_a \geq \frac{\mu_{\text{wet}}}{\mu_{\text{ice}}} m_b = \boxed{90 \text{ kg}}$$

**Setter: Shanay Jindal,** shanay.jindal@sgphysicsleague.org

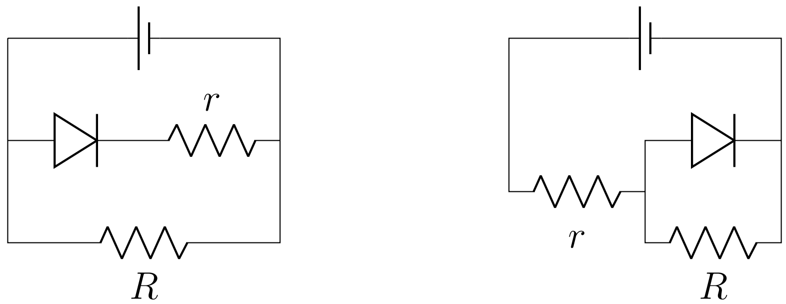
**Problem 3: Confused Diode** (3 points)

A battery is connected to an arrangement of a diode and two resistors of resistances $r$ and $R$, where $R > r$, with ideal wires. Initially, the current passing through the battery is $I$. When the battery's polarity is reversed, the current passing through the battery becomes $I' = 4I$. Find the ratio $R/r$.

*Leave your answer to 2 significant figures.*

---

*Solution:* Notice that both currents $I$ and $I'$ are non-zero and finite. Hence, there must always be a path for current to flow which bypasses the diode so that $I$ is non-zero, and there must be a resistor on the path so that $I$ is finite. Only two possible circuit arrangements satisfy this condition:




In the first arrangement, when the diode is forward-biased, the current $I_F$ in the circuit can be calculated by treating resistors $r$ and $R$ to be in parallel. When the diode is reverse-biased, the current $I_R$ in the circuit only passes through resistor $R$. Hence, we can write the following expressions for $I_F$ and $I_R$:

$$I_F = V\left(\frac{1}{r} + \frac{1}{R}\right)$$
$$I_R = \frac{V}{R}$$

In the second arrangement, the forward-biased current $I_F$ only passes through resistor $r$ as resistor $R$ is shorted. The reverse-biased current $I_R$ passes through resistors $r$ and $R$ in series. In this case, $I_F$ and $I_R$ are given by:

$$I_F = \frac{V}{r}$$
$$I_R = \frac{V}{r + R}$$

Surprisingly, the ratio $I_F/I_R$ is identical in both possible arrangements:

$$\frac{I_F}{I_R} = \frac{R + r}{r} = \frac{R}{r} + 1$$

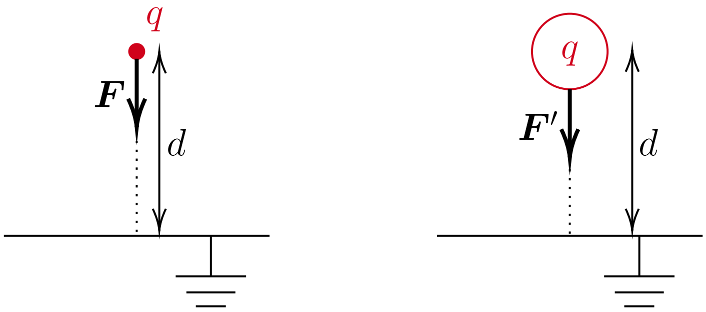
Since the ratio $I'/I = 4$ and $R > r$, the ratio $R/r$ is given by $R/r = \boxed{3.0}$.

**Setter: Christopher Ong,** chris.ong@sgphysicsleague.org

## Problem E: Ball vs. Point (2 points)

*Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.*

A point charge $+q$ is located a distance $d$ away from an infinite grounded conducting plate. The charge experiences a force $F$ towards the plate. Now, consider a conducting ball also with charge $+q$ whose centre is a distance $d$ away from an identical plate. The ball experiences a force $F'$ towards the plate. Which of the following is true?




(1) $F > F'$

(2) $F < F'$

(3) $F = F'$

---

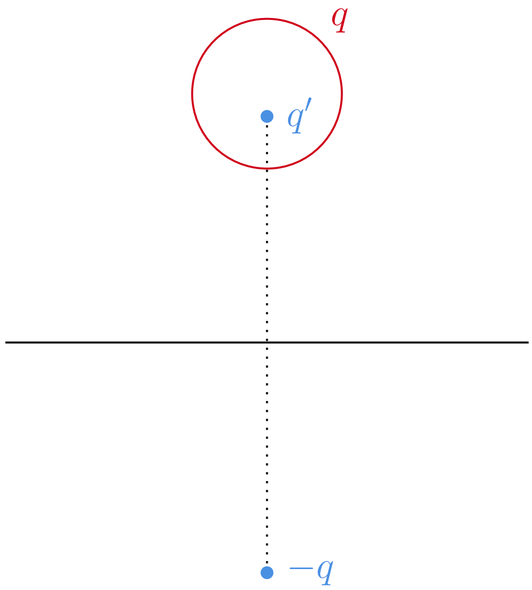
*Solution:* In both cases, the force is caused by negative charges being attracted onto the plate from the ground. For the case of the conducting ball, these charges on the plate will polarise the ball, with positive charges moving towards the plate and negative charges moving away. This causes the net electrostatic forces of attraction to increase. Thus, the answer is $\boxed{(2) F < F'}$.

---

*Alternative solution:* We may also use the method of images to solve this problem. When a charge is placed near a conducting surface, charges within the surface may rearrange themselves due to electrostatic forces of attraction and repulsion with the external charge. This rearrangement ensures that the surface of the grounded conducting plate remains at a uniform zero potential. The method of images places an imaginary "charge" somewhere in the conducting surface such that this boundary condition is satisfied. Though this is not the actual physical arrangement of charges in the conductor, the uniqueness theorem states that the electric field solution is unique if the boundary conditions are satisfied, thus we can use this image charge to solve for the electric field due to the rearrangement of charges in the conductor.

In the first case, a single point image charge $-q$ directly opposite the external charge is formed, resulting in an attractive force.

However, in the case of a spherical conductor, there are an infinite number of point image charges required. We may construct them sequentially: the first one is of charge $-q$, placed opposite to the first charge. Due to the fact that the external spherical ball is itself conducting, another image charge $q' = q\frac{R}{d}$ is formed within the ball to ensure that the surface of the ball is equipotential, increasing the attractive force between the ball and the plane. The image charge $q'$ then induces yet another negative image charge opposite the plane, and the process then repeats indefinitely. However, only the first two charges shown in the diagram are required to observe that $\boxed{(2) F < F'}$.




**Setter: Shanay Jindal,** shanay.jindal@sgphysicsleague.org

**Problem 4: Falling Pendulum** (3 points)

Roger is holding a toy that consists of a pendulum bob connected by a massless string of length $L = 0.40$ m to the ceiling of a sealed case. The bob oscillates with amplitude $\theta_0 = 0.251$ rad.

At time $t_0 = 0$ s, the bob is at angular position $\theta = \theta_0$. At this instant, Roger releases the case, allowing it to fall freely for $t_f = 0.15$ s. Determine the displacement $s$ of the bob from time $t_0$ to $t_f$. Neglect resistive forces.

*Leave your answer to 2 significant figures in units of* m.

---

*Solution:* Since the bob starts at its maximum angular displacement, its instantaneous velocity is initially zero. Both the bob and the case are released at zero velocity, and experience equal gravitational acceleration. Hence, there is no tension in the string during the fall, and the bob will accelerate vertically downwards without oscillation.

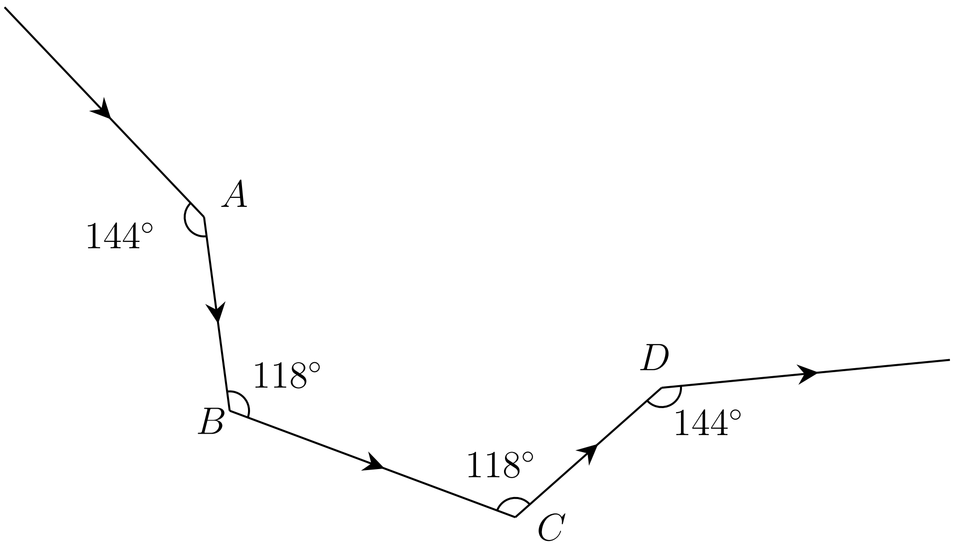
The displacement of the bob can thus be found with a simple kinematics equation:

$$s = \frac{1}{2}g(t_f - t_0)^2 \approx \boxed{0.11 \text{ m}}$$

**Setter: Huang Ziwen, ziwen.huang@sgphysicsleague.org**

**Problem 5: Prism Ray Tracing** (4 points)

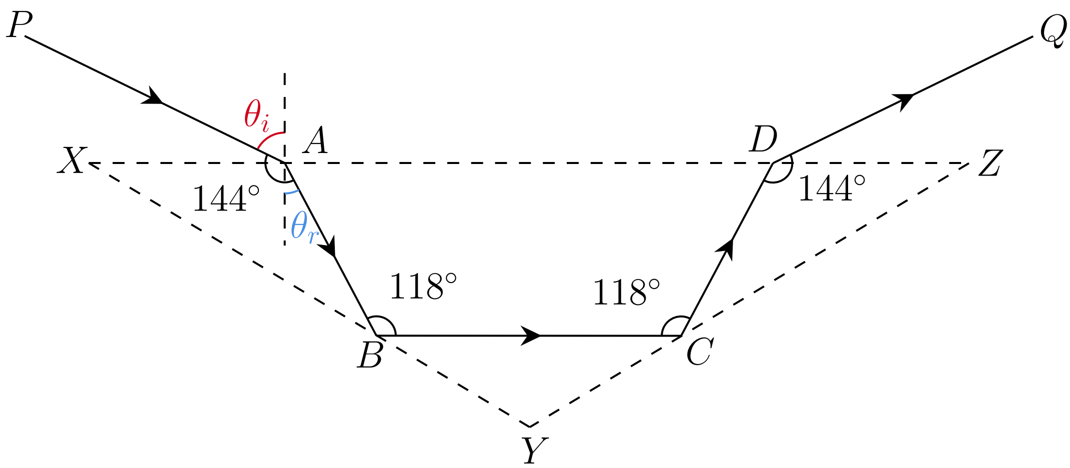
A laser beam travels from air through a transparent triangular prism with unknown orientation, forming the following path. The beam is deflected at exactly 4 points $A$, $B$, $C$ and $D$, with path lengths $AB = CD$. The entire system can be taken as two-dimensional, with the beam and triangular faces of the prism all within the plane of the page. Find $n$, the refractive index of the prism.




*Leave your answer to 3 significant figures.*

---

*Solution:* Since the beam enters the prism exactly once, the first deflection at point $A$ must be where the beam enters the prism, and point $D$ must be where the beam exits. The light is hence refracted at these two points. Points $B$ and $C$ must then be where the beam undergoes total internal reflection. Furthermore, since $AB = CD$, by symmetry, the cross-section of the prism is also an isosceles triangle.




We note that $AD$ is parallel to $BC$. Hence, $\angle XAB = \angle ABC = 118.0°$. Thus, the angle of refraction is $\theta_r = 118° - 90° = 28°$.

Next, note that $\angle PAX = \angle PAB - \angle XAB = 144° - 118° = 26°$. Hence, the angle of incidence is $\theta_i = 90^\circ - 26^\circ = 64^\circ$.

From Snell's Law, we know that:

$$\begin{aligned} 1.00 \sin\theta_i &= n \sin\theta_r \\ n &= \frac{\sin\theta_i}{\sin\theta_r} \\ &\approx \boxed{1.91} \end{aligned}$$

**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

**Problem 6: Tilting Glass** (3 points)

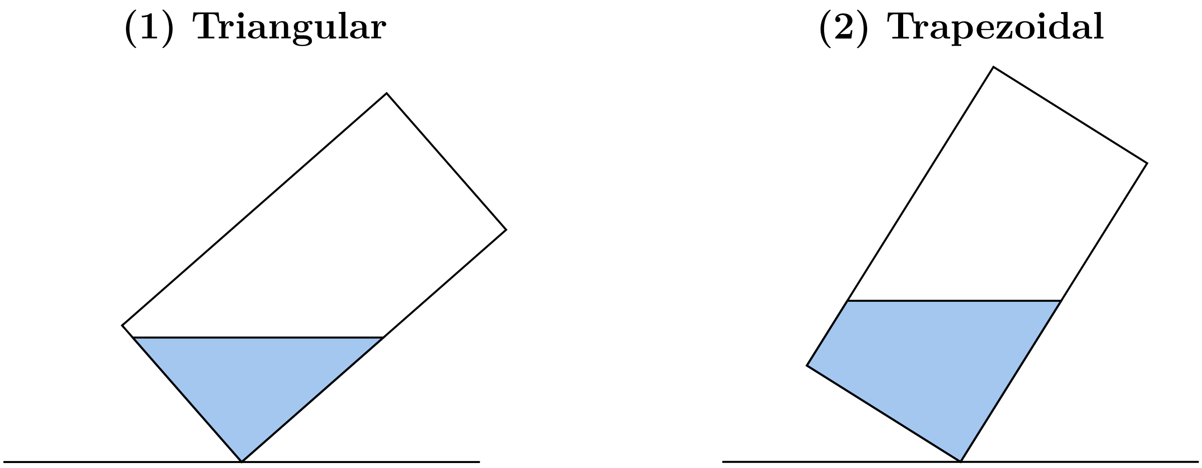
A light cuboidal glass with a square base of height $h = 50$ cm and side length $l = 10$ cm is placed on a flat surface. The top surface of the glass is open, allowing the glass to be filled with some water, then tilted slowly to an angle $\theta$ without any water spilling. Here, $\theta$ is defined as the angle between the base of the glass and the horizontal. Given that the volume of water in the glass can be varied, what is the maximum tilt angle $\theta_{\text{max}}$ of the glass before it topples?

You may assume the glass's mass is negligible compared to the water's mass.

*Leave your answer to 2 significant figures in units of degrees.*

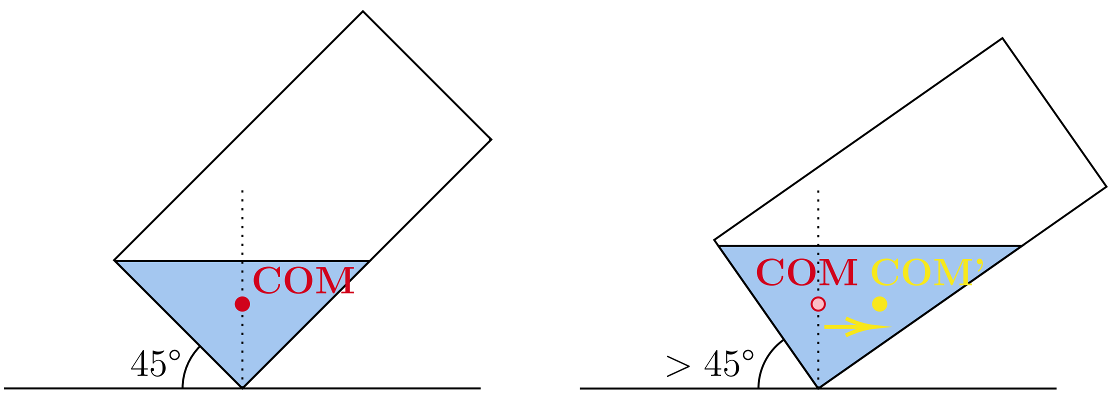
---

*Solution:* Firstly, we note that the shape of the water in the glass can vary depending on its tilt angle $\theta$ and volume of water:

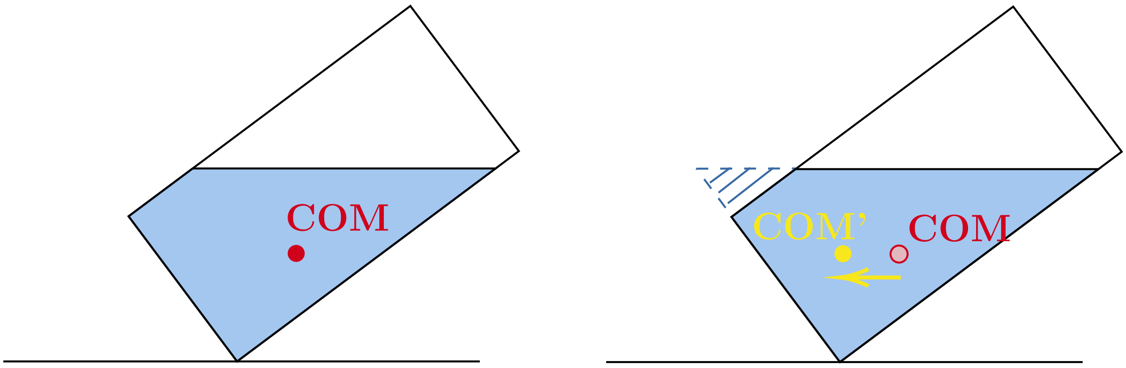



We require the centre of mass to be as far left as possible so that gravity exerts an anticlockwise restoring torque. We first consider the triangular water shape.

For the tilt angle to be maximised, the centre of mass of the water should be directly above the corner of the glass. This occurs when the triangle is isosceles, i.e. $\theta = 45°$. If we were to increase $\theta$ any further, we can observe that the centre of mass will shift to the right, tipping the glass over:




In the latter trapezoidal case, we see that if we were to keep the tilt angle $\theta$ constant and extend its area such that a triangle is formed, the trapezium's centre of mass is further to the right than the triangle's:




Hence, the trapezoidal shape need not be considered as it will have a lower critical angle than the triangular case.

The maximum tilt angle is thus independent of the dimensions of the glass and will always be $\theta_{\max} = \boxed{45^\circ}$.

**Setter: Huang Zehan, zehan.huang@sgphysicsleague.org**

**Problem 7: Hot Bulb** (3 points)

Physicist S has found an incandescent light bulb which is approximately spherical with radius $R = 2.5$ cm. For reasons best known only to herself, she paints the bulb with black paint such that all light is absorbed by its now opaque glass casing, depriving it of its only purpose in life. She then connects it to a power source of voltage $V = 120$ V. A long time later, her friend passes by and unfortunately finds that the surface of the bulb is now at a temperature $T = 1300$ K. Find the current $I$ drawn by the bulb.

*Leave your answer to 3 significant figures in units of* A.

---

*Solution:* The absorptivity of a material is equal to its emissivity, so the black bulb emits radiation like an ideal black body. The power the bulb emits is then given by the Stefan-Boltzmann Law $P = (4\pi R^2)\sigma T^4$. Since the bulb has been left alone for a long time, it has come to equilibrium, so the power radiated is equal to the electrical power drawn:

$$4\pi R^2 \sigma T^4 = VI$$

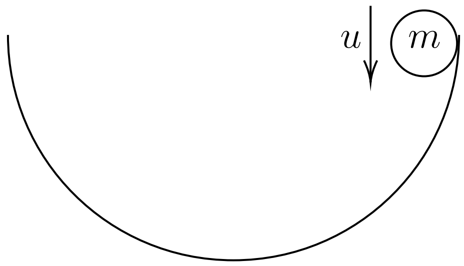
Hence, we obtain:

$$I = \frac{4\pi\sigma R^2 T^4}{V} \approx \boxed{10.6 \text{ A}}$$

**Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org**

**Problem 8: An Average Semicircle** (3 points)

Consider a smooth semicircular track in the vertical plane. A mass $m = 1.0$ kg enters one end of the track with an initial downward velocity $u = 3.0$ m s$^{-1}$, and takes time $t = 5.0$ s to reach the other end. Find the magnitude of the time-averaged normal force, $\left|\langle\vec{N}\rangle\right|$, exerted on the mass during its motion along the track.




*Leave your answer to 2 significant figures in units of* N.

---

*Solution:* By conservation of energy, the ball emerges from the other end of the track with upward velocity $u$. Since the initial momentum of the ball is $mu$ downwards and the final momentum is $mu$ upwards, the change in momentum of the ball throughout its motion is given by $\Delta p = 2mu$ upwards.

We can equate the average rate of change of momentum of the mass to the average resultant force, as given by Newton's Second Law:

$$\frac{\Delta p}{t} = \left|\langle\vec{N}\rangle\right| - mg$$

Hence, we can solve for $\left|\langle\vec{N}\rangle\right|$:

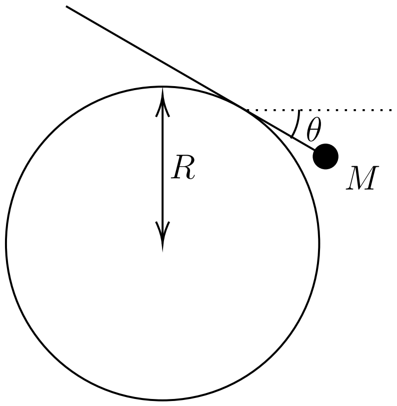
$$\left|\langle\vec{N}\rangle\right| = mg + \frac{2mu}{t} \approx \boxed{11 \text{ N}}$$

**Setter: Christopher Ong, chris.ong@sgphysicsleague.org**

## Problem 9: Slant Bar (4 points)

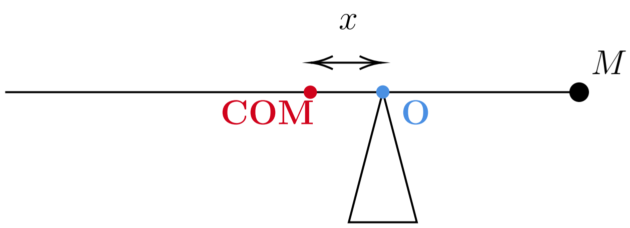
Roger is creating an ornament that consists of a uniform bar of mass $m = 0.20$ kg and length $L = 2.0$ m balancing horizontally on top of a horizontal cylinder of radius $R = 1.5$ m, with their centres vertically aligned. He then adds a decorative element of mass $M$ to one end of the bar, causing the bar to rotate without slipping on the cylinder surface. At equilibrium, the bar is inclined at an angle $\theta = 30°$ to the horizontal. The cylinder remains stationary and does not rotate throughout. Determine the mass $M$.

*Leave your answer to 2 significant figures in units of* kg.




---

*Solution:*

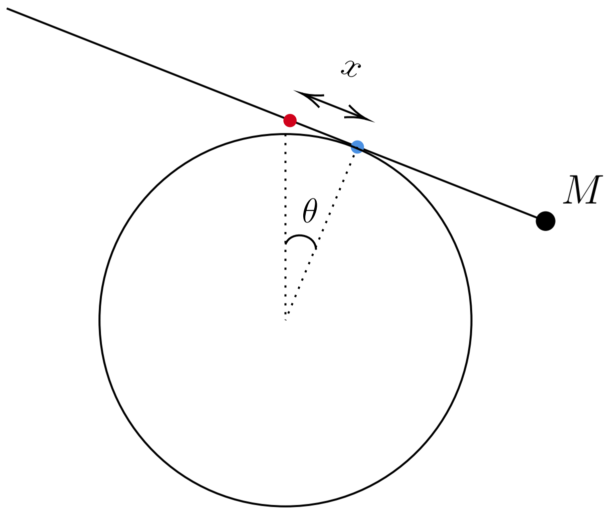



By imposing the condition that the bar does not slip when rotating, we can find $x$:

$$x = R\theta$$

where $\theta$ is in radians. We equate the clockwise moment due to mass $M$ with the anti-clockwise moment due to the mass of the rod. Balancing moments about the contact point, we have:

$$Mg\left(\frac{L}{2} - x\right)\cos\theta = mgx\cos\theta$$




We solve for $M$ and substitute numerical values to obtain:

$$M = \frac{mx}{\frac{L}{2} - x}$$
$$= \frac{mR\theta}{\frac{L}{2} - R\theta}$$
$$\approx \boxed{0.73 \text{ kg}}$$

**Setter: Huang Ziwen, ziwen.huang@sgphysicsleague.org**

**Problem 10: Single to Double** (4 points)

A monochromatic laser beam is directed towards a single slit of width $a = 0.30$ mm. A screen is placed a distance $D \gg a$ away from the slit, and the intensity of the central maximum on the screen is $I_0$. Guangyuan covers part of the slit with an opaque film of width $b = 0.12$ mm. Now, the intensity of the central maximum is $\alpha I_0$. Find $\alpha$.

*Leave your answer to 2 significant figures.*


---

*Solution:* Let the amplitude at the central maximum before placing the opaque film be $A_0$, where $I_0 = k{A_0}^2$.

By Huygens' Principle, every point on the wavefront passing through the original slit is itself a source of circular wavelets, and these wavelets interfere to form a new wavefront. If we were to divide the light passing through the slit into infinitesimally small sources, the amplitude of light due to each source sum to form the total amplitude at the screen. Given that all the tiny sources of light undergo constructive interference at the central maximum, the total amplitude at the screen is proportional to the number of sources emitting light, which is proportional to the width of the slit.

After placing the opaque film, we cover some of these sources. The amplitude at the central maximum due to the remaining part of the slit is now:

$$\frac{a-b}{a}A_0 = 0.60A_0$$

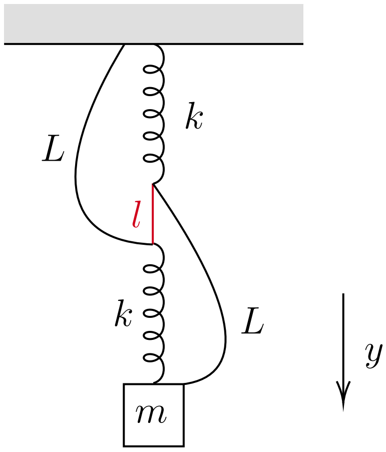
The intensity at the central maximum is thus $k(0.6A_0)^2 = 0.36k{A_0}^2 = 0.36I_0$.

$$\alpha = \boxed{0.36}$$

**Setter: Roger Zhang,** roger.zhang@sgphysicsleague.org

### Problem 11: Strings and Springs (4 points)

Two springs of stiffness $k = 8$ N m$^{-1}$ are connected by a string of length $l = 0.5$ m. A mass $m = 1$ kg is hung from the ceiling using the connected springs. Additionally, two initially slack strings of length $L = 2$ m are attached as shown below. The system is initially in equilibrium.



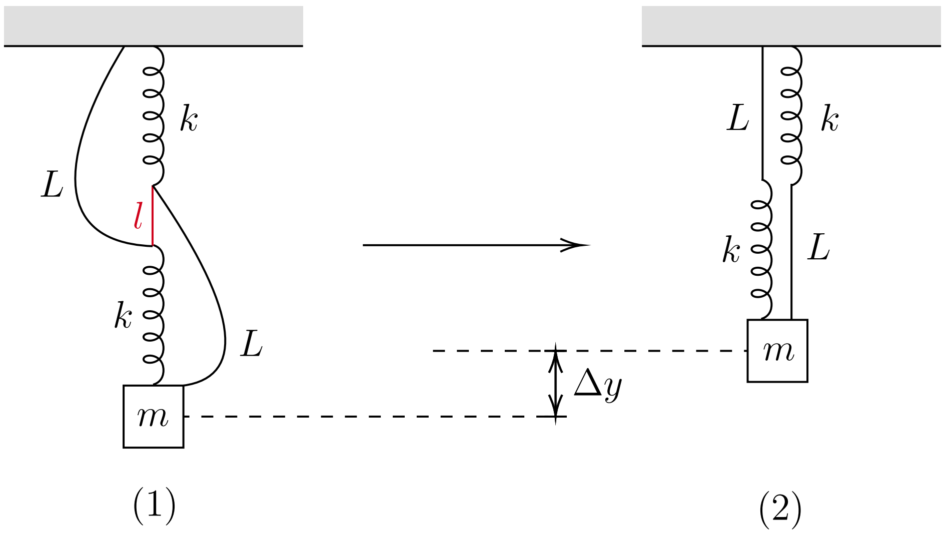

Assuming that the springs have zero rest length, find the **signed** change in height $\Delta y$ of the mass after $l$ is cut and the system reaches equilibrium.

*Leave your answer to 2 significant figures in units of* m.

*Leave a positive answer if you think the mass falls, and a negative answer if you think the mass rises.*

---

*Solution:* Once $l$ is cut, the two strings of length $L$ become taut, causing the spring setup to change from series to parallel. Accordingly, the effective spring constant experienced by the mass changes from $\frac{k}{2}$ to $2k$.



In each of the two cases, both springs share the same extension. Let the extension of each spring before and after $l$ is cut be $x_1$ and $x_2$ respectively. For both states to be in static equilibrium, the extensions are given by:

$$\left(\frac{k}{2}\right)(2x_1) = mg \Rightarrow x_1 = \frac{mg}{k} \tag{1}$$

$$(2k)(x_2) = mg \Rightarrow x_2 = \frac{mg}{2k} \tag{2}$$

Now, let $y_1$ and $y_2$ be the heights of the mass with respect to the ceiling before and after $l$ is cut respectively. We can express them as:

$$y_1 = 2x_1 + l$$
$$y_2 = x_2 + L$$

Thus, the change in height $\Delta y$ is given by:

$$\begin{aligned}\Delta y &= y_2 - y_1 \\ &= (L - l) - \frac{3mg}{2k} \\ &\approx \boxed{-0.34 \text{ m}}\end{aligned}$$

In other words, the mas *rose* by 0.34 m after the string is cut![1]

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**

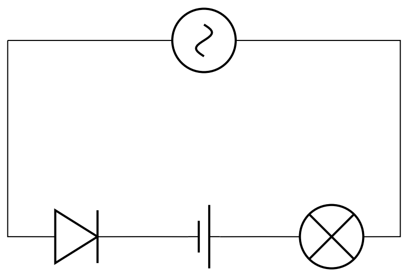
[1]Credits to Cohen and Horowitz for this ingenious setup: Cohen, Joel E. and Paul Horowitz. "Paradoxical behaviour of mechanical and electrical networks." Nature 352 (1991): 699-701.

**Problem 12: AC/DC** (3 points)

An AC voltage source is connected to an ideal diode, a DC voltage source, and a light bulb in series. The AC source has peak voltage $V_0 = 8$ V, while the DC voltage source has constant voltage $U = 4$ V. The bulb lights up when the potential difference across it is non-zero. Over a long time, determine the fraction of time during which the bulb is lit.

*Leave your answer to 2 significant figures.*

*Your answer should be between 0 and 1.*




---

*Solution:* The total source voltage due to the two sources is given by $V = V_0 \sin \omega t + U$. When $V > 0$, the diode allows current to flow through the bulb. Considering a single period, we want to determine the times $t = t_0$ such that $V = 0$:

$$V_0 \sin \omega t_0 + U = 0$$
$$\sin \omega t_0 = -\frac{U}{V_0}$$
$$= -\frac{1}{2}$$

There are two solutions to this equation within the range 0 to $2\pi$:

$$\omega t_0 = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The bulb will not be lit between these two times as $\sin \omega t < -\frac{1}{2}$. The fraction of time during which it is lit over a period is then given by:

$$\frac{\frac{7\pi}{6} + \frac{\pi}{6}}{2\pi} = \frac{2}{3} \approx \boxed{0.67}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Problem 13: FFFFFF**

Fëanor, as High King of the Noldorin elves, wishes to create a system of units for his people to use. Since Fëanor also has a very large ego, he chooses to use 6 units that all start with the letter F:

| Unit (abbreviation) | In SI units |
|---|---|
| fathom (f) | $1\ \text{f} = 1.83\ \text{m}$ |
| frequency of middle F ($\mathsf{F_4}$) | $1\ \mathsf{F_4} = 349\ \text{Hz}$ |
| Franklin (Fr) | $1\ \text{Fr} = 3.34 \times 10^{-10}\ \text{C}$ |
| Farad (F) | $1\ \text{F} = 1\ \text{F}$ |
| Faraday's constant ($F$) | $1\ F = 9.65 \times 10^{4}\ \text{C mol}^{-1}$ |
| Fahrenheit (°F) | $1\ °\text{F} = 0.556\ \text{K}$ |

Fëanor calls this his FFFFFF system. Fëanor also does not like prefixes (such as kilo- or milli-), and has banned their use.

(a) Fëanor wants to know the mass of a U-235 nucleus, which has mass $m = 235$ u in SI units. Find the numerical value of $m$ in the FFFFFF system.

*Leave your answer to 3 significant figures in the appropriate units.* (3 points)

(b) Fëanor decides to create "Fëanor's constant", given by $\mathcal{F} = 2\varepsilon_0 hc/e^2$, where $e$ is the elementary charge. Find the numerical value of $\mathcal{F}$ in the FFFFFF system.

*Leave your answer to 3 significant figures in the appropriate units.* (2 points)

You may refer to the Data Sheet for all of the constants cited above.

---

*Solution:*

(a) Within the unusual FFFFFF system, the dimensions of mass only appear in the unit Farad:

$$\text{F} = \text{kg}^{-1}\ \text{m}^{-2}\ \text{s}^{2}\ \text{C}^{2}$$

Therefore, the numerical value of $m$ is:

$$\begin{aligned}
m &= 235\left(1.66 \times 10^{-27}\ \text{kg}\right) \\
&= 235 \cdot 1.66 \times 10^{-27}\ \text{F}^{-1}\ \text{m}^{-2}\ \text{s}^{2}\ \text{C}^{2} \\
&= 235\left(1.66 \times 10^{-27}\ \text{F}^{-1}\right)\left(\frac{1}{1\ \text{m}^2}\right)\left(\frac{1}{1\ \text{Hz}^2}\right)\left(1\ \text{C}^2\right) \\
&= 235\left(1.66 \times 10^{-27}\ \text{F}^{-1}\right)\left(\frac{1}{1\ \text{m}^2}\right)\left(\frac{1.83\ \text{m}}{1\ \text{f}}\right)^2\left(\frac{1}{1\ \text{Hz}^2}\right)\left(\frac{349\ \text{Hz}}{1\ \mathsf{F_4}}\right)^2 \\
&\quad \left(1\ \text{C}^2\right)\left(\frac{1\ \text{Fr}}{3.34 \times 10^{-10}\ \text{C}}\right)^2
\end{aligned}$$

$$\approx \boxed{1.43\ \mathrm{f}^{-2}\ \mathsf{F_4}^{-2}\ \mathrm{Fr}^2\ \mathrm{F}^{-1}}$$

(b) We first calculate the value of $\mathcal{F}$ in SI units, and then convert to FFFFFF units. Note that converting to FFFFFF units first would lead to a lot more calculations needed for unit conversion.

$$\mathcal{F} = \frac{2\epsilon_0 hc}{e^2} \approx 138$$

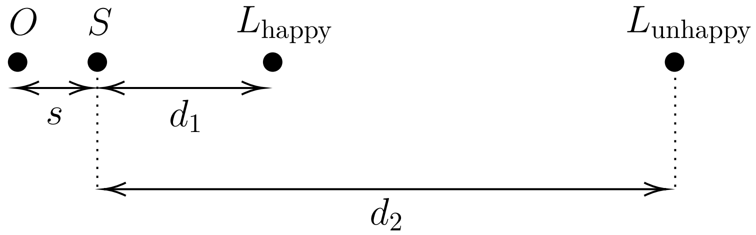
It turns out, Fëanor's new constant is dimensionless, and therefore has the same numerical value in any unit system. Thus, in the FFFFFF system, $\mathcal{F} \approx \boxed{138}$.

*Note:* The constant $\alpha = \frac{1}{\mathcal{F}} \approx \frac{1}{137}$ is known as the fine structure constant, and is commonly used in quantum electrodynamics. Since $\alpha$ is dimensionless, it is currently an open question whether the value of $\alpha$ can be derived from only mathematical constants. Unfortunately for Fëanor, $\alpha$ is much more popular in the literature than $\mathcal{F}$, and he has not revolutionised physics.

**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

### Problem 14: Controversial Concerto (4 points)

A *concerto* is performed by a soloist $S$ in front of an orchestra $O$. Here, we model the soloist and the orchestra as point sources positioned a distance $s = 10$ m apart. A happy listener, seated at distance $d_1 = 20$ m from the soloist, hears the soloist to be as loud as the orchestra. On the other hand, an unhappy listener, seated at distance $d_2 = 80$ m from the soloist, hears the soloist to be softer than the orchestra. Find the ratio of sound intensities of the soloist to the orchestra $I_s/I_o$, as heard by the unhappy listener. Neglect any acoustic effects from the walls of the concert hall.




*Leave your answer to 2 significant figures.*

---

*Solution:* Let the power of the sound emitted by the soloist and the orchestra be $P_s$ and $P_o$ respectively.

We first consider the happy listener. Since he hears the soloist and the orchestra to be equally loud, the sound intensities of the soloist and the orchestra must be equal at his position. We thus obtain the following relation between $P_s$ and $P_o$:

$$\frac{P_s}{d_1^2} = \frac{P_o}{(d_1 + s)^2} \implies P_o = \left(1 + \frac{s}{d_1}\right)^2 P_s$$

Now we consider the unhappy listener.[2] Let the sound intensities of the soloist and the orchestra at his position be $I_s$ and $I_o$ respectively. $I_s$ and $I_o$ are given by:

$$I_s = \frac{P_s}{d_2^2}$$
$$I_o = \frac{P_o}{(d_2 + s)^2}$$

The ratio $I_s/I_o$ is therefore given by:

$$\frac{I_s}{I_o} = \left(1 + \frac{s}{d_2}\right)^2 \frac{P_s}{P_o} = \left(\frac{1 + \frac{s}{d_2}}{1 + \frac{s}{d_1}}\right)^2 = \boxed{0.56}$$

**Setter: Christopher Ong,** chris.ong@sgphysicsleague.org

[2] I was once the unhappy listener in this problem where the orchestra seemed to overpower the soloist. As the physics here reveals, this is only an issue for those who booked cheap seats.

**Problem 15: Nuclear Fusion?**

In a fusion power experiment, a helium-4 nucleus undergoes fusion with a beryllium-8 nucleus. Their initial kinetic energies may be taken as negligible and their centre of mass stationary. Their fusion produces a carbon-12 nucleus and may also produce a gamma particle (a high-energy photon).

(a) Find the minimum wavelength $\lambda_{\text{min}}$ of the gamma particle, if it is produced.

*Leave your answer to 3 significant figures in units of* fm. (2 points)

(b) Find the maximum wavelength $\lambda_{\text{max}}$ of the gamma particle, if it is produced.

*Leave your answer to 3 significant figures in units of* fm. (2 points)

*Answer* 0 *if the wavelength can be arbitrarily small.*

*Answer* $-1$ *if the wavelength can be arbitrarily large or no gamma particle is produced.*

*Answer* $-2$ *to both parts if the fusion reaction is impossible or not enough information is given in the problem.*

Data:
Rest mass of a helium-4 nucleus and two electrons: $m_{\text{He}} = 4.002603$ u
Rest mass of a beryllium-8 nucleus and four electrons: $m_{\text{Be}} = 8.005305$ u

---

*Solution:* By the definition of the unified atomic mass unit, the mass of a carbon-12 nucleus and six electrons is precisely $m_{\text{C}} = 12$ u. Hence, the energy released through the fusion reaction is given by $E = (m_{\text{He}} + m_{\text{Be}} - m_{\text{C}})c^2 = (0.007908 \text{ u})c^2$.

This energy can only be released in the form of kinetic energy of the products. However, if the carbon-12 nucleus is the only product, it cannot have any kinetic energy as it must be stationary due to conservation of momentum. Thus, a gamma particle must be released to satisfy both conservation of energy and momentum.

We have previously calculated the mass defect of the reaction to be approximately three orders smaller than the mass of a carbon-12 nucleus. Hence, we can safely assume that the nucleus moves non-relativistically, with kinetic energy $K$ given by $K = \frac{p^2}{2m_C}$. Therefore, if we let the momentum of the gamma particle be $p$, we have $\frac{p^2}{2m_C} + pc = E$. Solving, we get $p \approx 3.93706 \times 10^{-21}$ N s as the only positive solution.

Since $p$ is uniquely determined by the conditions of the problem, there is only one value of $\lambda$ that the gamma particle can have.

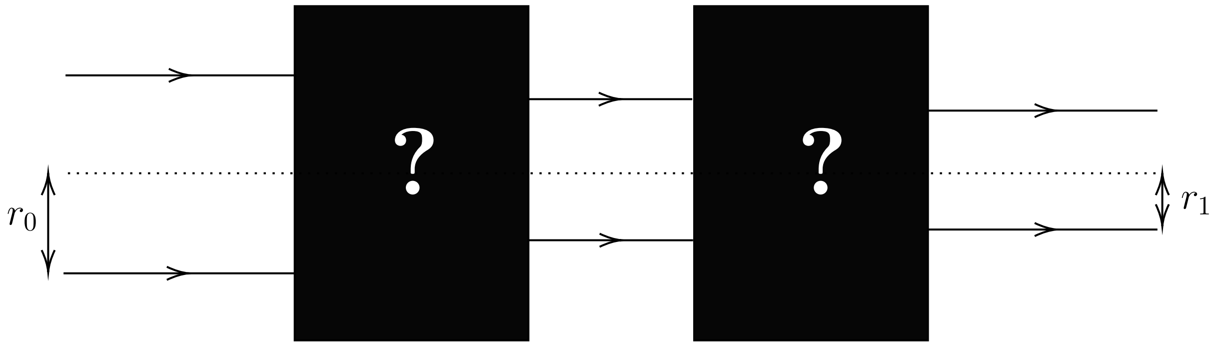
Hence, we get $\lambda_{\text{min}} = \lambda_{\text{max}} = \frac{h}{p} \approx \boxed{168 \text{ fm}}$, so our answers to both (a) and (b) are the same.

**Setter: Shen Xing Yang,** xingyang.shen@sgphysicsleague.org

**Problem 16: Always Parallel** (4 points)

Bob has four thin convex lenses of focal lengths 20 cm, 10 cm, 5 cm and 2 cm. He places all the lenses in two boxes (with at least one lens in each box), ensuring that their centres are aligned along a common principal axis. He then directs a light beam of radius $r_0 = 10$ cm through the centre of the first lens, parallel to the optical axis. To his surprise, he observes that the light rays are always parallel outside the boxes.

Given that he can arrange the lenses in any order fulfilling the constraints above, what is the smallest radius of the emergent beam $r_1$ he can obtain?

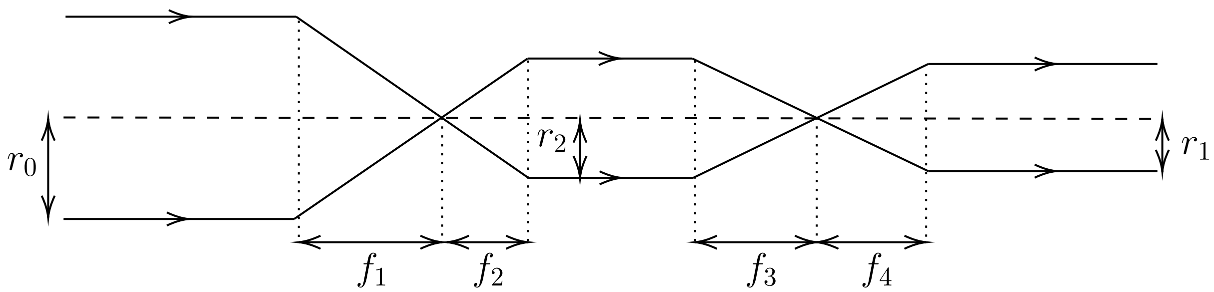



*Leave your answer to 2 significant figures in units of* cm.

---

*Solution:* Since parallel rays passing through a single lens will be refracted and the boxes are not empty, the only possible combination that allows the light rays to remain parallel after emerging from each box is two lenses in each box.

Let $f_1, f_2, f_3, f_4$ be the focal lengths of the four lenses in the order light passes through them. As the rays emerging from the first box are parallel, the rays within the first box must pass through the foci of both the first and the second lens. Hence, the first and second lens must be separated by a distance $f_1 + f_2$. Similarly, the third and fourth lens are separated by a distance $f_3 + f_4$.




Let $r_2$ be the radius of the beam between the two boxes. Using similar triangles to find the relationship between $r_0$ and $r_1$, we have:

$$\frac{r_0}{f_1} = \frac{r_2}{f_2}$$
$$\frac{r_2}{f_3} = \frac{r_1}{f_4}$$

Rearranging the equations, we have:

$$\frac{r_1}{r_0} = \frac{f_2 f_4}{f_1 f_3}$$

To minimise $r_1$, we let $f_2$ and $f_4$ be 2 cm and 5 cm, and let $f_1$ and $f_3$ be 10 cm and 20 cm to obtain:

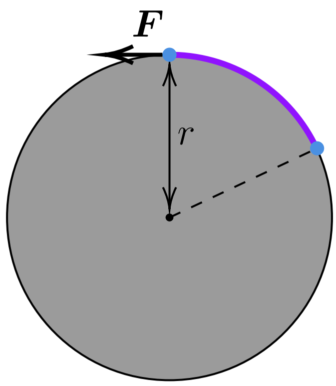
$$\begin{aligned} r_1 &= \frac{f_2 f_4}{f_1 f_3} r_0 \\ &= \boxed{0.50 \text{ cm}} \end{aligned}$$

**Setter: Li Yichen, yichen.li@sgphysicsleague.org**

**Problem 17: Pulling Strings** (4 points)

During a pre-delivery inspection of an airplane, Dave the overworked technician accidentally leaves a thin inextensible string on one of the plane's engines. When he finally notices the string sliding down the engine, Dave swiftly reaches for it and pulls on the top end of the string with a horizontal force $F$ to hold it in place.

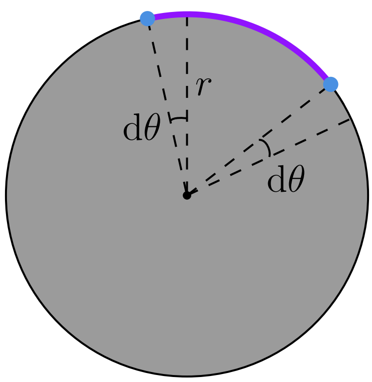
The vertical cross-section of the engine is a circle of radius $r = 1.06$ m. The top end of the string rests precisely at the top of the circle, and the string lies in the same plane as the circle. The string has length $\frac{\pi}{3}r$ and mass $m = 1.32$ kg. Find the magnitude of $F$ so that the string remains stationary. Neglect friction.




*Leave your answer to 2 significant figures in units of* N.

---

*Solution:* We can adopt the virtual work method to solve this problem. Suppose the force $F$ causes the top end of the string to move by an angular displacement $\mathrm{d}\theta$ counterclockwise i.e. a linear displacement $r\,\mathrm{d}\theta$ in the same direction as $F$. The "virtual" work done by $F$ is therefore $Fr\,\mathrm{d}\theta$.




Due to the action of $F$, the centre of mass of the string is raised by a small height, and the string gains some gravitational potential energy. Comparing the initial and final positions of the string, we can see that the increase in gravitational potential energy of the string is equal to the work done by an external force in raising a small segment of the string with a length of $r\,\mathrm{d}\theta$ from the bottom end to the top end, by a height $h = \left(1 - \cos\frac{\pi}{3}\right)r = \frac{r}{2}$. The mass of this small segment is $m\frac{\mathrm{d}\theta}{\pi/3}$. Hence, the increase in gravitational potential energy of the string is given by $\frac{r}{2}gm\frac{\mathrm{d}\theta}{\pi/3}$.

Equating the work done to the increase in gravitational potential energy, we have:

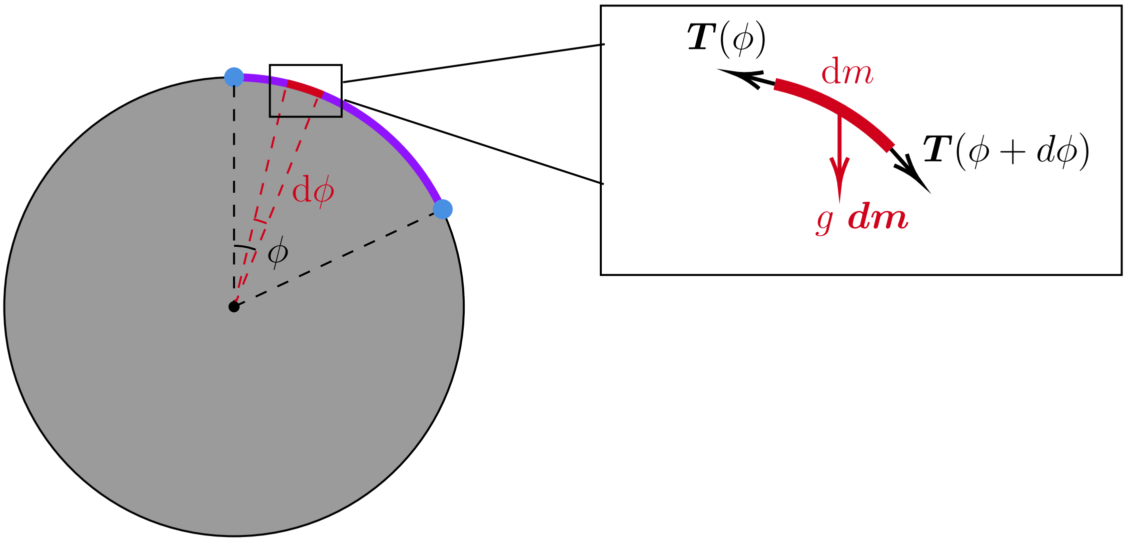
$$Fr\,\mathrm{d}\theta = \frac{r}{2}gm\frac{\mathrm{d}\theta}{\pi/3}$$

By simplifying the expression above, we obtain:

$$F = \frac{3}{2\pi}mg \approx \boxed{6.2\text{ N}}$$

---

*Alternative solution 1:*




We can also consider a small segment of the string that spans an angle $\mathrm{d}\phi$, where $\phi$ is the angular position of the top end of this small segment with respect to the top end of the string. Since this small segment is, well, infinitely small, the two tension forces acting on the ends of this small segment can be considered parallel, along the direction of the string segment. Also, because this small segment is in equilibrium, the resultant force along the direction of the string must be zero. Denoting the tension force acting on the top end of this small segment as $T(\phi)$ and that acting on the bottom end as $T(\phi + \mathrm{d}\phi)$, it follows that:

$$\begin{aligned} T(\phi) &= T(\phi + \mathrm{d}\phi) + \mathrm{d}mg\sin\phi \\ T(\phi) - T(\phi + \mathrm{d}\phi) &= \mathrm{d}mg\sin\phi \end{aligned}$$

where $\mathrm{d}m$ is the mass of this small segment. We can rewrite $T(\phi) - T(\phi + \mathrm{d}\phi)$ as $-\mathrm{d}T$ and $\mathrm{d}m$ as $m\frac{\mathrm{d}\phi}{\pi/3}$:

$$-\mathrm{d}T = \frac{3mg\sin\phi}{\pi}\mathrm{d}\phi$$

Integrating from $\phi = 0$ to $\phi = \frac{\pi}{3}$, we have:

$$T(0) - T\left(\frac{\pi}{3}\right) = \int_0^{\pi/3} \frac{3mg\sin\phi}{\pi}\,\mathrm{d}\phi = \frac{3}{2\pi}mg$$

We further note that the tension force at the bottom end of the string, $T\left(\frac{\pi}{3}\right)$, is 0 N. Hence, the applied force $F$, which is equal in magnitude to the tension force at the top of the string, $T(0)$, is given by:

$$F = \frac{3}{2\pi}mg \approx \boxed{6.2\text{ N}}$$

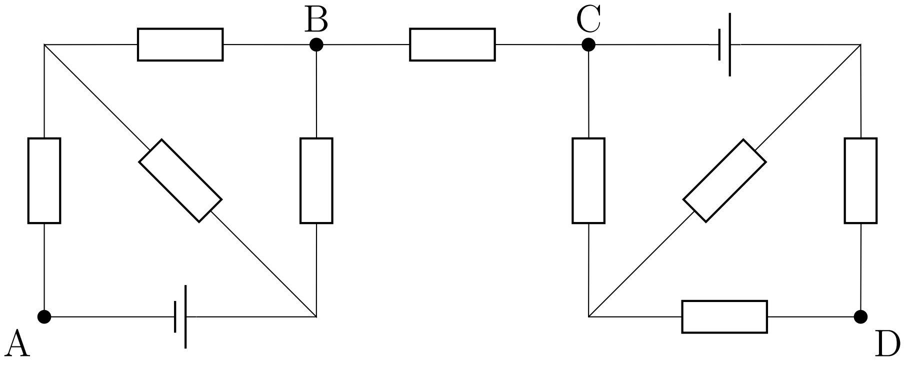
---

*Alternative solution 2:* One can also solve this problem by considering the balance of torques about the centre of the vertical cross-section. However, note that the centre of mass of the string is not at its geometric midpoint – it is somewhere outside the string, along the perpendicular bisector of the line segment connecting its two endpoints.

**Setter: Liu Yueyang,** yueyang.liu@sgphysicsleague.org

**Problem 18: Circuit Conundrum** (4 points)

Shanay constructs a circuit with batteries each of voltage $V = 10$ V and resistors each of resistance $R = 1\ \Omega$:




Find the potential difference $V_{DA} = V_D - V_A$.

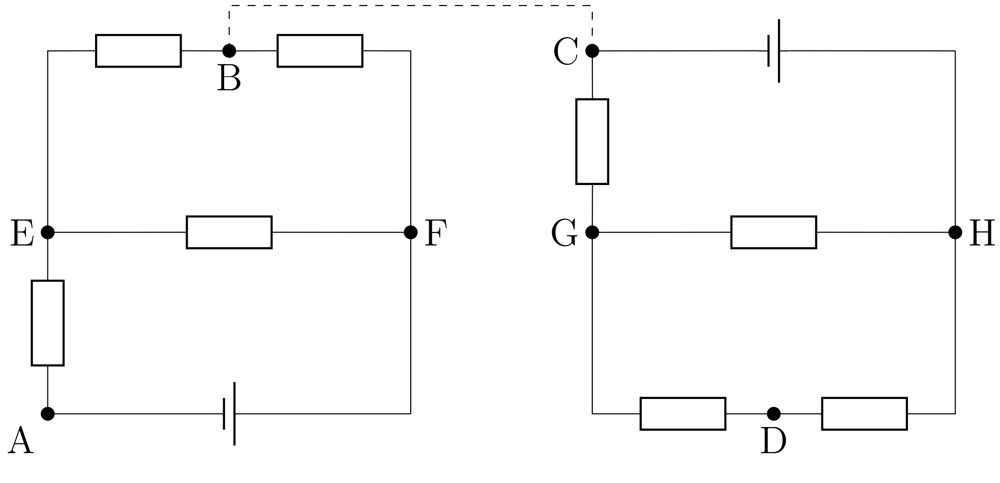
*Leave your answer to 2 significant figures in units of* V.

---

*Solution:* A key observation is required to simplify this circuit: the points B and C have no current flowing between them, and are hence the same potential.

To justify this, we first consider the fact that current must flow in a closed loop, due to conservation of charge. However, BC cannot form part of any closed loop in this setup, hence no current can flow through that path.

Now, we proceed to solve each half of the circuit independently. We may redraw the circuit in such a manner:




The effective resistance $R_{\text{EF}}$ is given by:

$$R_{\text{EF}} = \left(\frac{1}{1} + \frac{1}{2}\right)^{-1} = \frac{2}{3}\ \Omega$$

Let us use $A$ as a reference point and set $V_A = 0$ (this choice does not matter as potential differences are relative). By applying the potential divider rule, we have:

$$\begin{aligned} V_C = V_B &= V_A + 10 - \frac{1}{2} \cdot \frac{R_{\text{EF}}}{1 + R_{\text{EF}}}(10) \\ &= 8 \text{ V} \end{aligned}$$

Similarly, the resistance $R_{\text{GH}} = \frac{2}{3}\ \Omega$ and:

$$\begin{aligned} V_D &= V_C + 10 - \frac{1}{2} \cdot \frac{R_{\text{GH}}}{1 + R_{\text{GH}}}(10) \\ &= 16 \text{ V} \end{aligned}$$

Hence, $V_{DA} = \boxed{16 \text{ V}}$.

**Setter: Paul Seow, paul.seow@sgphysicsleague.org**

**Problem 19: It's Getting Hot In Here** (4 points)

After the SPhL servers shut down during SPhL 2023, Ziwen decides to shut himself in an airtight room of constant volume $V = 30.0\ \text{m}^3$. At $t = 0$ s, the room is initially at atmospheric pressure $P_0$ and temperature $T_0 = 30.0\,^\circ\text{C}$. As he feels the room is too hot, he turns on the air-conditioner. The air-conditioner feeds air of temperature $T_1 = 20.0^\circ\ \text{C}$ into the room at a constant rate $r = 7.50\ \text{mol s}^{-1}$ starting at $t = 0$ s. Find the temperature $T_a$ of the room after time $t = 300$ s.

Assume that air is a diatomic ideal gas and that the air in the room is always at thermal equilibrium. Ignore the heat from Ziwen's body and assume the walls of the room are perfectly insulating. Further assume that the air conditioner does the work to bring the air into the room, not the air itself.

*Leave your answer to 3 significant figures in units of* °C.

---

*Solution:* Firstly, we find the initial amount of air in the room $n_0$:

$$n_0 = \frac{P_0 V}{R T_0}$$

Since the work done to bring the air in is provided by the air-conditioner, and no heat is changed due to the insulating walls, we can apply conservation of internal energy of the air:

$$\frac{5}{2} n_0 R T_0 + \frac{5}{2} r t R T_1 = \frac{5}{2}(n_0 + rt) R T$$

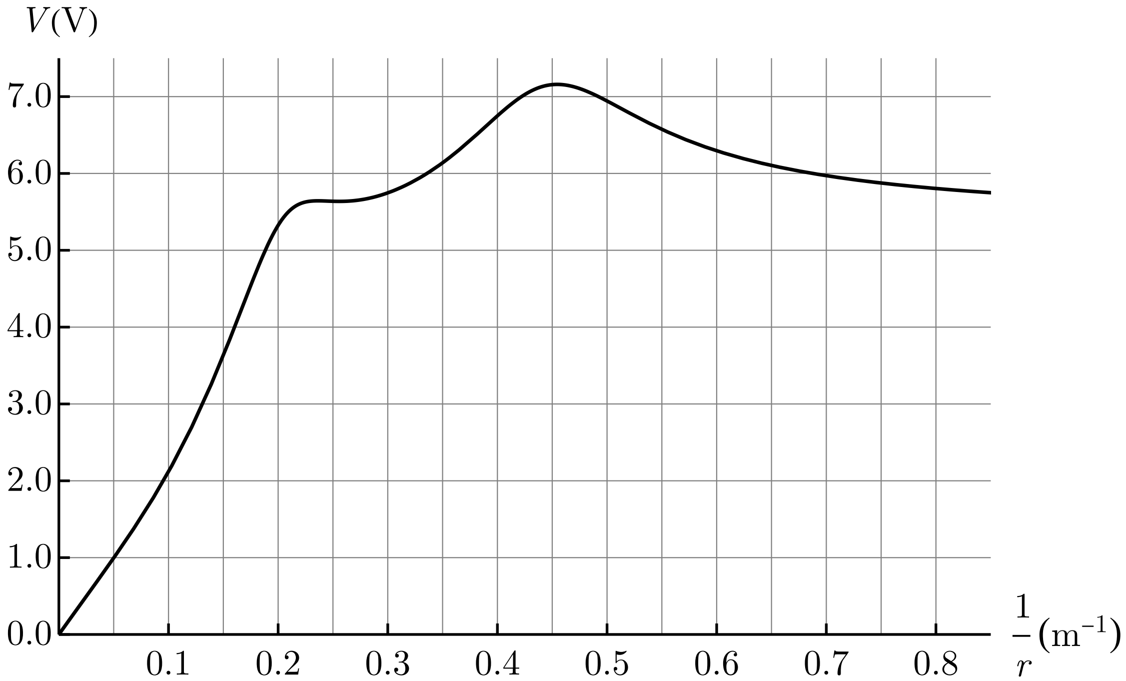
Solving for $T$, we have:

$$\begin{aligned} T &= \frac{n_0 T_0 + r t T_1}{n_0 + rt} \\ &= \frac{P_0 V T_0 + r t R T_0 T_1}{P_0 V + r t R T_0} \\ T_a &\approx 296.63\ \text{K} \\ &\approx \boxed{23.5\,^\circ\text{C}} \end{aligned}$$

**Setter: Huang Ziwen, ziwen.huang@sgphysicsleague.org**

**Problem 20: Black Box Potential** (4 points)

Consider a finite arrangement of charges that you know nothing about except for the following graph of the electric potential $V$ against $1/r$, where $r$ is the distance along an arbitrary axis starting somewhere near the charges. Determine the total charge $\sum Q$ present in the arrangement.




*Leave your answer to 2 significant figures in units of* nC.

---

*Solution:* Consider the potential $V$ at a very large distance $r$ away from the arrangement of charges. Since the distances between charges become very small compared to $r$, the arrangement of charges will "look" like a point charge with charge $\sum Q$. Hence, $V$ can be approximated for large $r$ as:

$$V \approx \frac{\sum Q}{4\pi\varepsilon_0}\frac{1}{r}$$

On the graph of $V$ against $1/r$, the gradient $k$ at the origin (where $1/r \to 0$ implies that $r$ is very large) is therefore given by:

$$k = \frac{\sum Q}{4\pi\varepsilon_0}$$

Since the tangent to the graph at the origin passes through points $(0.0, 0.0)$ and $(0.05, 1.0)$, we can deduce that $k = 1.0/0.05 = 20$ V m. As such, we obtain $\sum Q = 4\pi\varepsilon_0 k \approx \boxed{2.2 \text{ nC}}$.

---

*Alternative solution:* This problem can be more solved rigorously using the multipole expansion. For any charge distribution with charge density $\rho(x, y, z)$, the multipole expansion of the potential $V$ can be expressed in polar coordinates as powers of $1/r$ by:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{\int \rho \; \mathrm{d}V'}{r} + \frac{\int \rho r' \cos\theta' \; \mathrm{d}V'}{r^2} + \frac{\int \rho r'^2 \left(3\cos^2\theta' - 1\right)/2 \; \mathrm{d}V'}{r^3} + \cdots\right]$$

where all integrals are taken over the volume $\mathrm{d}V'$ of space in which charges reside. The first term (the monopole term) survives for large $r$, while all other terms (the dipole term, quadrupole term, and so on) vanish. To make this even more concrete, let us make the substitution $u = 1/r$ in accordance with the graph axes:

$$V = \frac{1}{4\pi\varepsilon_0}\left[u\int \rho \; \mathrm{d}V' + u^2 \int \rho r' \cos\theta' \; \mathrm{d}V' + u^3 \int \rho r'^2 \left(3\cos^2\theta' - 1\right)/2 \; \mathrm{d}V' + \cdots\right]$$

The gradient of the graph is thus given by:

$$\frac{\mathrm{d}V}{\mathrm{d}u} = \frac{1}{4\pi\varepsilon_0}\left[\int \rho \; \mathrm{d}V' + 2u\int \rho r' \cos\theta' \; \mathrm{d}V' + 3u^2 \int \rho r'^2 \left(3\cos^2\theta' - 1\right)/2 \; \mathrm{d}V' + \cdots\right]$$

Evaluating the gradient at the origin ($u = 0$), we obtain:

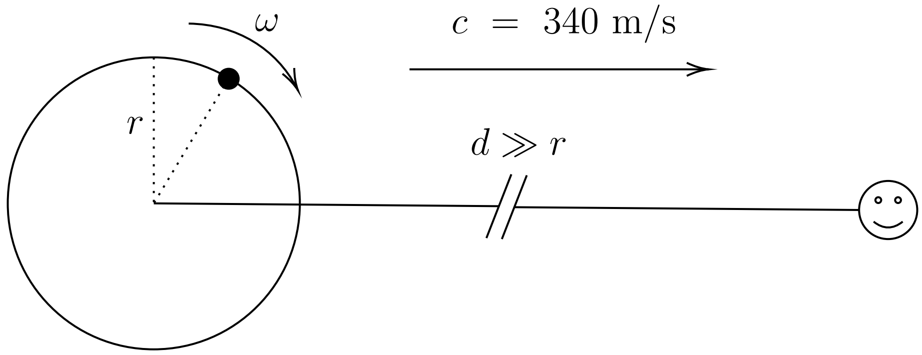
$$\left.\frac{\mathrm{d}V}{\mathrm{d}u}\right|_{u=0} = \frac{\int \rho \; \mathrm{d}V'}{4\pi\varepsilon_0}$$

Noting that $\int \rho \; \mathrm{d}V' = \sum Q$, we thus have a gradient of $\sum Q/4\pi\varepsilon_0$ at the origin, which agrees with the approximation we made in our previous method.

**Setter: Christopher Ong,** chris.ong@sgphysicsleague.org

**Problem 21: Siren Plane** (4 points)

A plane flying in circles of radius $r = 25$ m at $\omega = 6$ rad s$^{-1}$ emits a steady sound wave of frequency $f_s = 200$ Hz. Instead of the steady frequency emitted by the plane, an observer at a far distance hears a sound which oscillates between two frequencies, a higher $f_{\text{max}}$ and a lower $f_{\text{min}}$.




Assume that the speed of sound in air is $c = 340$ m s$^{-1}$. Find $|f_{\text{max}} - f_{\text{min}}|$.

*Leave your answer to 3 significant figures in units of* Hz.

---

*Solution:* The oscillating frequency is due to the Doppler shift caused by the movement of the plane.

As the observer is far away, the component of the plane's velocity directed towards the observer is approximately given by:

$$v_x \approx r\omega\cos(\omega t)$$

Thus, the frequency heard by the observer is given by:

$$f_o(t) = \frac{c}{c - r\omega\cos(\omega t)}\, f_s$$

Since $-1 \leq \cos(\omega t) \leq 1$, we can find the maximum and minimum frequencies:

$$f_{\text{max}} = \frac{c}{c - r\omega}\, f_s$$
$$f_{\text{min}} = \frac{c}{c + r\omega}\, f_s$$

Hence, our answer is given by

$$|f_{\text{max}} - f_{\text{min}}| = \frac{2cr\omega}{c^2 - (r\omega)^2}\, f_s \approx \boxed{219 \text{ Hz}}$$

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**

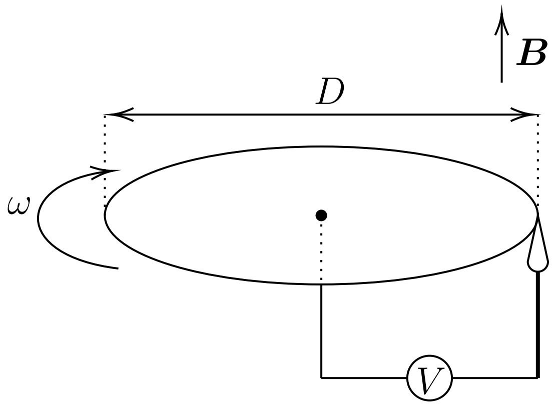
## Problem 22: Conducting Plate (4 points)

A horizontal circular plate with diameter $D = 13.0$ m, constructed from an ideal conductor, is rotating clockwise (viewed from above) at an angular velocity $\omega$ about a vertical axis passing through its centre. A uniform magnetic field $B = 3.0 \times 10^{-7}$ T points vertically upwards.

A voltmeter is connected to the plate such that one probe is fixed to the centre of the plate and the other is fixed slightly beyond the edge of the plate such that it remains stationary yet it is always in contact with the edge of the rotating plate. The reading on the voltmeter is $V = 0$. Find $\omega$, given that it is nonzero.

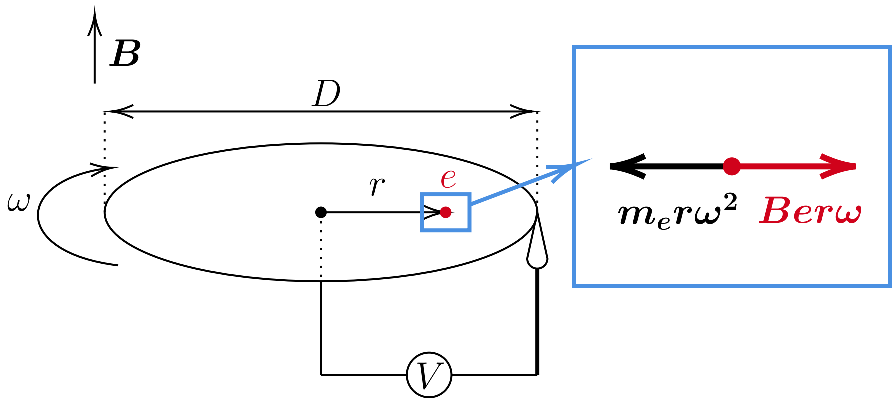
*Leave your answer to 3 significant figures in units of* rad s$^{-1}$.

*Leave a negative answer if you think the plate must be rotating counterclockwise.*




---

*Solution:* Consider an electron on the plate at a distance of $r$ from the centre. It is clear that it experiences a centre-directed magnetic force of $F = Ber\omega$, where $e$ is the electron charge. In order to avoid causing the electrons to move, which would create an electrostatic potential difference between the rim and the centre, this magnetic force must provide precisely the centripetal force for the rotation, which is $F = m_e r\omega^2$ classically.



Equating these two forces, we get:

$$\omega = \frac{eB}{m_e} \approx \boxed{52700 \text{ rad s}^{-1}}$$

A quick check verifies that even electrons on the rim are moving significantly below the speed of light (at about 0.1% of it, in fact). Hence, the classical approximation is valid and the above answer is correct.

**Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org**

**Problem 23: Interstellar** (5 points)

Hoping to set up a new habitable home for the continuation of humanity, Cooper and Brand decide to establish a new colony on a distant planet. To facilitate further scientific research, they launch two identical satellites, named TARS and CASE, into space. TARS and CASE orbit around the planet in coplanar circular orbits both in the clockwise direction, with periods $T_{\text{TARS}} = 96$ days and $T_{\text{CASE}} = 144$ days respectively. At $t = 0$, the centres of mass of TARS, CASE and the planet are collinear. After how much time $t_1$ will the centre-to-centre separation between TARS and CASE increase at the greatest rate? Answer with the minimum possible value of $t_1$.

*Leave your answer to 2 significant figures in units of days.*

---

*Solution:* Firstly, let us recall Kepler's Third Law, which states that for objects orbiting around the same planet:

$$T^2 \propto a^3$$

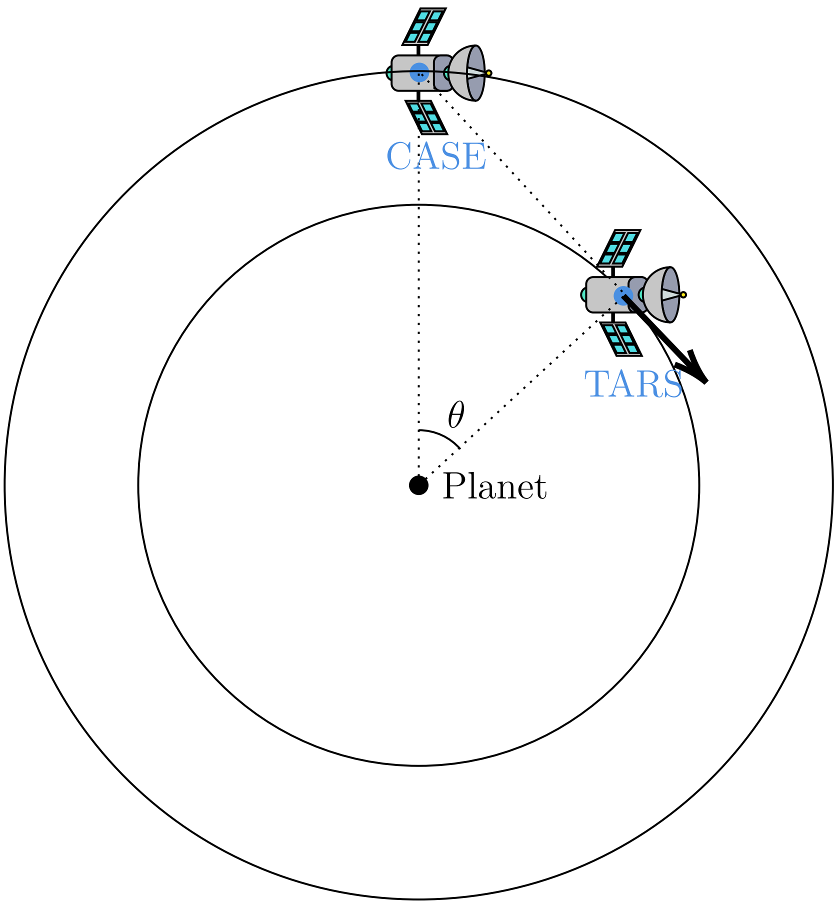
where $T$ is the orbital period and $a$ is the length of the semimajor axis of the orbit. In this case, as the orbits of TARS and CASE are both circular, $a$ is simply given by the radius of their orbits. It follows that:

$$\left(\frac{T_{\text{TARS}}}{T_{\text{CASE}}}\right)^2 = \left(\frac{r_{\text{TARS}}}{r_{\text{CASE}}}\right)^3$$

$$\frac{r_{\text{TARS}}}{r_{\text{CASE}}} = \left(\frac{T_{\text{TARS}}}{T_{\text{CASE}}}\right)^{\frac{2}{3}}$$

As the period of TARS is smaller than that of CASE, the radius of orbit of TARS must be smaller, and its angular velocity must be greater than that of CASE. Let us now consider the relative motion of the two satellites using a rotating frame of reference from the perspective of CASE. In this reference frame, CASE and the planet are both at rest, while TARS is rotating clockwise.

We note that the centre-to-centre separation between TARS and CASE increases at the highest rate when the tangential velocity vector of TARS produced passes through CASE. At this instant, the line joining the planet and TARS, the line joining the planet and CASE, and the line joining TARS and CASE form a right-angled triangle, as shown in the figure below:




Using geometry, the angle $\theta$ between the line joining the planet and TARS and the line joining the planet and CASE is given by:

$$\theta = \cos^{-1}\left(\frac{r_{\text{TARS}}}{r_{\text{CASE}}}\right) = \cos^{-1}\left(\frac{T_{\text{TARS}}}{T_{\text{CASE}}}\right)^{\frac{2}{3}}$$

This means that at this instant, the difference between the angular displacements of TARS and CASE is $\theta$. We can determine the difference $\Delta\omega$ between the angular speeds of TARS and CASE to be:

$$\Delta\omega = \frac{2\pi}{T_{\text{TARS}}} - \frac{2\pi}{T_{\text{CASE}}}$$

The time $t_1$ taken for the angular displacements of TARS and CASE to first differ by $\theta$ is thus:

$$t_1 = \frac{\theta}{\Delta\omega} = \frac{\cos^{-1}\left(T_{\text{TARS}}/T_{\text{CASE}}\right)^{\frac{2}{3}}}{2\pi/T_{\text{TARS}} - 2\pi/T_{\text{CASE}}} \approx \boxed{32 \text{ days}}$$

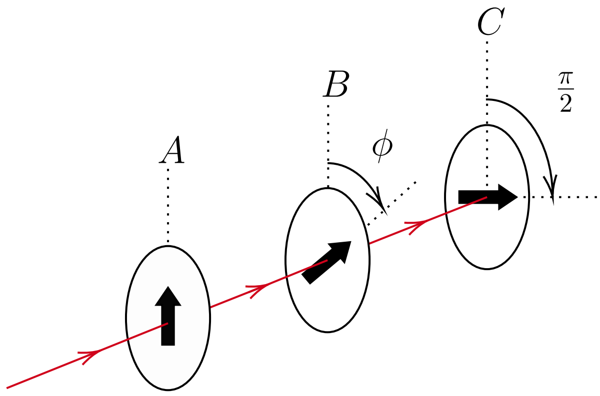
---

*Alternative solution:* One can also attempt to write down the expression for the centre-to-centre separation between TARS and CASE as a function of time $t$. From there, their relative velocity $v$ can be found, and the time at which its stationary values occur can be determined by setting $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$. Solving for $t$ and selecting the solution that yields $\frac{\mathrm{d}^2v}{\mathrm{d}t^2} < 0$, the same answer can be obtained.

**Setter: Liu Yueyang, yueyang.liu@sgphysicsleague.org**

**Problem 24: Wheel of Fortune** (4 points)

Alice and Bob play a physics-themed wheel of fortune. The game's setup consists of 3 ideal linear polarisers $A$, $B$ and $C$ arranged in a row. Polariser $A$ is fixed in the upright orientation, while polariser $C$ is fixed at angle $\frac{\pi}{2}$ rad from polariser $A$. Polariser $B$ is free to rotate. Unpolarised light of intensity $I_0$ enters polariser $A$, and the light intensity after the light passes through all 3 polarisers is $I$.




To play the game, each player spins polariser $B$. Once it stops spinning, the ratio $\alpha = I/I_0$ is the player's score. Alice plays first, and gets a score of $\alpha_A = 0.1$. What is the probability that Bob can get a higher score? You may assume that polariser $B$ is equally likely to be at any angle when it stops spinning.

*Leave your answer to 3 significant figures.*

---

*Solution:* When the unpolarised light passes through polariser $A$, the light becomes polarised and its intensity drops to $\frac{I_0}{2}$. Letting the final angle of polariser $B$ be $\phi$, we use Malus's Law to obtain the light intensity exiting $C$:

$$I = \frac{I_0}{2}\cos^2\phi\cos^2\left(\frac{\pi}{2}-\phi\right) = \frac{1}{2}\cos^2\phi\sin^2\phi I_0$$

Hence, the player's score is:

$$\alpha = \frac{1}{2}\cos^2\phi\sin^2\phi$$

We now express $\phi$ in terms of $\alpha$:

$$\begin{aligned} 2\alpha &= \cos^2\phi\sin^2\phi \\ \sqrt{2\alpha} &= \sin\phi\cos\phi \\ \sqrt{8\alpha} &= \sin 2\phi \\ \phi &= \frac{1}{2}\sin^{-1}\left(\sqrt{8\alpha}\right) \end{aligned}$$

Due to symmetry, we only need to consider $0 \leq \phi < \frac{\pi}{2}$. To obtain a score $\alpha_A$, the required angle $\phi_A$ is:

$$\phi_A = \frac{1}{2}\sin^{-1}\left(\sqrt{8\alpha_A}\right)$$

To obtain a higher score than $\alpha_A$, we must have $\phi_A < \phi < \frac{\pi}{2} - \phi_A$. Hence, the probability of the score exceeding $\alpha_A$ is given by the acceptable range of $\phi$ divided by the total angular range ($\pi/2$):

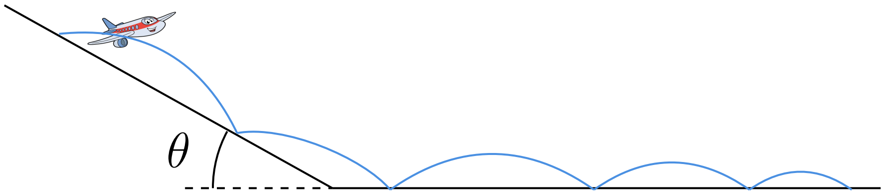
$$\begin{aligned} \mathrm{P}(\alpha \geq \alpha_A) &= \frac{\pi/2 - 2\phi_A}{\pi/2} \\ &= \frac{\pi/2 - \sin^{-1}\left(\sqrt{8\alpha_A}\right)}{\pi/2} \\ &\approx \boxed{0.295} \end{aligned}$$

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

## Problem 25: Boing Boing

A new airplane has been developed that cannot crash. Made from rubber polymers, it will just bounce. The craft was invented by Boeing, Boeing, Boeing.
- earlofdadjokes

The airplane, unfortunately, loses control and bounces exactly once on a mountain before falling onto the perfectly horizontal ground, where it bounces a few more times:




You are asked to reconstruct the path of the airplane based on the data from the flight data recorder (shown in the graph on the next page), which precisely shows you the magnitude of the airplane's velocity $|v|$ as a function of time. Unfortunately, the data is corrupted, and hence **the units for $|v|$ are arbitrary**. The units for time remain in seconds.

Assume that the mountain is a smooth slope angled at an angle $\theta$ to the ground, and that the coefficient of restitution $\eta$ between the airplane and the mountain is the same as that between the airplane and the ground. $t = 0$ occurs at an arbitrary time prior to the first collision. Neglect friction.

The coefficient of restitution $\eta$ between the airplane and each surface is a constant defined as the ratio of the magnitude of normal velocities of the airplane after and before the collision.

(a) Find $\eta$.

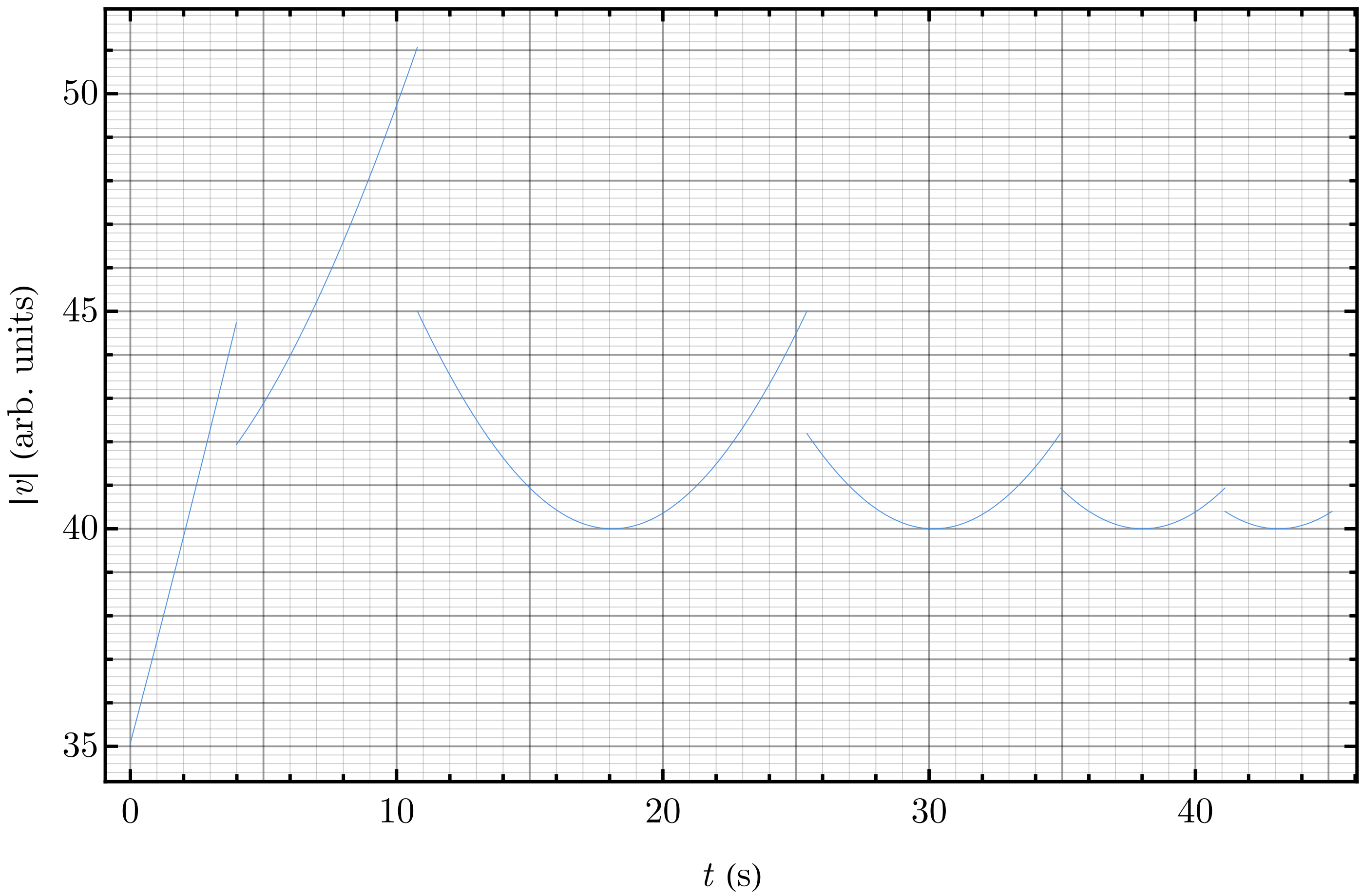
*Leave your answer to 2 significant figures.* (2 points)

(b) Find the conversion factor from the arbitrary units for velocity into m s$^{-1}$. Express your answer as $k$, where 1 arb. unit = $k$ m s$^{-1}$.

*Leave your answer to 2 significant figures.* (2 points)

(c) Find $\theta$.

*Leave your answer to 2 significant figures in units of degrees.* (3 points)




*Note for the computations:* You should try to keep a high precision in your calculations. You can zoom into the graph to find values.

---

*Solution:* Clearly, the discontinuities represent collisions. From the 2nd bounce onwards, the airplane is bouncing on the ground. Between bounces, it is moving in projectile motion.

(a) At the maximum height of each bounce on the ground, the airplane's velocity has no $y$-component, hence $|v|_{\text{min}} = v_x = 40$ as read from the graph. (Note: When velocities are given without units stated, they will be in the arbitrary units of the question.)

To find the coefficient of restitution, we need the ratio of $y$-velocities before and after a collision. The collision at $t \approx 25.5$ s is a good choice, as the velocity values are near the grid lines. Before the collision, $|v| \approx 45$, while after, $|v| \approx 42.2$. Hence:

$$\eta = \sqrt{\frac{45^2 - 40^2}{42.2^2 - 40^2}} \approx 0.652.$$

The exact value is $\eta = \boxed{0.65}$.

(b) We now use the time between bounces, which is best read off from the last part of the trajectory. While the time does not fall exactly on a grid line, we can see that the time between collisions is approximately 4 grid lines, or 4 seconds. The $|v|$ at the start is 40.4, corresponding to a $v_y$ of 5.67098. The time is given by:

$$t = \frac{2u_y}{g}$$

We then compute the velocity as $u_y = gt/2 \approx 19.62\ \text{ms}^{-1}$. The conversion factor is the ratio, which is approximately 3.46 here. The exact value is $k = \boxed{3.5}$.

(c) Considering the first collision, we split the initial velocity $\vec{\mathbf{u}}$ and the final velocity $\vec{\mathbf{v}}$ into their normal and tangential components. The normal component is multiplied by a factor $\eta$ after the collision, while the tangential component is conserved as there is no friction. We have:

$$\begin{aligned} v_n &= -\eta u_n \\ v_t &= u_t \end{aligned}$$

Additionally, we have the initial and final magnitudes of the velocity $|v_1|$ and $|v_2|$, which we can read off the graph:

$$\begin{aligned} u_n^2 + u_t^2 &= |v_1|^2 \approx (44.7)^2 \\ v_n^2 + v_t^2 &= |v_2|^2 \approx (42)^2 \end{aligned}$$

The goal is to solve for $v_n$ and $v_t$. Substituting the physical constraints gives:

$$\begin{aligned} \frac{1}{\eta^2} v_n^2 + v_t^2 &= |v_1|^2 \\ v_n^2 + v_t^2 &= |v_2|^2 \end{aligned}$$

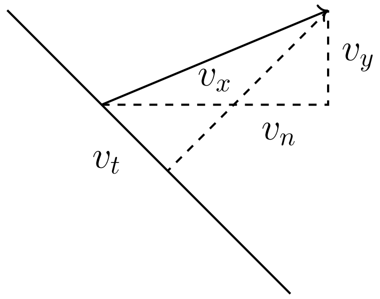
Subtracting the equations yields:

$$\left(\frac{1}{\eta^2} - 1\right) v_n^2 = \frac{1-\eta^2}{\eta^2} v_n^2 = |v_1|^2 - |v_2|^2.$$

The solutions are then:

$$\begin{aligned} |v_n| &= \sqrt{\frac{\eta^2}{1-\eta^2}\left(|v_1|^2 - |v_2|^2\right)} \approx 13.1514 \\ |v_t| &= \sqrt{\frac{|v_2|^2 - |v_1|^2\eta^2}{1-\eta^2}} \approx 39.8878 \end{aligned}$$

To find the angle, we simply compare the components $v_n$ and $v_t$ with the components in the $x - y$ coordinate system. This is summarised in the diagram below:




The angle $\theta$ obeys the equation:

$$v_t \cos\theta + v_n \sin\theta = v_x = 40$$

This equation is solvable analytically using the $R$ formula, or using computational tools. For example, one can graph the above expression as a function of $\theta$ in Desmos. By doing this, and substituting the values, we get an angle of $\theta \approx 35.9989^\circ \approx 36^\circ$. This is the exact value, which is $\theta = \boxed{36^\circ}$.

*Note on the accuracy:* It is, as promised, not necessary to use any more accurate measurement tool than the grid lines provided. All values in the solution were simply approximated, using the grid lines provided. However, it is necessary to choose the points to read off prudently. In particular, in the solution, points were chosen that lie almost exactly on the relevant grid line. In theory, it is possible to compensate for a poor choice by using computational tools to read off the graph exactly, but this is neither required nor optimal.

Additionally, for (c), it is optimal to solve the entire problem in terms of arbitrary units to prevent any rounding errors, and to exploit the incredibly nice value of $v_x$.

*Note on the problem statement:* Edits to the problem statement were made at the discretion of Guangyuan Chen to the original problem of Tan Jun Wei. The phrase "neglect friction" and the definition of the coefficient of restitution $\eta$ were not present in the original problem.

**Setter: Tan Jun Wei, junwei.tan@sgphysicsleague.org**

**Problem 26: Carbon-Free Christmas** (4 points)

It's the most wonderful time of the year again, and Paul wants to generate energy $E = 156$ MJ to illuminate his Christmas lights. To reduce his carbon footprint, Paul resorts to nuclear reactions and directs a brief beam of neutrons at a large sample of plutonium-239, causing an initial number $N_0 = 5.80 \times 10^{13}$ of plutonium-239 nuclei to undergo the fission reaction ${}^{239}_{94}\mathrm{Pu} + {}^{1}_{0}\mathrm{n} \to {}^{134}_{54}\mathrm{Xe} + {}^{103}_{40}\mathrm{Zr} + 3 \cdot {}^{1}_{0}\mathrm{n}$.

Paul notices that the neutrons produced go on to trigger further generations of the same fission reaction, each with $k = 1.02$ times as many fissions as the preceding generation, with an average time $T = 24.2\ \mu$s elapsed between two consecutive generations. After how much time $t$ since the first fission generation will Paul amass sufficient energy for his Christmas lights? You may assume that no other reactions take place in the sample, and that all the energy released is converted into electricity.

*Leave your answer to 2 significant figures in units of* ms.

Data:
Rest mass of a neutron, ${}^{1}_{0}\mathrm{n}$: $m_\mathrm{n} = 1.008665$ u
Rest mass of a zirconium-103 nucleus, ${}^{103}_{40}\mathrm{Zr}$: $m_\mathrm{Zr} = 102.926600$ u
Rest mass of a xenon-134 nucleus, ${}^{134}_{54}\mathrm{Xe}$: $m_\mathrm{Xe} = 133.905394$ u
Rest mass of a plutonium-239 nucleus, ${}^{239}_{94}\mathrm{Pu}$: $m_\mathrm{Pu} = 239.052157$ u

---

*Solution:* The number of fissions occurring in the 1st, 2nd, 3rd, ..., $r^\mathrm{th}$ generations are $N_0$, $N_0 k$, $N_0 k^2$, ..., $N_0 k^{r-1}$ respectively. We note that the terms of this sequence follow a geometric progression with common ratio $k$. Therefore, the total number of fissions $N_r$ that have occurred up to and including the $r^\mathrm{th}$ fission generation is given by:

$$N_r = N_0 + N_0 k + N_0 k^2 + \cdots + N_0 k^{r-1} = \frac{N_0\left(k^r - 1\right)}{k-1}$$

In each fission, a certain amount of energy $\Delta E$ is released as the total mass of the product particles is smaller than that of the reactant particles. This difference in mass $\Delta m$ is given by:

$$\begin{aligned}\Delta m &= m_\mathrm{Pu} + m_n - m_\mathrm{Xe} - m_\mathrm{Zr} - 3m_n \\ &= 0.202833 \text{ u}\end{aligned}$$

We then have $\Delta E = \Delta m c^2$. Hence, the total energy $E_r$ that has been released up to and including the $r^\mathrm{th}$ fission generation is given by:

$$E_r = \frac{N_0\left(k^r - 1\right)}{k-1}\Delta E$$

To find how many generations it takes for the fission reaction to generate this amount of energy, we equate $E_r$ to $E$ and solve for $r$ to obtain:

$$r = \ln\left(\frac{(k-1)E}{N_0 \Delta E} + 1\right) \Big/ \ln k \approx 377.839$$

This means that $\lceil r \rceil = 378$ fission generations are needed, which take place in $t = \lceil r - 1 \rceil T \approx \boxed{9.1 \text{ ms}}$ after the first fission generation.

**Setter: Liu Yueyang, yueyang.liu@sgphysicsleague.org**

## Problem 27: In All Directions

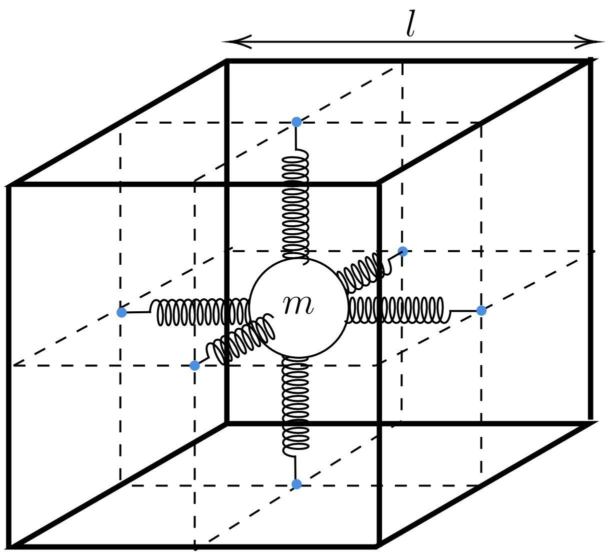
6 identical springs, with spring constant $k = 8\ \mathrm{N\ m^{-1}}$, are each attached to the centre of the faces of a cube with side length $l = 50$ cm. The other ends of each spring are all connected to a point mass $m = 3$ kg at the centre of the cube. Ignore the effects of gravity for the entire question.

(a) Given that the springs have zero rest length, find the period of small oscillations of the mass.

*Leave your answer to 3 significant figures in units of* s. (2 points)

(b) Given that the springs have a rest length of $\frac{l}{2}$, find the period of small oscillations of the mass.

*Leave your answer to 3 significant figures in units of* s. (3 points)




---

*Solution:*

(a) Let the vector from the one end of the spring to the initial position of the mass be $\vec{\mathbf{r}}_n$. The net force on the mass when it is displaced by $\vec{\mathbf{x}}$ becomes:

$$\vec{\mathbf{F}} = -k\left(\sum_{n=1}^{6} \vec{\mathbf{r}}_n + \vec{\mathbf{x}}\right) = -6k\vec{\mathbf{x}}$$

as $\sum_{n=1}^{6} \vec{\mathbf{r}}_n = 0$. Hence, the effective spring constant becomes $6k$. The period of oscillation $T$ is thus:

$$T = 2\pi\sqrt{\frac{m}{6k}} \approx \boxed{1.57\ \mathrm{s}}$$

(b) The change in length of the spring is given by the following when we use $x \ll l$:

$$\begin{aligned} |\vec{\mathbf{r}}_n + \vec{\mathbf{x}}| &= \sqrt{r_n^2 + 2\vec{\mathbf{x}} \cdot \vec{\mathbf{r}}_n + x^2} \\ &\approx |\vec{\mathbf{r}}_n| + \vec{\mathbf{x}} \cdot \hat{\mathbf{r}}_n \end{aligned}$$

Thus, the total force on the object is:

$$\begin{aligned} \vec{\mathbf{F}} &\approx -k \sum_{n=1}^{6} (\vec{\mathbf{x}} \cdot \hat{\mathbf{r}}_n)\hat{\mathbf{r}}_n \\ &= -2k\vec{\mathbf{x}} \end{aligned}$$

as $(\vec{\mathbf{x}} \cdot \hat{\mathbf{r}}_n)\hat{\mathbf{r}}_n$ are each components of $\vec{\mathbf{x}}$ in orthogonal directions, and $\vec{\mathbf{x}} = (\vec{\mathbf{x}} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\vec{\mathbf{x}} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\vec{\mathbf{x}} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$.

Hence, the effective spring constant becomes $2k$. The period of small oscillations is thus:

$$T = 2\pi\sqrt{\frac{m}{2k}} \approx \boxed{2.72 \text{ s}}$$

---

*Alternative solution:* Since the problem implies that there is a unique period of oscillation, we can consider an oscillation where the mass is simply displaced parallel to one pair of springs. The 4 springs orthogonal to the small displacement will have a second-order change in length, resulting in a negligible force exerted by these springs. Thus, the effective spring constant becomes $2k$, which is the same result as above.

**Setter: Shanay Jindal, shanay.jindal@sgphysicsleague.org**

### Problem 28: Confused Inductor

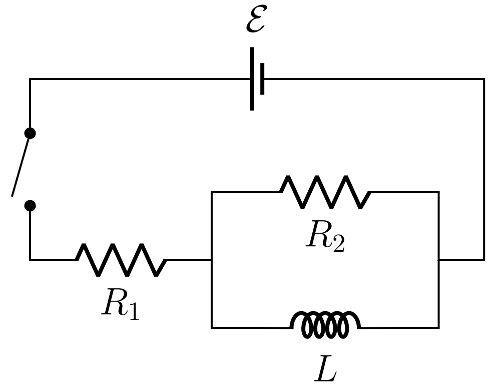
A battery of voltage $\mathcal{E} = 12$ V is connected to an arrangement of two resistors with resistances $R_1 = 20\ \Omega$ and $R_2 = 10\ \Omega$, and an ideal inductor as shown below. Initially, the switch has been open for a long time.

(a) The switch is now closed. Find the voltage $|V_L|$ across the inductor at the instant after the switch is closed.

*Leave your answer to 2 significant figures in units of* V. (2 points)

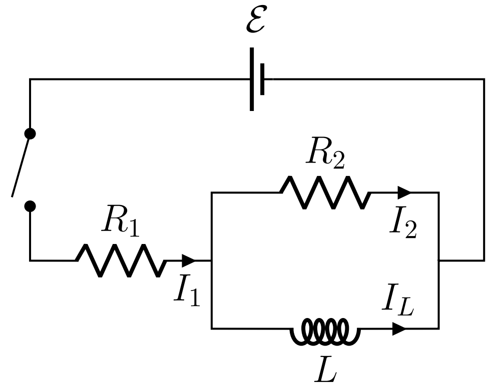
(b) After a long time, the switch is now opened again. Find the voltage $|V_L'|$ across the inductor at the instant after the switch is opened.

*Leave your answer to 2 significant figures in units of* V. (3 points)




---

*Solution:* Let us label the currents through $R_1$, $R_2$ and $L$ as $I_1$, $I_2$ and $I_L$ respectively, as shown below.




Recall that the voltage across an inductor is given by $V_L = L\frac{\mathrm{d}I_L}{\mathrm{d}t}$. This means that the current through an inductor cannot change instantaneously (otherwise, this results in $V_L \to \infty$, which is not physical). As such, whenever an instantaneous change occurs (e.g. a switch is opened or closed), the current through the inductor must still remain at its original value for that instant. On the other hand, the currents through the resistors may change instantaneously. This is an important fact that will be used throughout our solutions.

(a) Before the switch is closed, the circuit is open and thus there is no current anywhere, so $I_L = 0$. Right after the switch is closed, $I_L = 0$ since $I_L$ cannot change instantaneously. All of the current in the circuit thus flows through $R_1$ and $R_2$ in series:

$$I_1 = I_2 = \frac{\mathcal{E}}{R_1 + R_2}$$

Since the inductor is in parallel with $R_2$, the voltage $|V_L|$ across the inductor is equal to the voltage across $R_2$:

$$|V_L| = I_2 R_2 = \mathcal{E}\frac{R_2}{R_1 + R_2} = \boxed{4.0 \text{ V}}$$

(b) After the switch has been closed for a long time, the circuit must have reached a steady state, which means $\frac{\mathrm{d}I_L}{dt} = 0$. Hence, $V_L = 0$, which implies that the voltage across $R_2$ is also zero, and therefore $I_2 = 0$. In other words, the inductor has shorted $R_2$. The current thus only flows through $R_1$ and $L$ (which has no internal resistance) in series, and thus:

$$I_L = I_1 = \frac{\mathcal{E}}{R_1}$$

At the instant after the switch is opened, $I_L$ remains at $\frac{\mathcal{E}}{R_1}$ because, once again, it cannot change instantaneously. On the other hand, $I_1$ drops to zero instantly as the circuit is now open. As for $I_2$, by Kirchhoff's Current Law, its value must be given by:

$$I_2 = -I_L = -\frac{\mathcal{E}}{R_1}$$

The voltage $|V_L'|$ across the inductor is thus given by:

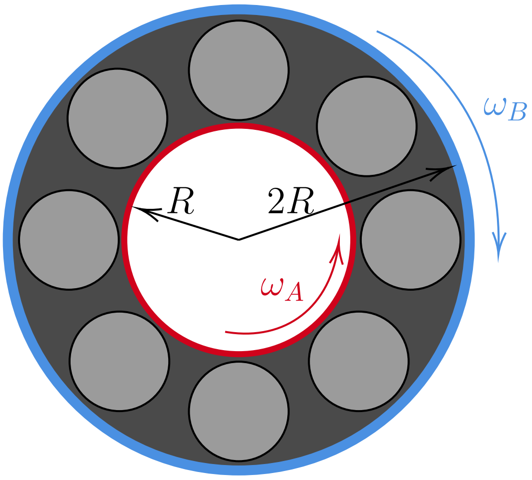
$$|V_L'| = |I_2| R_2 = \mathcal{E}\frac{R_2}{R_1} = \boxed{6.0 \text{ V}}$$

**Setter: Christopher Ong,** chris.ong@sgphysicsleague.org

**Problem 29: Ball Bearing** (5 points)

A ball bearing has inner radius $R$ and outer radius $2R$, with 8 balls of equal diameter stuck between the two cylindrical surfaces. Suppose the inner cylinder A is rotating counter-clockwise at angular velocity $\omega_A = 35$ rad s$^{-1}$, and the outer cylinder is rotating clockwise at angular velocity $\omega_B = 10$ rad s$^{-1}$, both measured in the lab frame. There is no slipping between the ball bearing and the two surfaces. Let $t_0$, $t_A$ and $t_B$ denote the time taken for the balls to complete one revolution about the centre of the ball bearing in the lab frame, inner cylinder frame (A) and outer cylinder frame (B) respectively. Let the ratio $t_0 : t_A : t_B$ be $O : A : B$, where $\gcd(O, A, B) = 1$. Find $\overline{OAB}$.

*Leave your answer as an integer. For example, if the ratio is* $2 : 4 : 8 \equiv 1 : 2 : 4$*, enter* 124 *as your answer.*




---

*Solution:* First, we analyse the bottom-most ball. Let the velocity of the centre of mass and angular velocity about the centre of mass of the ball be $v$ and $\omega$ in the lab frame respectively (defining rightwards and clockwise as positive respectively).

Then, by writing the no slip conditions at the inner and outer cylinders, we get:

$$v + \omega\frac{R}{2} = \omega_A R$$
$$v - \omega\frac{R}{2} = -2\omega_B R$$

Solving simultaneously, we obtain $\omega = \omega_A + 2\omega_B$ and $v = \frac{R}{2}(\omega_A - 2\omega_B)$.

The time taken for one revolution around the centre of the whole ball bearing **in the lab frame** is thus:

$$t_0 = \frac{2\pi(\frac{3}{2}R)}{v} = \frac{6\pi}{|\omega_A - 2\omega_B|}$$

Given that the velocity of the ball $v_0$ is known in the lab frame, we can compute the velocities $v_A$ and $v_B$ in the inner and outer cylinder frame:

$$v_A = v - \frac{3}{2}R\omega_A = -R(\omega_A + \omega_B)$$
$$v_B = v + \frac{3}{2}R\omega_B = \frac{R}{2}(\omega_A + \omega_B)$$

Dividing the distance travelled by the ball in one revolution by its velocity for each reference frame, we obtain:

$$\begin{aligned} t_0 : t_A : t_B &= \frac{6\pi}{|\omega_A - 2\omega_B|} : \frac{3\pi}{|\omega_A + \omega_B|} : \frac{6\pi}{|\omega_A + \omega_B|} \\ &= 2\frac{|\omega_A + \omega_B|}{|\omega_A - 2\omega_B|} : 1 : 2 \\ &= 6 : 1 : 2 \end{aligned}$$

The answer is thus $\boxed{612}$.

---

*Alternative solution:* Let $\omega$ be the angular velocity with which the balls rotate about the centre of the ball bearing. Then in the frame rotating at $\omega$ about the centre of the ball bearing, the balls would spin in place. By equating the velocity at the top and bottom of the balls, we have:

$$R(\omega_A - \omega_0) = 2R(\omega_B - \omega_0) \Rightarrow \omega_0 = \frac{1}{3}(\omega_A - 2\omega_B)$$

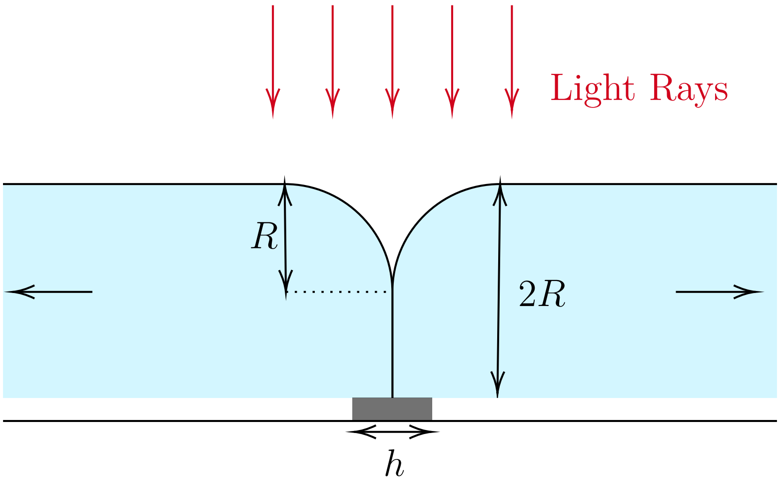
From this, we can obtain the ratio as:

$$\frac{1}{\omega_0} : \frac{1}{\omega_A - \omega_0} : \frac{1}{\omega_B + \omega_0} = 6 : 1 : 2$$

**Setter: Li Xinrui, xinrui.li@sgphysicsleague.org**

**Problem 30: Casting Shade** (4 points)

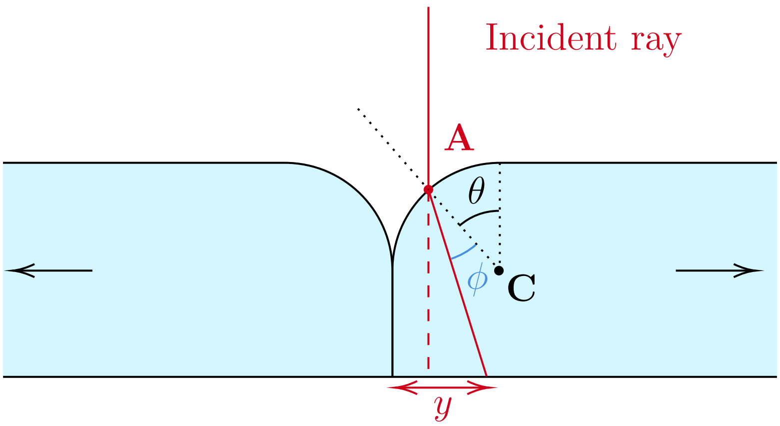
Two identical glass blocks with circular corners of radius of curvature $R = 10$ cm cut out are placed touching each other, as shown in the diagram. The blocks have height $2R$ and a long width. Parallel vertical light rays are shone onto the blocks, leaving a dark region of width $h$ centred where the glass blocks meet. Given that the refractive index of the glass is $n = 1.5$, determine $h$.




*Leave your answer to 3 significant figures in units of* cm.

---

*Solution:* Let us consider a single light ray first as shown in the diagram below. Define point $A$ as the point where the ray enters the glass, and point $C$ as the centre of the circle making up the circular corner. Define $\theta$ as the angle between the line $AC$ and the vertical line passing through $C$.




By Snell's Law, we can compute the angle of refraction $\phi$:

$$\sin\theta = n\sin\phi$$
$$\phi = \sin^{-1}\left(\frac{\sin\theta}{n}\right)$$

The angle of the refracted ray to the vertical is given by $\theta - \phi$. We let the horizontal distance between the contact axis of the glass and the position where the refracted ray reaches the bottom of the glass be $y$. This distance can be written as the sum of the horizontal distances to the left and right of the red dotted line in the diagram. $R(1 - \sin\theta)$ is the distance from the contact axis to the incident ray produced, and $R(1+\cos\theta)\tan(\theta - \phi)$ is the horizontal component of the refracted ray's displacement in glass. We thus have:

$$y = R\left[(1 - \sin\theta) + (1 + \cos\theta)\tan(\theta - \phi)\right]$$

We can plot the function $y(\theta)$ in Desmos and observe that there exists a minimum at $y_{\text{min}} \approx 8.09$ cm, which means that no light rays reach the region where $y$ is smaller than this value. Thus, we get $h = 2y_{\text{min}} \approx \boxed{16.2 \text{ cm}}$.

**Setter: Shanay Jindal, shanay.jindal@sgphysicsleague.org**

**Problem 31: Not a Relativity Problem**

Physicist S is on a rocket moving away from Earth at velocity $v = 0.8c$. Mission Control on Earth waits for one day to pass on Earth after she departs before sending a message to her. As it is their first message to her, the message header is "Mission Control 0". Afterwards:

- When Physicist S receives the message with header "Mission Control $X$" for some natural number $X$, she immediately sends a message back to Mission Control with header "Physicist S $X$".
- When Mission Control receives the message with header "Physicist S $X$" for some natural number $X$, they immediately send a message back to Physicist S with header "Mission Control $X + 1$".

Assume that all messages travel at the speed of light $c$.

(a) After 1000 years have passed on Earth since the departure of Physicist S, the agency funding the mission and operating Mission Control closed down. What is the number in the message header of the final message from Mission Control received by Physicist S?

*Leave your answer as an integer.* (3 points)

(b) According to special relativity, it should be impossible for two observers who are both non-accelerating to determine which of them is stationary. Hence or otherwise, how many days does Physicist S think has passed since her departure before she receives the first message with header "Mission Control 0"?

*Leave your answer to 3 significant figures in units of days.* (2 points)

---

*Solution:* While this problem at first appears to be a special relativity problem, no prior knowledge of special relativity is actually needed to solve it! It can be entirely solved using just knowledge of constant-velocity classical kinematics.

(a) Let us define the position of Mission Control, which remains on Earth, as $x_M = 0$, and let Physicist $S$ depart at time $t = 0$. Therefore, the position of Physicist S is $x_S = vt$. The message, which is sent one day later at time $t = t_0$, has position $x_{mM} = c(t - t_0) = ct - ct_0$. For convenience, we may use units of days for time and units of light-days (the distance light travels in a day) for distance, such that $c = 1$ and $t_0 = 1$. Then we have $x_S = 0.8t$ and $x_{mM} = t - 1$, and the message reaches Physicist S when $x_S = x_{mM}$. Solving this equation, we get $t = 5$, at which point $x_S = x_{mM} = 4$.

Physicist S then immediately sends a message back to Mission Control, which has position $x_{mS} = 5 - (t - 4) = 9 - t$. Mission Control receives this message when $x_{mS} = x_M$. Solving that equation, we get $t = 9$.

Now, of course, we may keep repeating this process until the agency running Mission Control closes down, but we actually observe that this equation is scale-invariant. Therefore, since Mission Control sends message "Mission Control 0" at time $t = t_0$ and gets a reply at time $t_1 = 9t_0$, the message "Mission Control 1" that they send at time $t = t_1$ will get a reply at time $t_2 = 9t_1 = 9^2 t_0$. In general, they will send the message "Mission Control X" at time $t_X = 9^X t_0$ until they close down.

Whether we take a year to have 365 days, 365.25 days, or even 366 days, the largest power of 9 that is less than the number of days in 1000 years is $9^5 = 59049$, since $9^6 = 531441$ is slightly longer than any reasonable definition of 1000 Earth years barring unusual cosmic events. Hence, the answer is $X = \boxed{5}$.

(b) Consider this: Mission Control sends a message at time $t_{0M}$ on Earth. Physicist S receives it at time $t_{0S}$ in her ship and immediately responds. Then, Mission Control receives her reply at time $t_{1M} = 9t_{0M}$ on Earth. To Mission Control, Physicist S is moving at velocity $v$ away from them, and her clock reads $t_{0S}$ when she receives their message, sent when Mission Control's clock reads $t_{0M}$. To Physicist S, on the other hand, Mission Control is moving at velocity $v$ away from her, and their clock reads $t_{1M}$ when they receive her message, sent when her clock reads $t_{0S}$.

According to special relativity, it should not be possible for Physicist S to tell that she is moving and that Mission Control is stationary. In both the frame of Mission Control and Physicist S, the time $t = 0$ is agreed upon, but the scale at which time progresses is different by a constant factor (as $v$ is constant). Therefore, to relate the time when two events occur in the two frames, we equate the ratio of the two times in each frame to obtain $\frac{t_{1M}}{t_{0S}} = \frac{t_{0S}}{t_{0M}}$. Rearranging, we have:

$$t_{0S}^2 = t_{0M} t_{1M} = 9t_{0M}^2 \implies t_{0S} = 3t_{0M} = \boxed{3.00 \text{ days}}$$

---

*Alternative solution:* We can use the formula for proper time $\tau = \sqrt{t^2 - x^2}$ to solve this using the values from part (a). Recall that $t = 5$ and $x = 4$, so $\tau = \sqrt{5^2 - 4^2} = \boxed{3.00 \text{ days}}$.

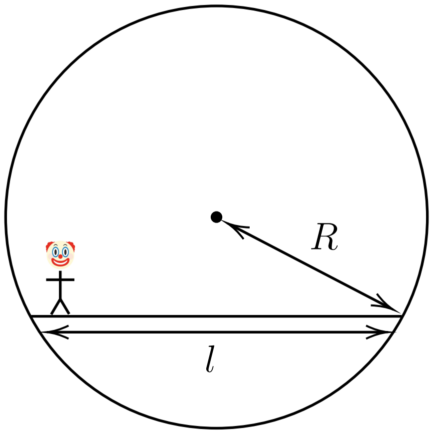
**Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org**

**Problem 32: Tightrope Walk** (5 points)

A rod of length $l = 0.7$ m is placed within a vertical, fixed circular hoop of radius $R = 0.5$ m. The rod is initially horizontal, with both ends touching the hoop. The ends of the rod are free to slide along the hoop with no friction, but both ends must always contact the hoop.

A circus clown of mass $m = 70$ kg, wishes to walk across this rod from one end to the other, starting from rest. To make his act more impressive, the clown wants to ensure that the rod remains *completely stationary* throughout his walk. How long will the clown take to complete his act?

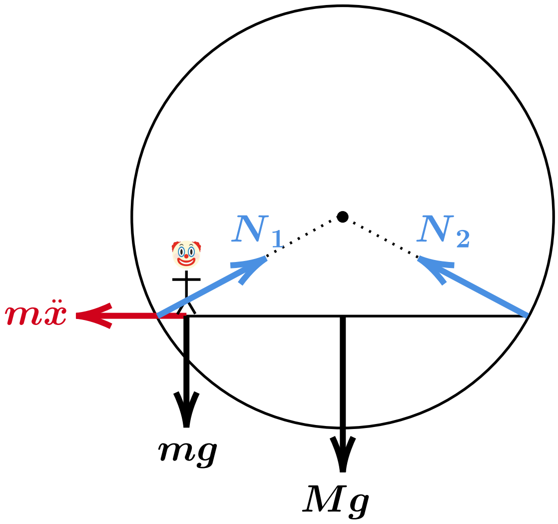
*Leave your answer to 2 significant figures in units of* s.




---

*Solution:* We first draw a free body diagram on the rod. We have the normal forces $N_1$ and $N_2$ from the loop acting on the rod, pointing perpendicular to the loop along the radial directions. The weight of the rod $Mg$ also acts vertically through its centre. Furthermore, the contact forces due to the clown include the downward vertical normal force due to the weight $mg$ of the clown, and a horizontal frictional force.

Suppose the clown moves along the rod with some horizontal acceleration $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, where $x$ is the displacement of the clown from the centre of the rod. The horizontal force acting on the clown has magnitude $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, in its direction of motion. By Newton's Third Law, the horizontal force acting on the rod has this same magnitude, and points in the opposite direction.




We now consider a torque balance of the rod. Taking the centre of the hoop as our pivot point eliminates the normal forces, as well as the weight of the rod itself, since they all pass through the pivot point. The vertical and horizontal contact forces on the rod must exert equal and opposite torques for the clown to remain in rotational equilibrium:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\sqrt{R^2-(l/2)^2} = mgx$$

Simplifying this equation, we get an equation in the form of a simple harmonic motion:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{g}{\sqrt{R^2-(l/2)^2}}x$$

By applying initial conditions $x(0) = -\frac{l}{2}$ and $x'(0) = 0$ and solving, we obtain:

$$x(t) = -\frac{l}{2}\cos(\omega t)$$

Here, the angular frequency of oscillation $\omega$ is given by:

$$\omega = \sqrt{\frac{g}{\sqrt{R^2-(l/2)^2}}}$$

At time $t_f$ when the clown reaches the other end of the rod, we have:

$$-\frac{l}{2}\cos\omega t_f = \frac{l}{2}$$
$$cos(\omega t_f) = -1$$

Hence, solving for $t_f$, we obtain:

$$\begin{aligned} t_f &= \frac{\pi}{\omega} \\ &= \frac{\pi}{\sqrt{\frac{g}{\sqrt{R^2-(l/2)^2}}}} \\ &\approx \boxed{0.60 \text{ s}} \end{aligned}$$

**Setter: James He,** james.he@sgphysicsleague.org

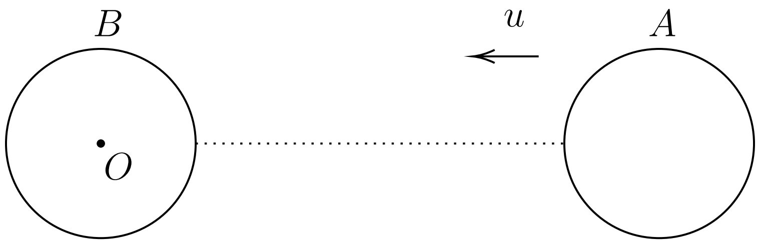
**Problem 33: Fluffball Interactions** (5 points)

Yueyang loves watching adorable cat videos where furry felines saunter towards each other and snuggle up in the most endearing ways. Below are two frames of a video[3] Yueyang thoroughly enjoys.



In an attempt to model this interaction, Yueyang treats each cat as a uniformly charged insulating sphere of radius $r = 10$ cm, mass $m = 4.0$ kg, and charge $q = +0.50\ \mu$C. Now, consider the case as shown in the diagram below, where cat A, with an initial velocity $u = 0.10$ m s$^{-1}$ to the left, approaches cat B from infinitely far away along the straight line joining the centres of the two cats. Cat B is initially stationary with its centre at the origin $O$.

Yueyang expects the centre-to-centre separation between the two cats to be minimal at a point in time during the interaction. At this instant, what is the displacement $s$ of the centre of cat B from the origin $O$? Take the leftward direction to be positive and neglect friction.




*Leave your answer to 2 significant figures in units of* m.

---

*Solution:* Let us observe this interaction in the zero-momentum reference frame, which is moving to the left at a constant velocity $u/2$ with reference to the lab frame. Here, cat A has an initial velocity $u/2$ to the left, and cat B has an initial velocity $u/2$ to the right. Due to the repulsive electric force between the two cats, their speeds will both decrease as they approach each other. In fact, since the two cats have the same mass, their decelerations will share the same magnitude throughout, and there will be a point in time when they are both momentarily at rest. At this instant, $d$ is minimal,

[3]These frames were taken from a YouTube video by studio GNYANG.

and both cats have zero kinetic energy in the zero-momentum reference frame. We note that friction is neglected and we assume that the cats do not collide. Thus, the total energy of the two-cat system is conserved and the initial kinetic energy of the two cats will have all converted to the electric potential energy of the system at this instant. It follows that:

$$\frac{1}{2}m\left(\frac{u}{2}\right)^2 + \frac{1}{2}m\left(\frac{u}{2}\right)^2 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d_{\text{min}}}$$

where $d_{\text{min}}$ is the centre-to-centre separation between the two cats at this instant, or the distance of closest approach. Solving the above equation, we obtain:

$$d_{\text{min}} = \frac{q^2}{\pi\epsilon_0 m u^2} \approx 0.224795 \text{ m}$$

*Note:* We see that $d_{\text{min}}$ is larger than $2r = 0.20$ m. This shows that even when the centre-to-centre separation between the two cats is minimal, the two cats do not collide, ruling out the possibility of any energy loss to irreversible deformation of the two cats.

After this instant of closest approach, $d$ will begin increasing again under the continuous action of the electric force. Let us move back to the lab frame to find the displacement of cat B. An important observation is that the interaction is symmetric in time with respect to the instant of closest approach.

More concretely, suppose that at time $t = 0$, cat A is at negative infinity and cat B is at the origin, and at time $t = t_1$, the two cats are at their distance of closest approach. Then, due to the time symmetry about time $t_1$, cat A will be at the origin at time $t = 2t_1$ while cat B will be at positive infinity.

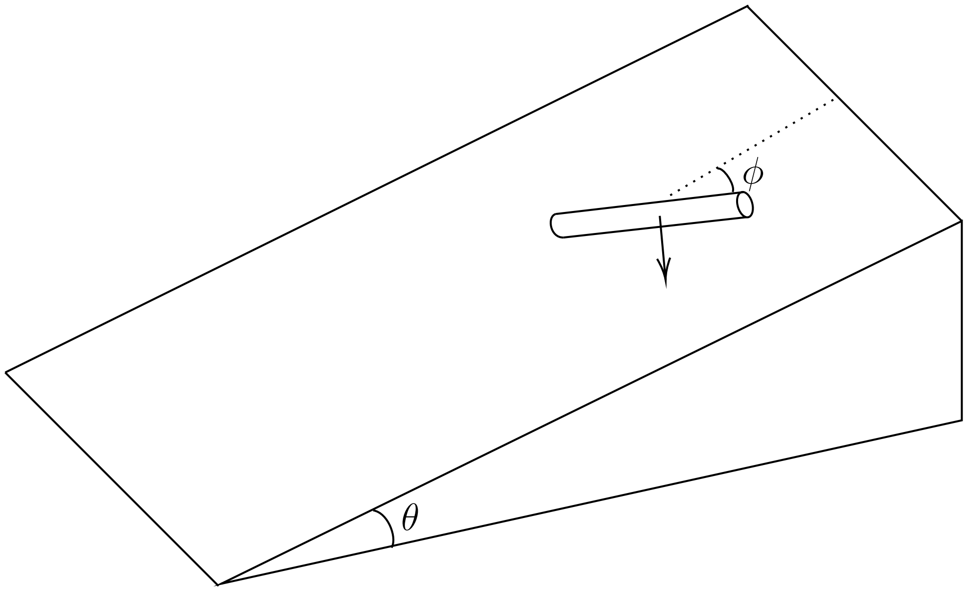
This means that at time $t_1$, the origin $O$ must be exactly at the midpoint of the line joining the centres of the two cats. Hence, at this instant, the displacement of cat B from the origin $O$ is simply half of the centre-to-centre separation between the two cats, i.e., $d_{\text{min}}/2 \approx \boxed{0.11 \text{ m}}$, in the leftward direction.

Of course, one can also tackle this problem by solving differential equations characterizing the motions of the two cats, but in Yueyang's humble opinion, the approach presented above more elegantly captures the interaction between the graceful felines.

**Setter: Liu Yueyang,** yueyang.liu@sgphysicsleague.org

**Problem 34: Double Slanted** (5 points)

A uniform solid cylinder is placed on a slope fixed to the ground and inclined at an angle $\theta = 15°$ from the horizontal. The axis of the cylinder makes an angle $\phi = 30°$ with the direction of inclination. Given that the cylinder rolls without slipping, find the minimum coefficient of static friction $\mu$ between the cylinder and the slope.

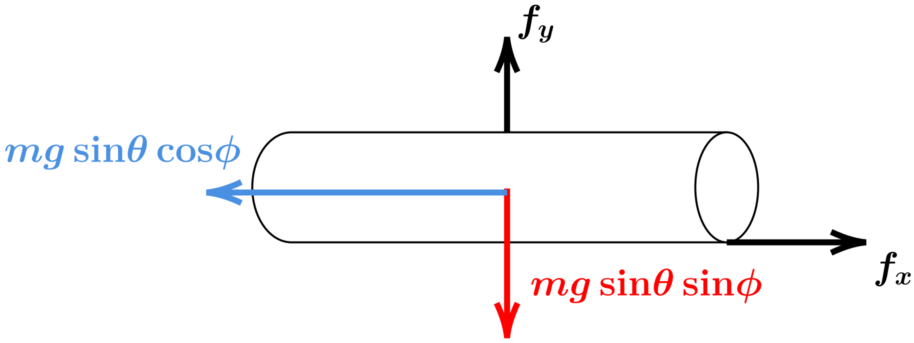



*Leave your answer to 3 significant figures.*

---

*Solution:* Let the $x$ and $y$ axes be directed along the plane of the slope, with the $x$-axis parallel to the cylinder axis, and the $y$-axis perpendicular to the cylinder axis.

The forces on the cylinder resolved along the $x$ and $y$ axes are shown below, where $f_x$ and $f_y$ denote the components of friction along the $x$ and $y$ axes respectively.




For the cylinder to roll without slipping, its translational motion must purely be in the $y$-direction. Balancing forces in the $x$-direction, we have:

$$mg \sin\theta \cos\phi - f_x = 0$$

Applying Newton's Second Law in the $y$-direction, we obtain an expression for the cylinder's translational acceleration $a_y$ in the $y$ direction:

$$mg\sin\theta\sin\phi - f_y = ma_y$$

The only torque exerted on the cylinder comes from the $y$-component of friction. The rotational form of Newton's Second Law gives us an expression for the angular acceleration $\alpha$ of the cylinder about its own axis:

$$\tau = I\alpha \implies rf_y = \frac{1}{2}mr^2\alpha$$

Applying the non-slip condition $a_y = r\alpha$, we can solve these equations for $f_x$ and $f_y$:

$$f_x = mg\sin\theta\cos\phi$$
$$f_y = \frac{1}{3}mg\sin\theta\sin\phi$$

The total friction $f$ exerted on the cylinder is therefore given by:

$$f = \sqrt{f_x^2 + f_y^2} = mg\sin\theta\sqrt{\cos^2\phi + \frac{1}{9}\sin^2\phi}$$

The normal force exerted on the cylinder is given by $N = mg\cos\theta$, by balancing forces perpendicular to the slope. The condition $f \leq \mu N$ thus gives us an expression for the minimum value of $\mu$:

$$\mu \geq \frac{f}{N} = \tan\theta\sqrt{\cos^2\phi + \frac{1}{9}\sin^2\phi} \approx \boxed{0.236}$$

**Setter: Wong Yee Hern, yeehern.wong@sgphysicsleague.org**

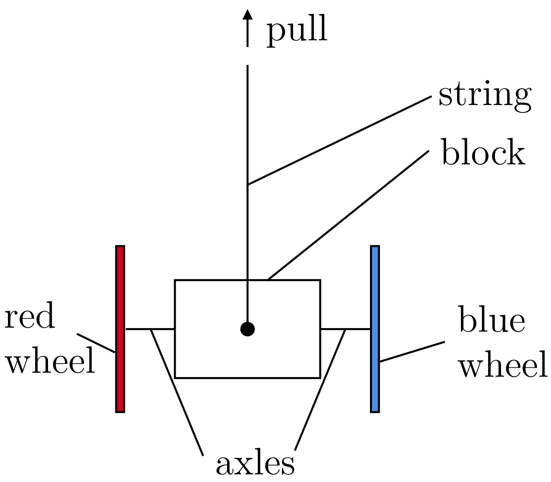
**Problem 35: Tracing an Ellipse** (3 points)

Niko has a toy consisting of a wooden block attached to two thin circular wheels and a string. The wheels (labelled in red and blue) have the same radius and are free to rotate about their axles which lie on a common axis. The axles of the two wheels are not connected, allowing the wheels to spin independently of each other. The two wheels are fixed a distance $l = 1.0$ m apart.

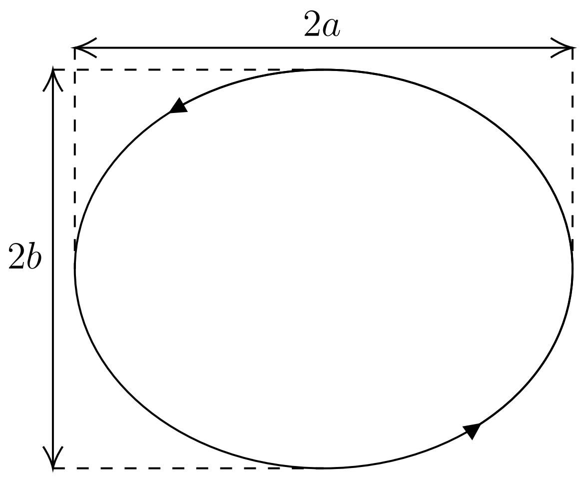
Niko then pulls the string, such that the point of the red wheel contacting the ground traces out exactly one anti-clockwise loop around an ellipse. The ellipse has semi-major axis $a = 50$ m and semi-minor axis $b = 40$ m. At all times, the axles of the toy are perpendicular to the direction of motion.

Let the distance travelled by the centres of the red and blue wheels be $d_r$ and $d_b$ respectively. Find the difference between these two distances $\Delta d = d_b - d_r$.

*Leave your answer to 3 significant figures in units of* m.



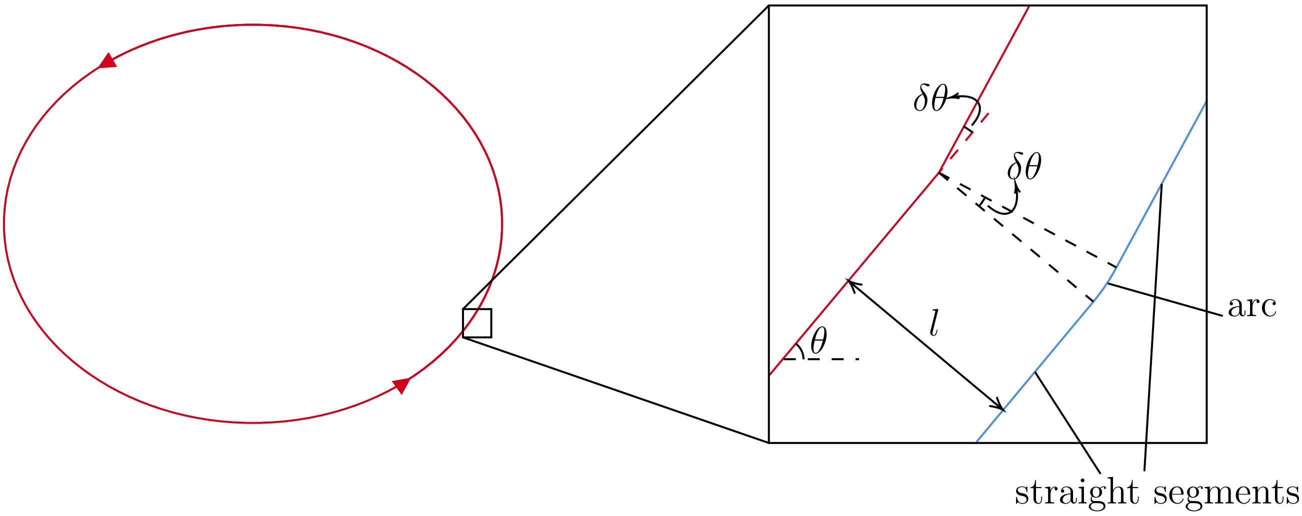

a) The toy (top view)




b) The path of the red wheel

---

*Solution:* Consider the ellipse to be made out of small line segments, with angle $\delta\theta$ between each line segment. For each of these red line segments, draw a blue line segment of the same length parallel to the red segment, a distance $l$ away. These blue segments will be separated by small gaps, each with length $\approx l\,\delta\theta$. To complete the blue wheel's path, these segments have to be filled in. Hence, for every distance $l$ the red wheel travels, the blue wheel travels a distance $l + l\,\delta\theta$.




Taking the sum of the extra distance covered by the red wheel around the entire ellipse, the total extra distance is:

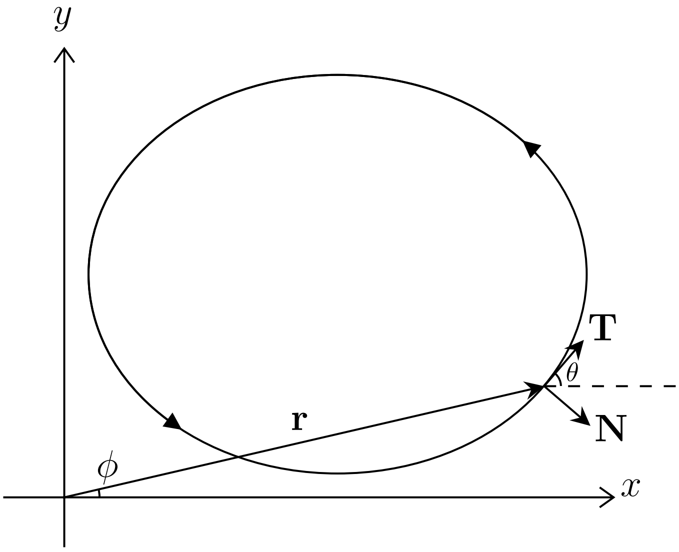
$$\sum l\,\delta\theta = l\sum\delta\theta$$

Since $\sum\delta\theta = 2\pi$, we have:

$$\Delta d = 2\pi l \approx \boxed{6.28\text{ m}}$$

---

*Alternative solution:* Let the red path be $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ for time $0 \leq t \leq t_0$, where $t_0$ is the time taken for 1 full loop.




The velocity vector is $\mathbf{r}'(t) = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}$, and hence, the unit tangent vector is:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

Therefore, the unit vector[4] $\mathbf{N}$ is a $\frac{\pi}{2}$ rotation of $\mathbf{T}$:

$$\mathbf{N} = \frac{\dot{y}\hat{\mathbf{i}} - \dot{x}\hat{\mathbf{j}}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

We now compute $\mathbf{N}'$:

$$\begin{aligned}
\mathbf{N}' &= \frac{\left(\ddot{y}\hat{\mathbf{i}} - \ddot{x}\hat{\mathbf{j}}\right)\sqrt{\dot{x}^2 + \dot{y}^2} - \frac{\left(\dot{y}\hat{\mathbf{i}} - \dot{x}\hat{\mathbf{j}}\right)(\dot{x}\ddot{x} + \dot{y}\ddot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}}}{\dot{x}^2 + \dot{y}^2} \\
&= \frac{\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \\
&= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\mathbf{T}
\end{aligned}$$

Next, compute $\theta'$ using implicit differentiation:

$$\begin{aligned}
\tan\theta &= \frac{\dot{y}}{\dot{x}} \\
\theta' \sec^2\theta &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \\
\theta'\left(1 + \tan^2\theta\right) &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \\
\theta'\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right) &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2} \\
\theta' &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}
\end{aligned}$$

Therefore, we get $\mathbf{N}' = \theta'\mathbf{T}$. The blue wheel's path follows the vector $\mathbf{b} = \mathbf{r} + l\mathbf{N}$, so its speed is:

$$|\mathbf{b}'| = |\mathbf{r}' + l\mathbf{N}'| = ||\mathbf{r}'|\,\mathbf{T} + l\theta'\mathbf{T}| = ||\mathbf{r}'| + l\theta'|$$

since $|\mathbf{T}| = 1$. Clearly, since $\theta$ is increasing at all times, $\theta' > 0$ and thus:

$$|\mathbf{b}'| = |\mathbf{r}'| + l\theta'$$

The length of the blue path $L_B$ in terms of the length of the red path $L_R$ is therefore:

$$\begin{aligned}
L_B &= \int_0^{t_0} |\mathbf{b}'| \,\mathrm{d}t \\
&= \int_0^{t_0} |\mathbf{r}'| \,\mathrm{d}t + \int_0^{t_0} l\theta' \,\mathrm{d}t \\
&= L_R + l\,[\theta]_0^{t_0}
\end{aligned}$$

[4]Technically, the unit normal vector is usually defined to be in the opposite direction, but we choose this direction for convenience

$$= L_R + 2\pi l$$

since $\theta$ increases from 0 to $2\pi$ across the loop. Hence, the extra distance travelled is $2\pi l \approx \boxed{6.28\text{ m}}$.

*Note:* It is interesting that the path does not really matter, as long as it remains a closed simple convex loop (hence, the perimeter of the ellipse is fortunately irrelevant). As such, it is possible for contestants to guess the answer based on trying some simpler loops (e.g. circle or squares). If the loop is non-convex, the line of reasoning in both solutions can fail. Solution 1 breaks entirely, but for solution 2, the final result can still hold, if the radius of curvature of the concave part does not become less than $l$ (otherwise, the line where we go from $||\mathbf{r}'| + l\theta'|$ to $|\mathbf{r}'| + l\theta'$ will break). Finally, the path of the blue wheel is not an ellipse, and assuming so leads to a wrong answer.

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

**Problem 36: Hot Balls** (5 points)

Bob has a pair of identical solid balls. Bob gives Alice one of his balls. Initially, Alice's ball is at temperature $T_a = 200$ K and Bob's ball is at temperature $T_b = 300$ K. They want their balls to be at the same temperature but they can only transfer heat between balls via an ideal heat pump. What will the maximal final temperature $T_f$ of the two balls be after equilibrium is reached?

*Leave your answer to 3 significant figures in units of* K.

---

*Solution:* We define the changing temperatures of Alice's and Bob's balls to be $T_1$ and $T_2$ respectively, and the heat capacity of the balls to be $c$.

An ideal heat pump is a heat engine operating in a reversible Carnot cycle. The efficiency at which such a pump can output heat $Q_{\text{out}}$ given some heat input $Q_{\text{in}}$ is given by $\eta = \frac{Q_{\text{out}}}{Q_{\text{in}}} = \frac{T_1}{T_2}$, where $T_2 > T_1$. Since Bob's ball is initially at a higher temperature, heat is transferred from his ball to Alice's ball with this Carnot efficiency:

$$\begin{aligned} c\,\mathrm{d}T_1 &= -\eta c\,\mathrm{d}T_2 \\ &= -\frac{T_1}{T_2} c\,\mathrm{d}T_2 \end{aligned}$$

We can separate variables and integrate to obtain:

$$\begin{aligned} \frac{\mathrm{d}T_1}{T_1} &= -\frac{\mathrm{d}T_2}{T_2} \\ \ln T_1 &= -\ln T_2 + \text{const.} \\ T_1 T_2 &= \text{const.} \end{aligned}$$

With this quantity conserved, we can obtain our final answer as:

$$T_f = \sqrt{T_a T_b} \approx \boxed{245 \text{ K}}$$

*Note:* This solution was modified after the competition due to an error in the original solution. Instead of the efficiency of heat transfer $\frac{T_1}{T_2}$, we had mistakenly used the efficiency of work done $1 - \frac{T_1}{T_2}$. Upon solving this way, the answer obtained is $\boxed{215 \text{ K}}$. Note that this answer is **physically incorrect**.

Another issue arises in the ambiguity of the phrase "ideal heat pump" in the problem statement. This can be interpreted as a heat pump constructed such that the maximum possible final temperature $T_f$ can be reached. With this in mind, we would want to minimise the work done by the pump and maximise the heat output. This is done by setting the heat transfer efficiency to $\eta = 1$, meaning that the heat from the hot ball is fully transferred to the cold ball. We can check that this does not violate the Second

Law of Thermodynamics, as the infinitesimal entropy change $\mathrm{d}S = \frac{\mathrm{d}Q}{\mathrm{d}T_1} - \frac{\mathrm{d}Q}{\mathrm{d}T_2} > 0$. The answer obtained this way is $\boxed{250\ \mathrm{K}}$.

The main solution presented is the closest to the problem setter's original intentions, and is also physically correct. However, to credit all reasonable interpretations and to avoid punishing participants for the error in our original solution, **all three answers presented are marked as correct**.

**Setter: Shanay Jindal, shanay.jindal@sgphysicsleague.org**

**Problem 37: Water Bending** (5 points)

Consider the fictitious *SPhL* ocean somewhere on Earth, with a flat ocean floor a distance $D = 4000$ m below sea level. A cylindrical meteor with radius $R = 2000$ m and height $h = 100$ m lands onto the bottom of the ocean floor with its axis orientated vertically. As a result, the water level above the cylinder rises. Find $\delta$, the increase in sea level directly above the centre of the meteor as compared to regions far away, at equilibrium. The density of the meteor is $\rho_m = 5321$ kg m$^{-3}$.

*Leave your answer to 2 significant figures in units of* mm.

*Leave a negative answer if you think the sea level above the meteor falls.*

---

*Solution:* We note that the dimensions of the Earth are much larger than the dimensions of the meteor, hence the increase in sea level after the meteor landing due to conservation of water volume is negligible. The main cause of the increased water level directly above the meteor is due to gravity.

In order for the water surface to be at equilibrium, all points on the surface must experience the same gravitational potential. If that were not the case, there would be a net force on the water particles along the surface, causing a redistribution of the water until a new equilibrium is reached.

The potential at the water surface can be viewed as the sum of the potential due to the Earth and the potential due to the meteor and water in the ocean. The potential difference due to the Earth between the point above the meteor and points far away is given by $g\delta$, as the water surface above the meteor is higher by a distance $\delta$. The potential difference due to the meteor and water is given by the difference between the potential due to the cylindrical meteor and the potential due to a cylinder of water with the same dimensions.

Thus, we are now interested in finding the potential due to a cylindrical mass. Consider an infinitesimal ring of thickness $\mathrm{d}r$, with a mass $\mathrm{d}m = 2\pi r h \rho \, \mathrm{d}r$. Using the approximation $\delta, h \ll D$, the distance from the ring to the water surface above the centre of the cylinder is $\sqrt{D^2 + r^2}$. Thus the gravitational potential due to the cylinder at the water surface $V_C$ is given by:

$$\begin{aligned}
V_C &= -\int_0^{r=R} \frac{G \, \mathrm{d}m}{\sqrt{D^2 + r^2}} \\
&= -\int_0^R G\rho \frac{2\pi r h}{\sqrt{D^2 + r^2}} \, \mathrm{d}r \\
&= -2\pi G\rho h \sqrt{D^2 + r^2} \Big|_{r=0}^{r=R} \\
&= -2\pi G\rho h \left( \sqrt{D^2 + R^2} - D \right)
\end{aligned}$$

*Note:* The assumption $h \ll D$ only causes the integral to differ by less than 2%.

The potential difference $\Delta V_C$ due to the cylinders is thus:

$$\Delta V_C = -2\pi G(\rho_m - \rho_w)h\left(\sqrt{D^2 + R^2} - D\right)$$

In order for the surface of the water to be equipotential, the two potential differences must sum to 0, so $g\delta + \Delta V_C = 0$. We have:

$$g\delta - 2\pi G(\rho_m - \rho_w)h\left(\sqrt{D^2 + R^2} - D\right) = 0$$

Solving for $\delta$, we get:

$$\delta = \frac{2\pi G h(\rho_m - \rho_w)\left(\sqrt{D^2 + R^2} - D\right)}{g} \approx \boxed{8.7 \text{ mm}}$$

*Note:* We find that $\delta \ll D$, so our assumption holds.

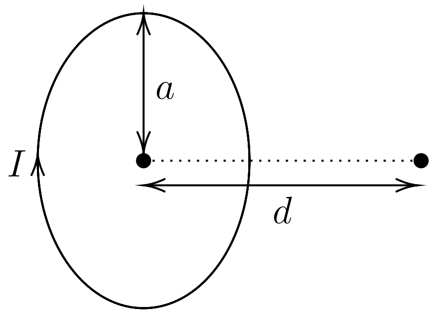
**Setter: Shanay Jindal,** shanay.jindal@sgphysicsleague.org

## Problem 38: Magnetic Charge

Consider, in analogue to an electric charge, a hypothetical *magnetic charge*, which creates a uniform magnetic field pointing radially outwards from itself. The magnitude of the magnetic field $B$ is given by $B = \frac{k_m q_m}{r^2}$, where $k_m = \frac{\mu_0}{4\pi}$, $q_m = 100$ A m is the magnetic charge, and $r$ is the distance from the charge.

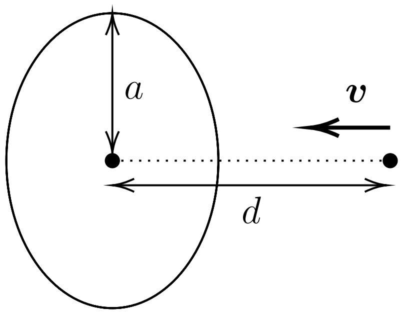
(a) A magnetic charge is placed a distance $d = 0.100$ m away from the centre of a circular wire loop of radius $a = 0.100$ m, along its axis of symmetry. A current $I = 10.0$ A flows through the loop. Find the magnitude of the force $F$ on the magnetic charge due to the loop.

*Leave your answer to 3 significant figures in units of* N. (3 points)




(b) Consider the same setup as in part (a), but let there be no current $I$ initially flowing through the wire loop. The magnetic charge now moves towards the wire loop with speed $v = 1000$ m s$^{-1}$, along its axis of symmetry. Find the magnitude of the instantaneous force $F$ on the charge due to the loop. The electrical resistance of the wire loop is $R = 1.00 \times 10^{-3}$ $\Omega$.

*Leave your answer to 3 significant figures in units of* N. (3 points)




---

*Solution:*

(a) Let us find the force acting on the ring due to the charge first. Recall that $F = BIL$, where $I$ is the current, $B$ is the magnetic field component perpendicular to $I$, and $L$ is the length of the wire. Using the given equation, we have:

$$B = \frac{k_m q_m}{r^2} = \frac{k_m q_m}{d^2 + a^2}$$

Since the components of force acting along the plane of the wire loop cancel out, the net force on the loop is directed along its axis:

$$\begin{aligned} F &= \frac{k_m q_m}{d^2+a^2} I(2\pi a) \frac{a}{\sqrt{d^2+a^2}} \\ &= \frac{2\pi k_m q_m I a^2}{(d^2+a^2)^{3/2}} \end{aligned}$$

By Newton's Third Law, the force acting on the charge due to the ring will be equal and opposite in magnitude. Hence, upon substituting numerical values, our final answer is $\boxed{F \approx 0.00222 \text{ N}}$.

---

*Alternative solution:* Another equally valid approach is to extend the analogy between electric and magnetic charges further. Just like how $F = qE$ for electric charges, $F = q_m B$ for magnetic charges. By the Biot-Savart Law, the magnetic field at the magnetic charge due to the wire loop is given by:

$$B_m = \frac{\mu_0 I R^2}{2(d^2+a^2)^{3/2}}$$

Hence, we have:

$$F = \frac{\mu_0 q_m I a^2}{2(d^2+a^2)^{3/2}}$$

which is the same result as our original solution.

(b) The key idea to solving this part is the *changing magnetic flux* through the ring. As the charge moves, a greater fraction of magnetic field lines emitted by the charge pass through the ring. Hence, the magnetic flux through the ring increases, and by Faraday's Law, this induces a current in the ring. This current then interacts with the magnetic charge, similar to part (a), inducing a force on the charge.

The flux through the ring $\Phi$ can be expressed in terms of the solid angle $\Omega$ covered by the ring from the perspective of the charge:

$$\Phi = k_m q_m \Omega$$

where $\Omega = 2\pi\left(1 - \frac{d}{\sqrt{d^2+a^2}}\right)$.

The EMF $\mathcal{E}$ in the ring is given by Faraday's Law:

$$\begin{aligned} \mathcal{E} &= -\frac{\mathrm{d}}{\mathrm{d}t}\left[2\pi k_m q_m \left(1 - \frac{d}{\sqrt{d^2+a^2}}\right)\right] \\ &= -\frac{2\pi k_m q_m v a^2}{(d^2+a^2)^{3/2}} \end{aligned}$$

Hence, the magnitude of current $I$ induced in the ring is:

$$\begin{aligned} I &= \frac{|\mathcal{E}|}{R} \\ &= \frac{2\pi k_m q_m v a^2}{R(d^2+a^2)^{3/2}} \end{aligned}$$

Using the same expression we derived in part (a), the magnitude of the instantaneous force is:

$$\begin{aligned} F &= \frac{2\pi k_m q_m a^2}{(d^2+a^2)^{3/2}} \frac{2\pi k_m q_m v a^2}{R(d^2+a^2)^{3/2}} \\ &= \frac{4\pi^2 k_m^2 q_m^2 a^4 v}{R(d^2+a^2)^3} \\ &\approx \boxed{0.0493 \text{ N}} \end{aligned}$$

**Setter: James He, james.he@sgphysicsleague.org**

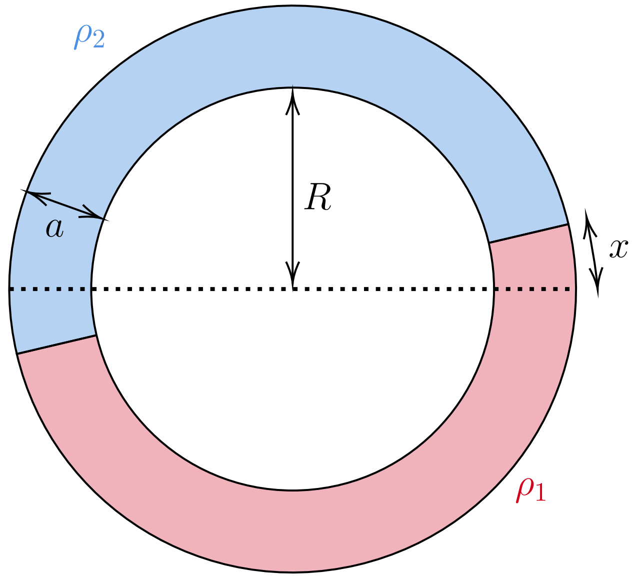
**Problem 39: Instability** (5 points)

A vertical circular O-tube in gravity has its lower half filled with a liquid of density $\rho_1 = 1000$ kg m$^{-3}$, and upper half filled with a liquid of density $\rho_2 = 2000$ kg m$^{-3}$. The radius of the circle $R = 1.00$ m is much larger than the radius of the tube $a$.

Initially, the system is at equilibrium. After a small initial perturbation of the fluid surface $x_0$, the fluid-fluid interface may begin to move. At small time-scales, this perturbation grows exponentially such that the perturbation $x$ is of the form $x = x_0 e^{\gamma t}$ where $\gamma$ is known as the growth rate. Find the value of $\gamma$.

Assume $x \ll a$ throughout the motion, and neglect any effects of viscosity, friction or surface tension.

*Leave your answer to 3 significant figures in units of* s$^{-1}$.




---

*Solution:* The problem setup is a simplification of the Rayleigh-Taylor instability: a layer of dense liquid on top of a layer of less dense liquid is an unstable configuration. While the interface between the two liquids may initially be flat and horizontal, small and random perturbations in the interface are *amplified*, causing large interface deformations.

We will approach this problem from an energy perspective. Consider a small upward displacement $x$ of the right interface. The change in gravitational potential energy is due to the change in height of the two small displaced fluid segments:

$$\begin{aligned}\Delta V &= -(\rho_2 \pi a^2 x) g x + (\rho_1 \pi a^2 x) g x \\ &= -\pi a^2 (\rho_2 - \rho_1) g x^2\end{aligned}$$

The kinetic energy of the fluid is due to the velocity of the entire fluid $\frac{\mathrm{d}x}{\mathrm{d}t}$. We can write the kinetic energy as follows:

$$T = \frac{1}{2}(\rho_1 + \rho_2)(\pi a^2)(\pi R)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$$

The total energy is constant, so by conservation of energy we have:

$$-\pi a^2(\rho_2 - \rho_1)gx^2 + \frac{1}{2}(\rho_1 + \rho_2)\pi^2 a^2 R\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \text{const.}$$

Taking a time derivative of this expression, we get:

$$-\pi a^2(\rho_2 - \rho_1)g\left(2x\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{1}{2}(\rho_1 + \rho_2)\pi^2 a^2 R\left(2\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right) = 0$$

Rearranging this equation, we get a differential equation in $x$:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\frac{2g}{\pi R}x$$

We are almost done. Substituting $x = x_0 e^{\gamma t}$ as hinted in the question statement ($x_0$ is some arbitrary constant):

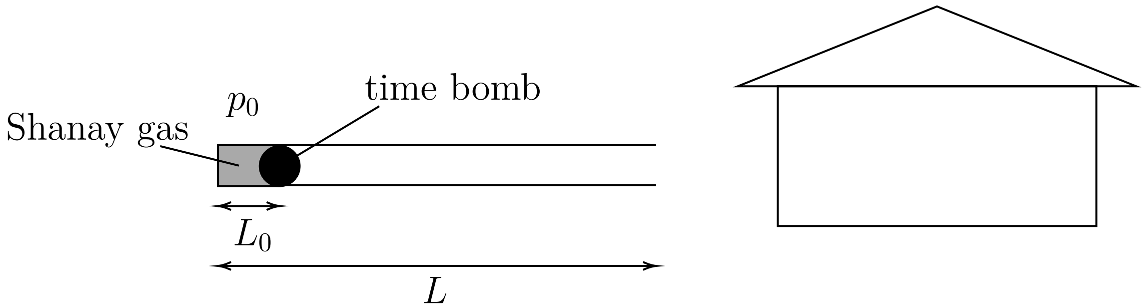
$$x_0\gamma^2 e^{\gamma t} = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\frac{2g}{\pi R}x_0 e^{\gamma t}$$

$$\gamma = \sqrt{\frac{2}{\pi}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\frac{g}{R}} \approx \boxed{1.44\ \mathrm{s}^{-1}}$$

**Setter: James He, james.he@sgphysicsleague.org**

**Problem 40: Return to Sender** (5 points)

Shanay constructs a cannon and points it at your house, intending to launch a time bomb of mass $m = 6$ kg. The cannon shoots by initially pressurizing some Shanay gas of adiabatic index $\gamma = 2$ to pressure $p_0 = 1000\, p_{\text{atm}}$ and then allowing it to expand rapidly. Take $p_{\text{atm}} = 1.01 \times 10^5$ Pa as the pressure of the air to the right of the bomb, and throughout the atmosphere. The bomb initially sits at a distance $L_0 = 0.01$ m away from the left end of the cannon, and is spherical with radius $r \ll L_0$.




You know that if the cannon is of a minimum length $L$, the bomb will eventually return towards Shanay (exploding him). Thus, you quickly extend the cannon while Shanay is away. Find the value of $L$.

You may assume that the bomb and the walls of the cannon are perfectly insulating and the bomb fits snugly in the barrel while moving with no friction along the length of the cannon. You may further assume that Shanay gas behaves like an ideal gas.

*Leave your answer to 3 significant figures in units of* m.

---

*Solution:* Let the instantaneous pressure of the Shanay gas be $p$. As long as $p > p_{\text{atm}}$, the bomb will accelerate rightwards. As such, the bomb reaches its maximum velocity when $p = p_{\text{atm}}$ and begins to slow down after that, eventually coming to a stop. At that point, $p < p_{\text{atm}}$, which causes the bomb to accelerate back towards Shanay. Since no energy is lost, the bomb is able to return to its initial position.

Since the walls are perfectly insulating, no heat is lost to the atmosphere and the Shanay gas expands adiabatically. We can write:

$$p = p_0 \left( \frac{L_0}{x} \right)^{\gamma}$$

where $x$ is the distance from the left end of the cannon to the bomb.

Let $C_p$ and $C_v$ be the molar heat capacities of the Shanay gas at constant pressure and volume respectively. We know that $\gamma = C_p/C_v = 2$ and $C_p = C_v + R$. Thus $C_v = R$.

From this, we can express the change in internal energy $\Delta U$ of the gas in terms of the amount $n$ of the gas and its change in temperature $\Delta T$:

$$\begin{aligned}\Delta U &= nC_v\Delta T \\ &= nR\Delta T \\ &= \Delta(pV)\end{aligned}$$

As the gas expands adiabatically, the change in heat $\Delta Q$ is zero. We can write an equation for conservation of energy, taking into account the kinetic energy $K$ of the bomb and the work done by the atmosphere on the gas $W$:

$$\Delta U = K + W$$

Since the pressure of the atmosphere is constant at $p_{\text{atm}}$, the work done by the atmosphere as the bomb travels a distance $\Delta L$ is $-p_{\text{atm}}A\Delta L$, where $A$ is the cross-sectional area of the bomb. When the bomb stops, $K = 0$ and $p = p_0\left(\frac{L_0}{L}\right)^\gamma$, so we have:

$$\begin{aligned}\Delta U &= W \\ \Delta(pV) &= -p_{\text{atm}}A\Delta L \\ \Rightarrow p_0\left(\frac{L_0}{L}\right)^\gamma L - p_0L_0 &= -p_{\text{atm}}(L - L_0)\end{aligned}$$

Substituting $\gamma = 2$ and rearranging, we obtain:

$$p_{\text{atm}}L^2 - (p_{\text{atm}} + p_0)L_0L + p_0L_0^2 = 0$$

Solving for the larger $L$, we find that $L = \boxed{10.0\text{ m}}$.

---

*Alternative solution:* We can then express the equation of motion of the bomb as:

$$\begin{aligned}m\ddot{x} &= A(p - p_{\text{atm}}) \\ &= A\left(p_0\left(\frac{L_0}{x}\right)^\gamma - p_{\text{atm}}\right)\end{aligned}$$

Substituting $\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$, we can integrate to obtain:

$$\begin{aligned}m\int_0^0 \dot{x}\,\mathrm{d}\dot{x} &= \int_{L_0}^{L} A\left(p_0\left(\frac{L_0}{x}\right)^\gamma - p_{\text{atm}}\right)\mathrm{d}x \\ 0 &= A\left[p_{\text{atm}}\left(L - L_0\right) - p_0\frac{1}{1-\gamma}L_0^\gamma\left(L^{1-\gamma} - L_0^{1-\gamma}\right)\right]\end{aligned}$$

We can re-express as the same equation in the previous solution by substituting $\gamma = 2$:

$$\begin{aligned}p_{\text{atm}}\left(L - L_0\right)L + p_0L_0^2\left(1 - \frac{L}{L_0}\right) &= 0 \\ p_{\text{atm}}L^2 - L_0(p_{\text{atm}} + p_0)L + p_0L_0^2 &= 0\end{aligned}$$

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**

**Problem 41: Interstellar (Continued)** (5 points)

At the beginning of the film *Interstellar*, the spacecraft Endurance travels from Earth to Saturn with the help of Mars. Mars has mass $M = 6.00 \times 10^{23}$ kg and orbits at velocity $V_i = 24$ km s$^{-1}$ around the Sun. Endurance has mass $m = 2 \times 10^6$ kg and its velocity is given by $v_i = 2V_i$. Endurance approaches Mars at an angle $\theta = 60°$ with respect to the direction of Mars' velocity. The trajectory of Endurance is chosen such that its final velocity after leaving Mars is perpendicular to its initial velocity, and is as large in magnitude as possible. Find the change in the kinetic energy $\Delta K$ of Mars through this process.

You may model both Endurance and Mars as point masses. The magnitude and direction of all velocities are given in the frame of the Sun. You may also assume that the orbit of Mars is always circular.

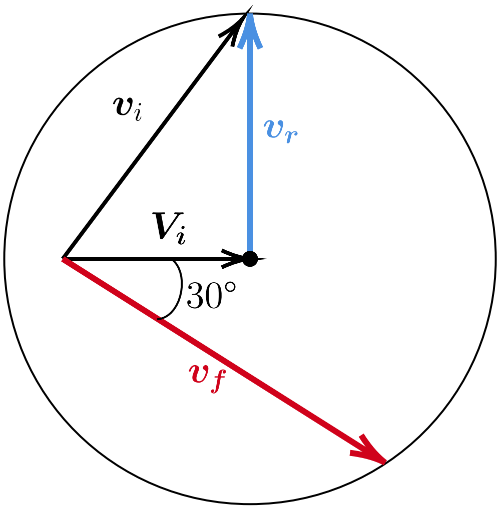
*Leave your answer to 3 significant figures in units of* TJ.

*Leave a negative answer if you think the kinetic energy decreases.*

---

*Solution:* As there is no net energy loss in the system when Endurance approaches Mars, we can model this scenario as an elastic collision.

Since the mass of Mars is much greater than the mass of Endurance, we shift our problem to Mars' reference frame (which approximately coincides with the centre of mass frame), where the relative speed of Endurance with respect to Mars is constant. It is then more intuitive to use a geometrical method to demonstrate the possible final velocities of Endurance.




In the diagram, $\mathbf{v_i}$ is the initial velocity of Endurance, and $\mathbf{V_i}$ is the initial velocity of Mars. The relative speed of Endurance with respect to Mars $\mathbf{v_r}$ may point in any direction from the tip of $\mathbf{V_i}$, the locus of all such points being a circle. Hence, the final velocity of Endurance, $\mathbf{v_f} = \mathbf{V_i} + \mathbf{v_r}$, must lie on this circle.

There are then two possible choices of $\mathbf{v_f}$ which are perpendicular to $\mathbf{v_i}$. We choose the one larger in magnitude, which is at an angle of $90° - \theta$ from $\mathbf{V_i}$. Using the Cosine Rule, we can determine the magnitude of the final velocity of Endurance $v_f$:

$$v_r^2 = V_i^2 + v_f^2 - 2V_i v_f \cos(90° - \theta)$$
$$v_f = \sqrt{v_r^2 - V_i^2 + 2V_i v_f \cos(90° - \theta)}$$

The increase in Endurance's kinetic energy, $\frac{1}{2}m(v_f^2 - v_i^2)$ is equal to the net loss of Mars' energy. A possible misconception is that Mars' final velocity must decrease because the kinetic energy of Endurance increases. However, we know that the relationship between the kinetic energy and the total energy of Mars in a stable orbit around the Sun is given by:

$$K = -\frac{1}{2}U$$
$$K + U = T$$
$$\implies K = -T$$
$$\Delta K = -\Delta T$$

where $K$ is the kinetic energy, $U$ is the potential energy, and $T$ is the total energy. As Mars' total energy decreases by $\frac{1}{2}m(v_f^2 - v_i^2)$, its kinetic energy will increase by the same amount. Hence we have:

$$\Delta K = \frac{1}{2}m(v_f^2 - v_i^2)$$
$$\approx 1.37 \times 10^{15} \text{ J}$$
$$= \boxed{1370 \text{ TJ}}$$

*Note:* There is potential ambiguity in the statement "Endurance approaches Mars at an angle $\theta = 60°$ with respect to the direction of Mars' velocity." The problem setter's intention is that the velocity vectors of Endurance and Mars are separated by an angle 60°, i.e. $\mathbf{v_i} \cdot \mathbf{V_i} = v_i V_i \cos 60°$. We still believe this to be the most direct interpretation of the problem statement, hence the solution above is still correct.

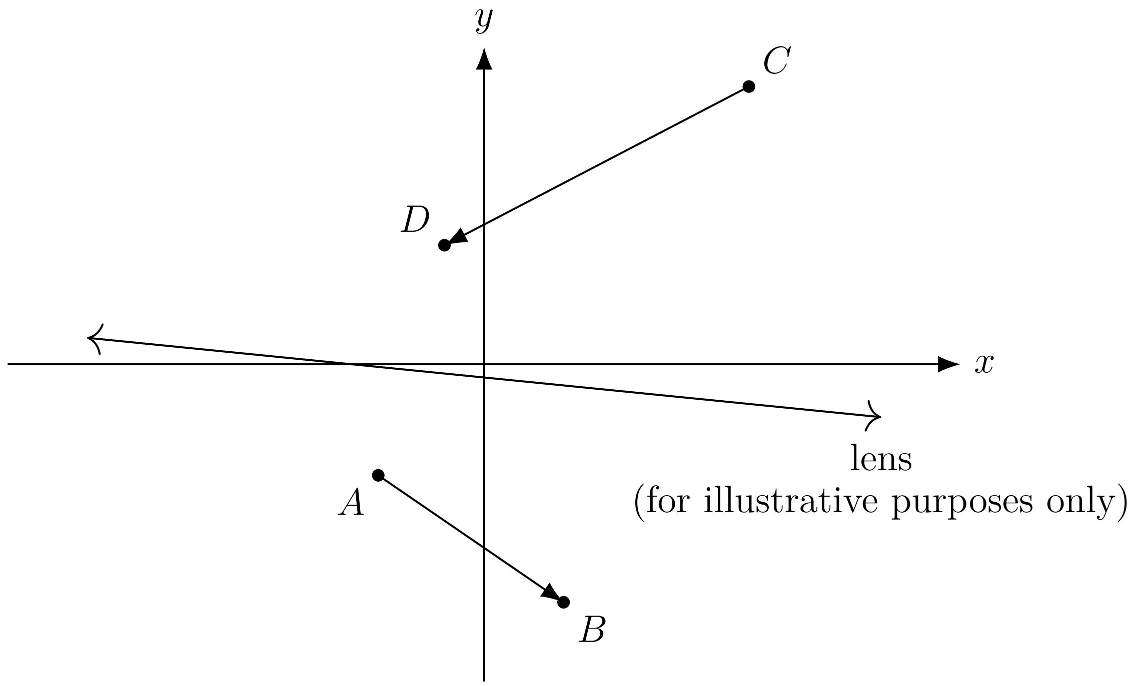
However, the direction constraint may be interpreted as a looser one, where only the lines containing the velocities of Endurance and Mars need to be separated by an angle 60°. This opens up a second possibility of the direction of Endurance's velocity, where it is moving against rather than along the direction of Mars, i.e. $\mathbf{v_i} \cdot \mathbf{V_i} = -v_i V_i \cos 60°$. With this interpretation in mind, we follow through with the same steps to obtain $\Delta K = \boxed{4610 \text{ TJ}}$, which is larger than our original value of $\Delta K$.

To reward participants for sound physical reasoning and avoid punishing them for this reasonable alternative interpretation of the problem statement, **both answers presented are marked as correct**.

**Setter: Li Yichen,** yichen.li@sgphysicsleague.org

**Problem 42: Object and Image** (5 points)

A convergent ideal thin lens is placed such that the plane of the lens is perpendicular to the plane of the paper, and the optical axis lies within the plane of the paper. Define points $A(-4.0, -4.2)$, $B(3.0, -9.0)$, $C(10.0, 10.5)$ and $D(-1.5, 4.5)$ within the plane of the paper. If the vector $\overrightarrow{AB}$ is used as an object for the lens, then the real image $\overrightarrow{CD}$ is formed. Find the focal length of the lens.

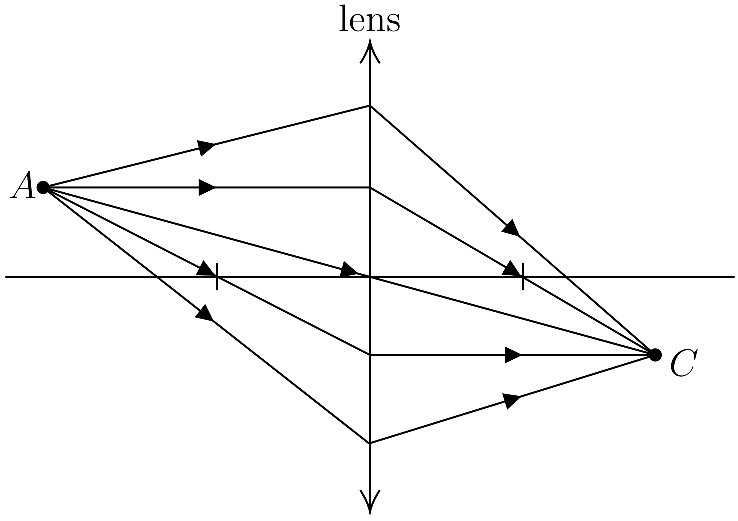



All coordinates are given in centimetres.

*Leave your answer to 2 significant figures in units of* cm.

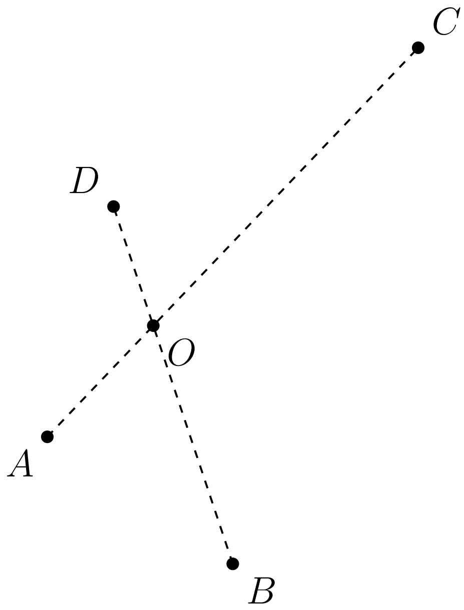
---

*Solution:* The image of point $A$ is point $C$. Hence, every light ray emitted from $A$ that intersects the lens will be deflected such that it passes through point $C$.

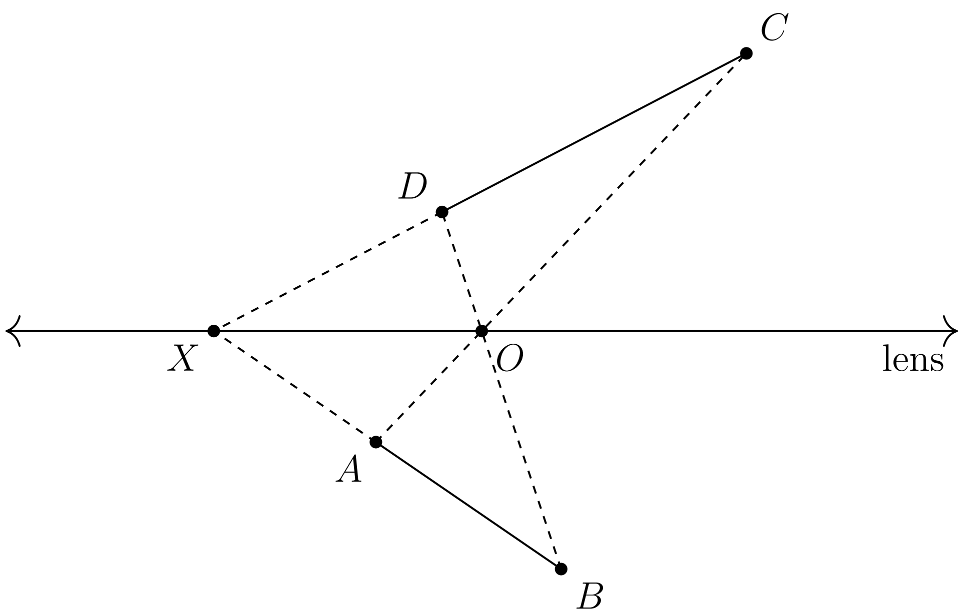


In particular, the light beam going from $A$ to $C$ in a straight line has to pass through the centre of the lens.

Similarly, the light beam going from $B$ to $D$ in a straight line also passes through the centre of the lens. Therefore, the centre of the lens (which we label as point $O$) lies on the intersection of lines $AC$ and $BD$.

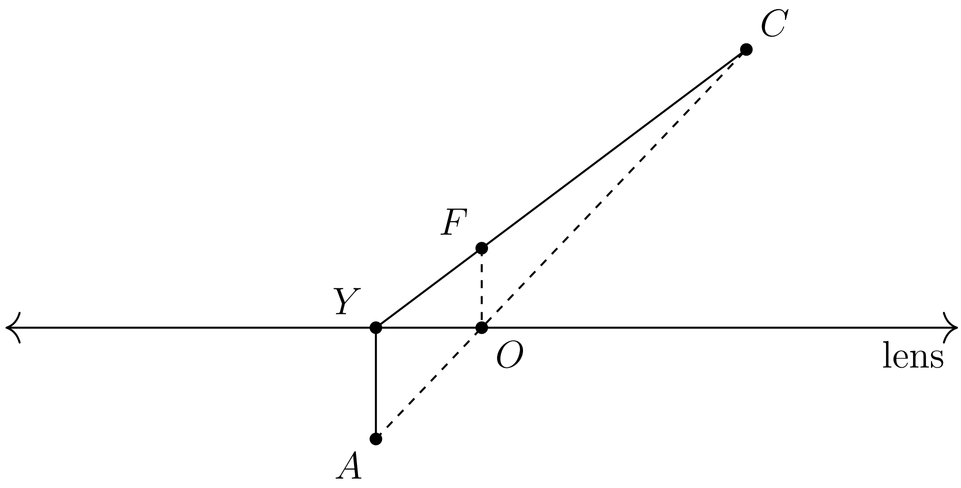



Next, consider the light ray in the direction of $\overrightarrow{BA}$, emanating from point $A$. Since this light ray passes through point $A$, the light ray must also necessarily pass through point $C$ after deflection. Similarly, as the ray passes through point $B$, the deflected ray also passes through point $D$. Since the light ray only gets deflected once, the deflected ray must pass through line $CD$. Therefore, the light ray must get deflected at point $X$, the intersection of lines $AB$ and $CD$.




Therefore, the lens lies along line $OX$.

With the position of the lens found, we can proceed to find the focal length. Consider points $A$ and $C$. A light ray emitting from point $A$ that is perpendicular to the lens has to pass through the focal point $F$ and the image point $C$ after deflection by the lens. Hence, we draw the perpendicular from point $A$ to the lens, and connect the foot of the perpendicular $Y$ to point $C$ with a line. The point where this line intersects the optical axis of the lens (the perpendicular to the lens at point $O$) is the focal point $F$, and the focal length is the length of $OF$.




Upon performing these construction steps (on Desmos or other graphing software), we obtain the coordinates of point $O(0.0, 0.0)$ and $X(0.0, 3.0)$. The focal length of the lens is then $\boxed{3.0 \text{ cm}}$.

---

*Alternative solution:* We may also choose to construct the lines analytically rather than graphically.

Equation of $AC$:

$$y - (-4.2) = \frac{10.5 - (-4.20)}{10.0 - (-4.00)}(x - (-4.0))$$
$$y = 1.05x$$

Equation of $BD$:

$$y - (-9.0) = \frac{4.5 - (-9.0)}{-1.5 - 3.0}(x - 3.0)$$
$$y = -3x$$

Clearly, $AC$ and $BD$ intersect at the origin $O$.

Equation of $AB$:

$$y-(-4.2)=\frac{-9.0-(-4.2)}{3.0-(-4.0)}(x-(-4.0))$$
$$y=-\frac{24}{35}x-\frac{243}{35}$$

Equation of $CD$:

$$y-10.5=\frac{10.5-4.5}{10.0-(-1.5)}(x-10.0)$$
$$y=\frac{12}{23}x+\frac{243}{46}$$

Solving these 2 equations, we obtain $X(-10.125,0)$. Hence, it appears the problem setter was lazy and decided to just set the $x$-axis as the lens and have the origin be its centre. Therefore, the foot of the perpendicular from $A$ to the lens is $Y(-4.0,0)$ and therefore, the focal length is just the $y$-intercept of $CY$:

Equation of $CY$:

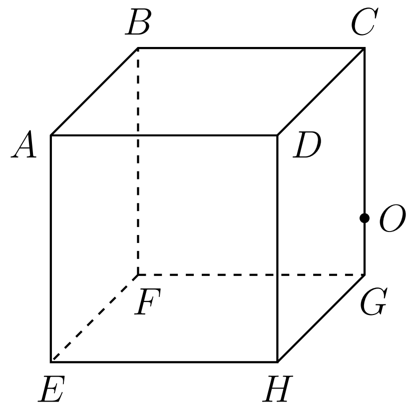
$$y=\frac{10.5-0}{10.0-(-4.0)}(x-(-4.0))$$
$$=0.75x+3.0$$

Therefore, the focal length of the lens is $\boxed{3.0\text{ cm}}$.

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

**Problem 43: Electric Cube** (7 points)

Paul constructs a cube structure $ABCDEFGH$ using 12 uniform, identical wires each with resistance $R = 1.0\ \Omega$, as shown in the figure below. While connecting it to an external circuit via points $A$ and $G$, Paul notices that the contact supposedly at $G$ unexpectedly slips towards $C$ along side $CG$. He denotes $O$ as the new contact point along side $CG$. Given that $O$ can be at any point along $CG$, what is the maximum effective resistance $R_{AO}$ between points $A$ and $O$?

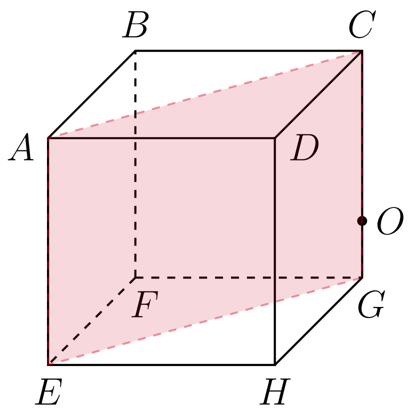


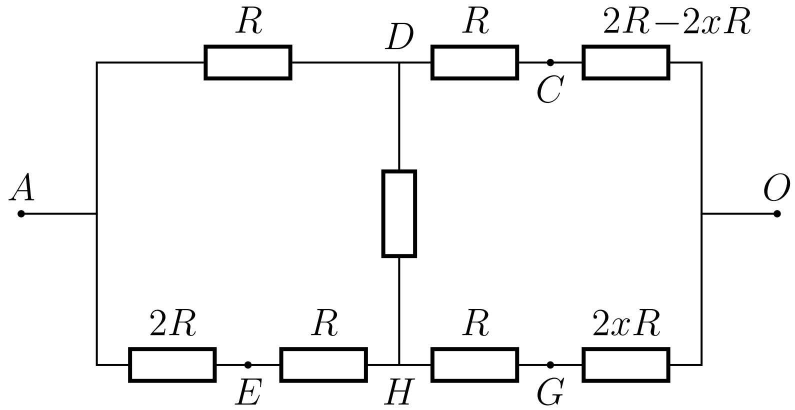

*Leave your answer to 2 significant figures in units of* $\Omega$.

---

*Solution:* Let us denote the ratio of the lengths $GO$ and $CG$ as $x$. Since all wires are uniform, the resistance between $G$ and $O$ is $xR$, and the resistance between $C$ and $O$ is $R - xR$.

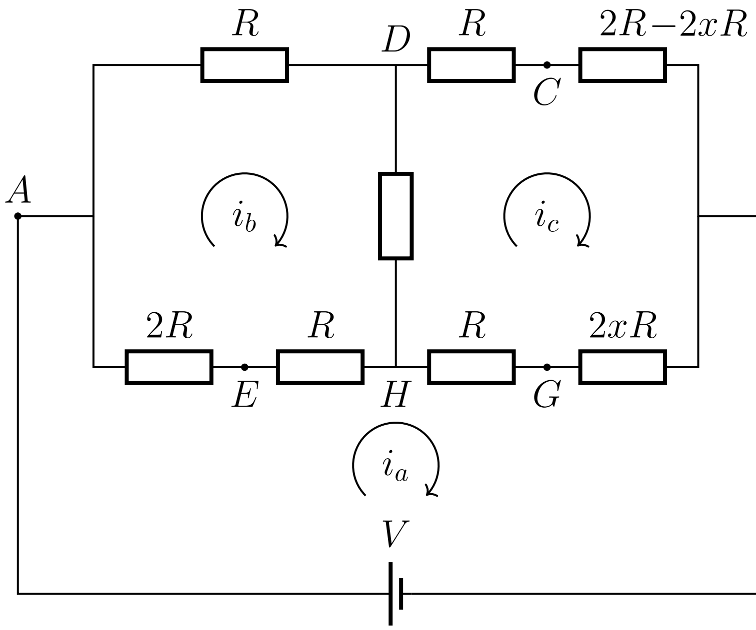
Observe that when connected to an external circuit via points $A$ and $O$, the cube structure is symmetrical with respect to plane $ACGE$, so points $(B, D)$ and $(F, H)$ are equipotential points. We can therefore consider the cube structure as two identical components connected in parallel. The equivalent structure of one such component is shown in the circuit diagram below.




Note that wires $AE$ and $CG$ are shared between the two components, so the cross-sectional areas of the two wires are halved in each component. This explains why in the figure above, the resistance between points $A$ and $E$ is $2R$ and not $R$. The same applies to the resistances between points $G$ and $O$ and between points $C$ and $O$.

From here on, there are a few ways to find the effective resistance of the cube structure. We first present the mesh current method. We begin by adding in an "imaginary" voltage source of electromotive force $V$ between points $A$ and $O$. Then, we may identify current "loops" within the circuit in a way that at least one loop current passes through every component in the circuit. One way to do so is as shown in the figure below.




Here, we label three current loops as shown in the diagram. WLOG, assume the current is flowing clockwise in all three loops. Applying Kirchhoff's Voltage Law to each loop, we can write:

$$(2R + R)(i_a - i_b) + (R + 2xR)(i_a - i_c) = V$$
$$Ri_b + R(i_b - i_c) + (2R + R)(i_b - i_a) = 0$$
$$(R + 2R - 2xR)i_c + (R + 2xR)(i_c - i_a) + R(i_c - i_b) = 0$$

Solving these three equations simultaneously, we obtain:

$$i_a = \frac{6V}{(10 + 4x - 5x^2)R}$$

Hence, the effective resistance $R'_{\text{eff}}$ of the component shown in the figure above is:

$$R'_{\text{eff}} = \frac{V}{i_a} = \frac{10 + 4x - 5x^2}{6}R$$

The effective resistance $R_{\text{eff}}$ of the cube structure between points $A$ and $O$ is half of

$R'_{\text{eff}}$, as there are two of these components in parallel. We have:

$$R_{\text{eff}} = \frac{R'_{\text{eff}}}{2} = \frac{10 + 4x - 5x^2}{12} R$$

We then differentiate the expression with respect to $x$:

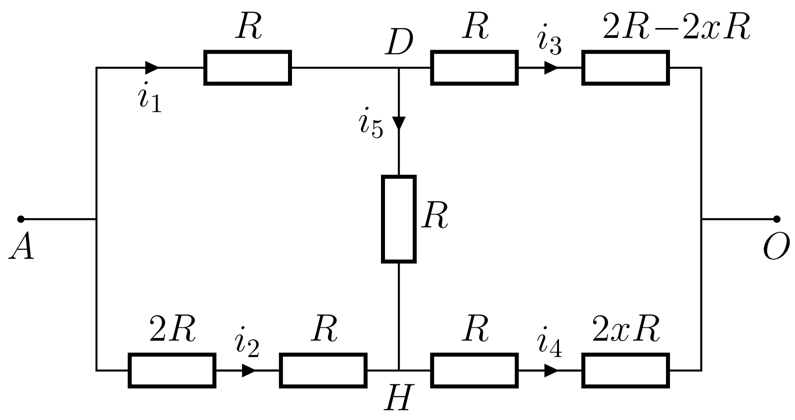
$$\frac{\mathrm{d}R_{\text{eff}}}{\mathrm{d}x} = \frac{2 - 5x}{6} R$$

Setting $\frac{\mathrm{d}R_{\text{eff}}}{\mathrm{d}x} = 0$, we obtain $x = 0.40$. We can verify that this value of $x$ yields the maximum value of $R_{\text{eff}}$ since $\frac{\mathrm{d}^2 R_{\text{eff}}}{\mathrm{d}x^2} = -\frac{5}{6}R < 0$ for all values of $x$. Substituting this value of $x$ into the equation for $R_{\text{eff}}$, we obtain:

$$R_{AO} = \boxed{0.90\ \Omega}$$

---

*Alternative solution:* We can also attempt to solve for the magnitudes of currents in all branches, as labelled in the figure below.




Using Kirchhoff's Laws, we can write:

$$i_1 = i_3 + i_5 \tag{1}$$
$$i_2 + i_5 = i_4 \tag{2}$$
$$i_1 R + i_5 R = i_2(2R + R) \tag{3}$$
$$i_4(R + 2xR) + i_5 R = i_3(R + 2R - 2xR) \tag{4}$$
$$i_1 R + i_3(R + 2R - 2xR) = V \tag{5}$$

where $V$ is the potential difference between points $A$ and $O$.

Using (1) to (4) to eliminate $i_4$ and $i_5$, it follows that:

$$i_2 = \frac{4 - x}{8 + x} i_1 \tag{6}$$
$$i_3 = \frac{4 + 5x}{8 + x} i_1 \tag{7}$$

Using (5) to (7) to eliminate $i_3$, it follows that:

$$i_1 = \frac{8+x}{20+8x-10x^2}\frac{V}{R}$$
$$i_2 = \frac{4-x}{20+8x-10x^2}\frac{V}{R}$$

Hence, the effective resistance $R'_{\text{eff}}$ of the component shown in the figure above is

$$R'_{\text{eff}} = \frac{V}{i_1+i_2} = \frac{10+4x-5x^2}{6}R$$

which is identical to that found via the first method.

**Setter: Liu Yueyang,** yueyang.liu@sgphysicsleague.org

## Problem 44: Heat Capacity

A perfectly conducting block with heat capacity $C = 455$ J K$^{-1}$ is connected to a heat source by a cylindrical rod of negligible heat capacity, length $L = 91.0$ cm, radius $r = 3.25$ cm and conductivity $k = 398$ W m$^{-1}$ K$^{-1}$. The heat source has temperature $T_s(t) = T_0 + \Delta T \sin(\omega t)$ at time $t$, where $T_0 = 299$ K, $\Delta T = 52.0$ K and $\omega = 0.0161$ s$^{-1}$. We may assume that the setup is given sufficiently long to reach steady-state before time $t = 0$ and that there is negligible heat loss to the surroundings.

(a) Find the maximum temperature $T_{\text{max}}$ of the block after time $t = 0$.

*Leave your answer to 3 significant figures in units of* K. (4 points)

(b) At what time $t_0$ does the block first reach this temperature after time $t = 0$?

*Leave your answer to 3 significant figures in units of* s. (2 points)

---

*Solution:* Let the temperature of the block be $T(t)$. The rate $\dot{q}$ at which heat is transferred across the cylindrical rod is given by:

$$\dot{q} = \frac{k\pi r^2}{L}(T_0 + \Delta T \sin(\omega t) - T(t))$$

As heat is transferred to the block, the rate at which its temperature changes $\frac{\mathrm{d}T}{\mathrm{d}t}$ is given by:

$$\dot{q} = C\frac{\mathrm{d}T(t)}{\mathrm{d}t}$$

We observe that this setup is mathematically equivalent to an RC circuit with $\dot{q}$ as the "current" and $T(t)$ as the "potential". The heat source is like an alternating voltage source of magnitude $\Delta T$ and angular frequency $\omega$, the rod is like a resistor with resistance $\frac{L}{\pi k r^2}$ and the block is like a capacitor with capacitance $C$. We may hence calculate the impedance $Z = \frac{L}{\pi k r^2} + \frac{1}{i\omega C}$.

From the voltage divider rule, the temperature $T(t)$ is then given by:

$$T(t) = T_0 + \Delta T \left|\frac{\frac{1}{i\omega C}}{Z}\right| \sin(\omega t + \phi)$$

where $\phi$ is the phase angle given by $\phi = \arg\left(\frac{\frac{1}{i\omega C}}{Z}\right) = -\tan^{-1}\left(\frac{\omega L C}{\pi k r^2}\right)$.

(a) $T_{\text{max}}$ occurs when $\sin(\omega t + \phi) = 1$:

$$T_{\text{max}} = T_0 + \Delta T \left|\frac{\frac{1}{i\omega C}}{Z}\right| \approx \boxed{309 \text{ K}}$$

(b) Since $\omega t + \phi = \frac{\pi}{2}$ when $T(t) = T_{\text{max}}$, $t_0 = \frac{1}{\omega}\left(\frac{\pi}{2} - \phi\right) \approx \boxed{183 \text{ s}}$.

**Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org**

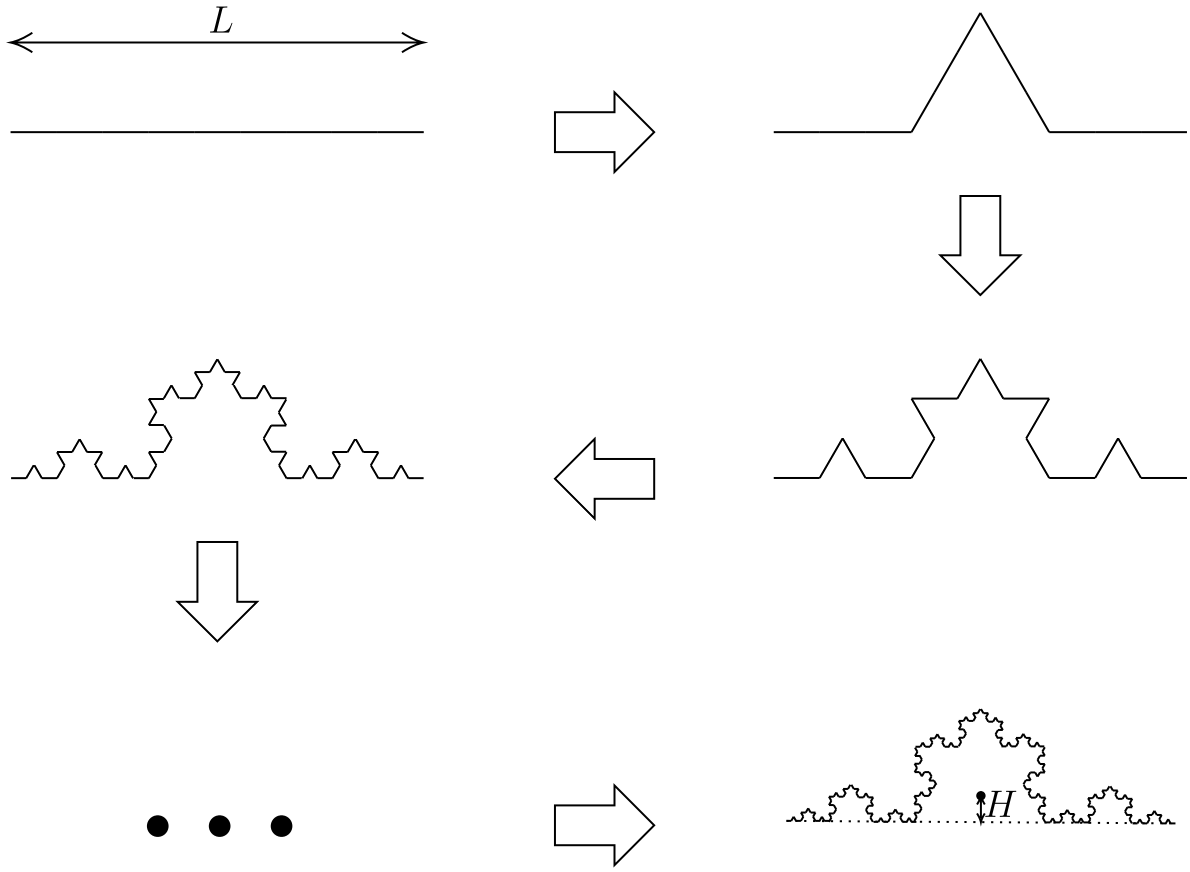
**Problem 45: Snowflake Fragment** (5 points)

A fractal is produced as follows:

1. Start with a uniform thin straight wire of length $L = 1.00$ m.
2. Cut out the middle third of the wire segment, and stretch it to twice its initial length. Its mass remains constant, distributed uniformly along its length (i.e. this new segment has half the mass per unit length of the original segment).
3. Bend the wire into two such that the two branches form an angle of 60°, and reattach it to the other twos segments, facing outwards.
4. Repeat Steps 2 and 3 for each straight segment of the wire recursively, down to arbitrarily small scales.

The resulting fractal-shaped wire has centre of mass at a distance $H$ from the original centre-of-mass position of the straight wire. Find the value of $H$.

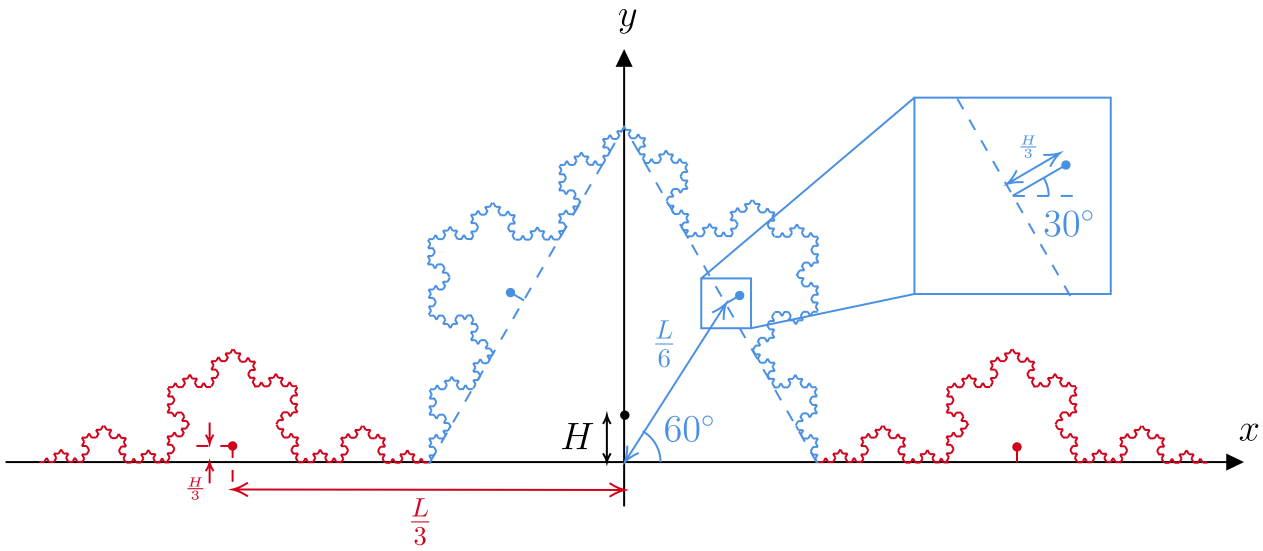
*Leave your answer to 3 significant figures in units of* cm.




---

*Solution:* Let the mass of the wire be $M$, a constant throughout the fractal creation process.

We define a coordinate system such that the $x$-axis points in the initial direction of the straight wire, with the origin at the centre of the initially straight wire. The centre of mass is therefore at $\mathbf{c} = H\hat{\mathbf{j}}$.




We split the fractal into 4 smaller segments, highlighted in red and blue. Note that the blue segments have mass $\frac{M}{6}$ while the red segments have mass $\frac{M}{3}$. Each one of the segments is similar to the original fractal, and have been scaled by a factor of $\frac{1}{3}$. Hence, the centres of mass of the red segments have position vectors:

$$\mathbf{r}_{\pm} = \pm\frac{L}{3}\,\hat{\mathbf{i}} + \frac{H}{3}\,\hat{\mathbf{j}}$$

Similarly, the centres of mass of the blue segments are at the same relative position within the segments. We can find their exact coordinates via trigonometry as shown in the figure above:

$$\begin{aligned}\mathbf{b}_{\pm} &= \pm\frac{L}{6}\cos 60^\circ\,\hat{\mathbf{i}} + \frac{L}{6}\sin 60^\circ\,\hat{\mathbf{j}} \pm \frac{H}{3}\cos 30^\circ\,\hat{\mathbf{i}} + \frac{H}{3}\sin 30^\circ\,\hat{\mathbf{j}} \\ &= \pm\left(\frac{L}{12} + \frac{\sqrt{3}H}{6}\right)\hat{\mathbf{i}} + \left(\frac{\sqrt{3}L}{12} + \frac{H}{6}\right)\hat{\mathbf{j}}\end{aligned}$$

We can apply the equation for centre of mass for the centres of mass of these four segments to obtain:

$$\begin{aligned}\mathbf{c} &= \frac{\frac{M}{3}\left(\mathbf{r}_+ + \mathbf{r}_-\right) + \frac{M}{6}\left(\mathbf{b}_+ + \mathbf{b}_-\right)}{M} \\ \mathbf{c} &= \frac{1}{3}\left(\mathbf{r}_+ + \mathbf{r}_-\right) + \frac{1}{6}\left(\mathbf{b}_+ + \mathbf{b}_-\right) \\ H\hat{\mathbf{j}} &= \frac{1}{3}\left(\frac{2H}{3}\right)\hat{\mathbf{j}} + \frac{1}{6}\left(\frac{\sqrt{3}L}{6} + \frac{H}{3}\right)\hat{\mathbf{j}} \\ 6H &= \frac{4H}{3} + \frac{\sqrt{3}L}{6} + \frac{H}{3}\end{aligned}$$

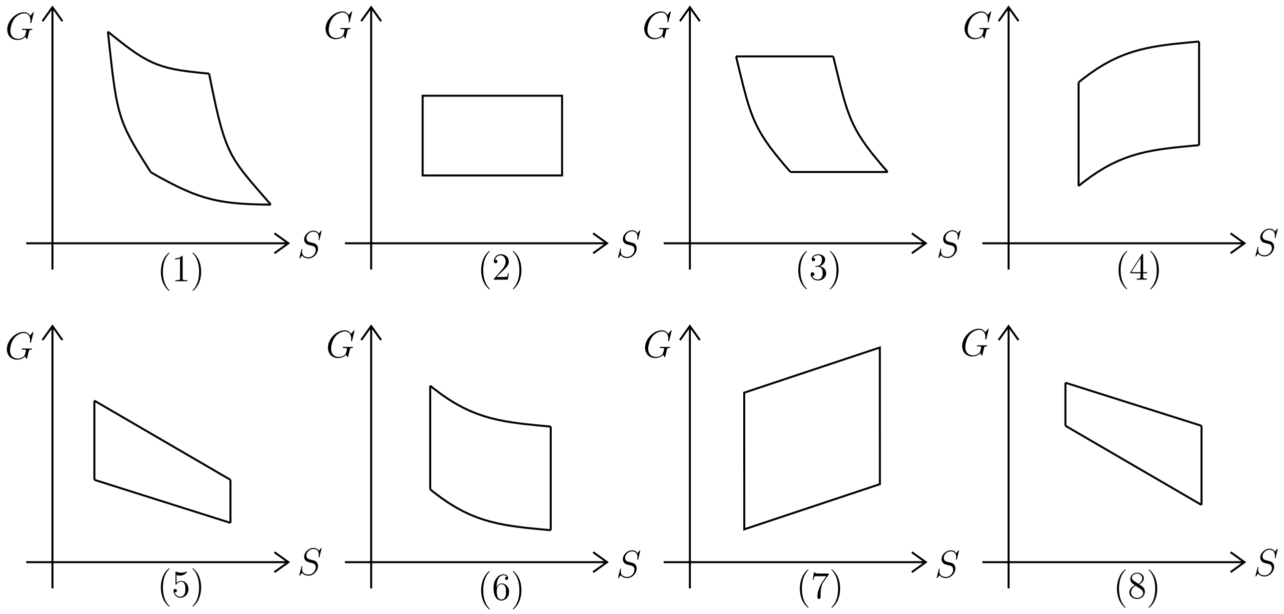
$$\frac{13H}{3} = \frac{\sqrt{3}L}{6}$$
$$H = \frac{\sqrt{3}}{26}L$$
$$\approx \boxed{6.66 \text{ cm}}$$

*Note:* This fractal is part of what is known as Koch's snowflake.

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

**Problem 46: Carnot's Strange Cycle**

(a) The Gibbs free energy of a gas (with internal energy $U$, pressure $p$, volume $V$, temperature $T$ and entropy $S$) is defined as $G = U + pV - TS$. For a Carnot engine using an ideal gas working between 2 large heat reservoirs, which $G$-$S$ diagram is the most accurate representation of the cycle, and in which direction? You may assume that $G$ and $S$ are defined such that they are always positive.




*Leave your answer as a non-zero integer from -8 to 8. If you think the cycle is clockwise, leave you answer as the option number. If you think the cycle is anti-clockwise, add a negative sign in front of your answer.* (3 points)

(b) Suppose the Carnot engine uses $n = 0.800$ mol of ideal monoatomic gas, between 2 large heat reservoirs at temperature $T_h = 400$ K and $T_c = 300$ K. The volume of the gas doubles during the isothermal expansion at temperature $T_h$. Furthermore, the maximum entropy of the system is three times the minimum entropy. What is the area $A$ enclosed by the $G$-$S$ graph?

*Leave your answer to 2 significant figures in units of* $\mathrm{J^2\ K^{-1}}$. (4 points)

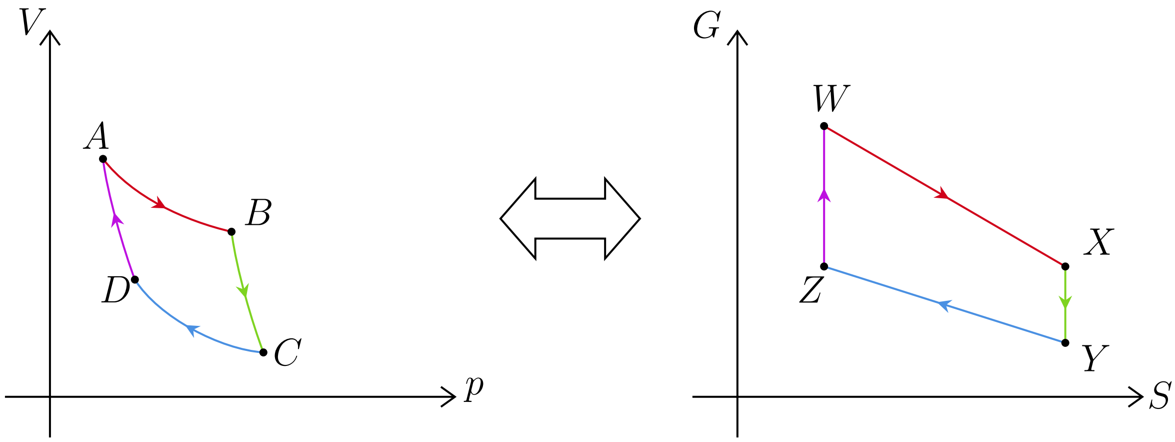
---

*Solution:*

(a) A Carnot cycle consists of 2 adiabatic processes, and 2 isothermal processes. We therefore have a few considerations:

- During the adiabatic processes, entropy $S$ is constant, since $\mathrm{d}S = \frac{\mathrm{d}Q}{T} = 0$. This corresponds to 2 straight vertical lines on the $G$-$S$ diagram.
- During the isothermal processes, internal energy $U$ is constant, and since $U$ is proportional to $pV$, $pV$ is also constant. Finally, since $T$ is also clearly constant, we have $G = \alpha - TS$ for some constant $\alpha$, i.e. Gibbs free energy

$G$ varies linearly with entropy $S$. This corresponds to 2 straight lines with negative gradients on the $G$-$S$ diagram. Hence, the correct diagram is either (5) or (8).




- Referring to the $pV$ diagram above, consider the adiabatic process $B \to C$. Entropy remains constant while the temperature decreases. For an ideal gas, we have:

$$\begin{aligned} G &= U + pV - TS \\ &= nc_V T + nRT - TS \\ &= (nc_V + nR - S)T \end{aligned}$$

  Hence, $G$ decreases in the process $B \to C$. Similarly, $G$ increases in the adiabatic process $D \to A$. Since the entropy is higher at $BC$ compared to $DA$, by the Second Law of Thermodynamics, the cycle goes clockwise. $BC$ corresponds to the vertical line on the right, while $DA$ corresponds to the vertical line on the left.

- Consider the isothermal processes $A \to B$ and $C \to D$. Since $G = \alpha - TS$ for some constant $\alpha$, the straight line corresponding to $A \to B$ is steeper than the straight line corresponding to $C \to D$. Therefore, diagram 5 is the correct answer.

Therefore, the final answer is $\boxed{5}$.

(b) $WXYZ$ is a trapezium. Hence, we only need the lengths of $WZ$, $XY$ and the height of the trapezium to find its area A.

- The height of the trapezium is the entropy change along process $AB$. The change in entropy can be calculated using the First Law of Thermodynamics:

$$\begin{aligned} \Delta S &= \frac{Q_h}{T_h} \\ &= \frac{1}{T_h}(\Delta U - W_{AB}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T_h}(-W_{AB}) \\
&= \frac{1}{T_h}\int_{V_A}^{V_B} p\,\mathrm{d}V \\
&= \frac{1}{T_h}\int_{V_A}^{V_B} \frac{nRT_h}{V}\,\mathrm{d}V \\
&= nR(\ln V_B - \ln V_A) \\
&= nR\ln\left(\frac{V_B}{V_A}\right)
\end{aligned}
$$

- The length of $WZ$ is the difference in Gibbs free energy between W and Z:

$$
\begin{aligned}
G_W - G_Z &= U_W - U_Z + p_W V_W - p_Z V_Z - T_W S_W + T_Z S_Z \\
&= \frac{3}{2}nR(T_A - T_D) + nR(T_A - T_D) - S_A(T_A - T_D) \\
&= \left(\frac{5}{2}nR - S_A\right)(T_A - T_D)
\end{aligned}
$$

- Similarly, the length of $XY$ is the difference in Gibbs free energy between X and Y:

$$
\begin{aligned}
G_X - G_Y &= U_X - U_Y + p_X V_X - p_Y V_Y - T_X S_X + T_Y S_Y \\
&= \frac{3}{2}nR(T_B - T_C) + nR(T_B - T_C) - S_B(T_B - T_C) \\
&= \left(\frac{5}{2}nR - S_B\right)(T_B - T_C)
\end{aligned}
$$

Since $S_B = 3S_A$ and $S_B - S_A = \Delta S$, we have $S_A = \frac{1}{2}\Delta S$, $S_B = \frac{3}{2}\Delta S$ and $S_A + S_B = 2\Delta S$. Using the fact that $T_A = T_B$ and $T_C = T_D$, we obtain:

$$
\begin{aligned}
A &= \frac{1}{2}\Delta S\left((G_W - G_Z) + (G_X - G_Y)\right) \\
&= \frac{1}{2}\Delta S\left(\left(\frac{5}{2}nR - S_A\right)(T_A - T_D) + \left(\frac{5}{2}nR - S_B\right)(T_A - T_D)\right) \\
&= \frac{1}{2}\Delta S(T_A - T_D)(5nR - (S_B + S_A)) \\
&= \frac{1}{2}\Delta S(T_A - T_D)(5nR - 2\Delta S) \\
&= \frac{1}{2}nR\ln\left(\frac{V_B}{V_A}\right)(T_A - T_D)\left(5nR - 2nR\ln\left(\frac{V_B}{V_A}\right)\right) \\
&= \frac{1}{2}n^2R^2(T_h - T_c)(5 - 2\ln 2)\ln 2 \\
&\approx \boxed{5500\ \mathrm{J^2\ K^{-1}}}
\end{aligned}
$$

**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

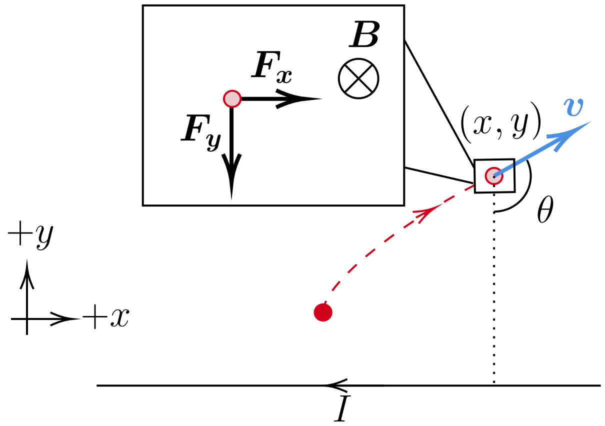
**Problem 47: Revenge of the Crazy Electron** (5 points)

An electron is projected with initial velocity $v_0 = 150$ m s$^{-1}$ at an initial distance $y_0 = 0.36$ m from an infinitely long wire carrying current $I = 0.0014$ A. The direction of initial velocity is perpendicular and directed away from the wire. Determine the maximum distance $y_{\max}$ of the electron from the wire.

*Leave your answer to 2 significant figures in units of* m.

---

*Solution:*




Let the electron have charge $-e$ and mass $m$. Define the $x$ and $y$ axes as positive rightwards and upwards respectively. Only the magnetic force $F_m$ acts on the electron, due to the magnetic field of the wire pointing into the page. We split the magnetic force into its $x$ and $y$ components, expressed in terms of the $x$ and $y$ velocity components $v_x$ and $v_y$ as follows:

$$F_x = Bev_y$$
$$F_y = -Bev_x$$

We can use our expression for $F_x$ to find $v_x$ as a function of $y$:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = Be\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$m\ \mathrm{d}v_x = \frac{\mu_0 I}{2\pi y}e\ \mathrm{d}y$$
$$\int_0^{v_x} \mathrm{d}v_x = \frac{\mu_0 I e}{2\pi m}\int_{y_0}^{y} \frac{1}{y}\,\mathrm{d}y$$
$$v_x = \frac{\mu_0 I e}{2\pi m}\ln\frac{y}{y_0}$$

Since the magnetic force does no work on the electron, the kinetic energy of the electron remains constant throughout its motion i.e. ${v_x}^2 + {v_y}^2 = {v_0}^2$. When the electron is at maximum distance from the wire, $v_y = 0$. Hence, $v_x = v_0$. We can use this deduction to solve for $y_{\max}$:

$$\begin{aligned} v_0 &= \frac{\mu_0 I e}{2\pi m} \ln \frac{y_{\max}}{y_0} \\ y_{\max} &= y_0 e^{\frac{2\pi m v_0}{\mu_0 I e}} \\ &\approx \boxed{7.6 \text{ m}} \end{aligned}$$

---

*Alternative solution:* We can also proceed via manual integration. Similarly to $F_x$ and $F_y$, we find $v_y$ as a function of $y$. We use the negative of $\frac{\mathrm{d}v_y}{\mathrm{d}t}$ since the magnetic force causes $v_y$ to decrease. We have:

$$\begin{aligned} -m\frac{\mathrm{d}v_y}{\mathrm{d}t} &= Bev_x \\ &= \frac{\mu_0 I e}{2\pi y}\left(\frac{\mu_0 I e}{2\pi m} \ln \frac{y}{y_0}\right) \\ &= \frac{{\mu_0}^2 I^2 e^2}{4\pi^2 m}\left(\frac{1}{y} \ln \frac{y}{y_0}\right) \end{aligned}$$

Using $\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = v_y \dfrac{\mathrm{d}v_y}{\mathrm{d}y}$, we obtain the following differential equation:

$$\begin{aligned} -v_y\frac{\mathrm{d}v_y}{\mathrm{d}y} &= \frac{{\mu_0}^2 I^2 e^2}{4\pi^2 m^2}\left(\frac{1}{y} \ln \frac{y}{y_0}\right) \\ -\int_{v_0}^{v_y} v_y \, \mathrm{d}v_y &= \frac{{\mu_0}^2 I^2 e^2}{4\pi^2 m^2} \int_{y_0}^{y} \frac{1}{y} \ln \frac{y}{y_0} \, \mathrm{d}y \\ \left[-\frac{{v_y}^2}{2}\right]_{v_y}^{v_0} &= \frac{{\mu_0}^2 I^2 e^2}{4\pi^2 m^2} \left[\frac{1}{2}\left(\ln \frac{y}{y_0}\right)^2\right]_{y_0}^{y} \\ -\left(\frac{{v_y}^2}{2} - \frac{{v_0}^2}{2}\right) &= \frac{{\mu_0}^2 I^2 e^2}{8\pi^2 m^2}\left(\ln \frac{y}{y_0}\right)^2 \end{aligned}$$

When the electron is furthest away from the wire, $v_y = 0$. We have:

$$\begin{aligned} {v_0}^2 &= \frac{{\mu_0}^2 I^2 e^2}{4\pi^2 m^2}\left(\ln \frac{y}{y_0}\right)^2 \\ \ln \frac{y}{y_0} &= \pm \frac{2\pi m v_0}{\mu_0 I e} \\ y &= y_0 e^{\pm \frac{2\pi m v_0}{\mu_0 I e}} \end{aligned}$$

To find the maximum distance from the wire, take the positive index to obtain:

$$\begin{aligned} y_{\max} &= y_0 e^{\frac{2\pi m v_0}{\mu_0 I e}} \\ &\approx \boxed{7.6 \text{ m}} \end{aligned}$$

**Setter: Roger Zhang, roger.zhang@sgphysicsleague.org**

## Problem 48: Hot Ring

Annatar has a ring of radius $R = 0.700$ m, situated in the $x$-$y$ plane with its centre at the origin (the $z$-direction points upwards). The ring has a total surface area of $A = 0.0300$ m$^2$ and a small cross-sectional area $S \ll A$. The ring is a perfect black body with temperature $T_0 = 10000$ K. A small thin disc with radius $r = 1.00$ mm $\ll R$ is located at a height $H = 1.75$ m directly above the centre of the ring (so the centre of the disc is at $(0, 0, H)$). The plane of the disc is parallel to the $x$-$y$ plane.

(a) Calculate the intensity of thermal radiation $I$ at the centre of the disc.

*Leave your answer to 2 significant figures in units of* kW m$^{-2}$. (2 points)

(b) Calculate the equilibrium temperature $T$ of the disc. Assume heat is lost from the disc only through radiation. The disc can be treated as a perfect heat conductor, and has emissivity $\varepsilon = 0.170$.

*Leave your answer to 2 significant figures in units of* K. (3 points)

(c) The disc is able to hover in position, due to the radiation pressure from the thermal radiation of the ring. Calculate the mass $m$ of the disc.

*Leave your answer to 2 significant figures in units of* ng. (3 points)

---

*Solution:*

(a) By the Stefan-Boltzmann Law, the power $P$ of the thermal radiation emitted by the ring is:

$$P = \sigma A T_0^4$$

The distance between every point on the ring and the disc's centre is $\sqrt{R^2 + H^2}$. Hence, the intensity of thermal radiation at the centre of the disc is:

$$\begin{aligned} I &= \frac{P}{4\pi\left(\sqrt{R^2+H^2}\right)^2} \\ &= \frac{\sigma A T_0^4}{4\pi\left(R^2+H^2\right)} \\ &\approx \boxed{380 \text{ kW m}^{-2}} \end{aligned}$$

(b) The power absorbed by the disc $P_{\text{in}}$ is:

$$\begin{aligned} P_{\text{in}} &= \varepsilon \pi r^2 I \frac{H}{\sqrt{R^2+H^2}} \\ &= \frac{\varepsilon \sigma A H r^2 T_0^4}{4\left(R^2+H^2\right)^{3/2}} \end{aligned}$$

The extra factor of $\frac{H}{\sqrt{R^2+H^2}}$ is because the photons are not travelling normal to the disc, so the effective area of the disc to the photons is $\pi r^2 \frac{H}{\sqrt{R^2+H^2}}$.

Let the equilibrium temperature of the disc be $T$. Since the disc has an upper and lower surface, the disc emits thermal radiation with power $P_{\text{out}} = \sigma\varepsilon(2\pi r^2)T^4$ At the disc's thermal equilibrium, we equate $P_{\text{in}}$ to $P_{\text{out}}$:

$$\begin{aligned}
P_{\text{in}} &= P_{\text{out}} \\
\frac{\varepsilon\sigma A H r^2 T_0^4}{4\left(R^2+H^2\right)^{3/2}} &= \sigma\varepsilon(2\pi r^2)T^4 \\
\frac{AHT_0^4}{8\pi\left(R^2+H^2\right)^{3/2}} &= T^4
\end{aligned}$$

Solving for $T$, we have:

$$\begin{aligned}
T &= \left(\frac{AH}{8\pi\left(R^2+H^2\right)^{3/2}}\right)^{1/4} T_0 \\
&\approx \boxed{1300 \text{ K}}
\end{aligned}$$

(c) For light of intensity $I$ shining normally onto a surface with emissivity $\varepsilon$, the intensity of light absorbed is $\varepsilon I$ and the intensity of light reflected is $(1-\varepsilon)I$.

For the absorbed light, if the light has wavelength $\lambda$, the number of photons absorbed per unit time per unit area is $\frac{\varepsilon I \lambda}{hc}$. Since each photon has momentum $\frac{h}{\lambda}$, the momentum transferred to the surface per unit time per unit area (i.e. pressure) is $\frac{\varepsilon I\lambda}{hc} \cdot \frac{h}{\lambda} = \frac{\varepsilon I}{c}$. Through a similar argument, the pressure from the reflected light is $\frac{2(1-\varepsilon)I}{c}$ (since each photon transfers $\frac{2h}{\lambda}$ of momentum). The total pressure on the surface of the disc is $\frac{\varepsilon I}{c} + \frac{2(1-\varepsilon)I}{c} = (2-\varepsilon)\frac{I}{c}$.

Back to the problem, the thermal radiation received by the disc is not normal to the surface of the disc. This results in the effective area of the disc being only $\frac{H}{\sqrt{R^2+H^2}}\pi r^2$, as seen in part (b). Furthermore, by symmetry, the $x$ and $y$ components of the radiation cancel out, so we only take into account the $z$-component of the radiation. This multiplies the radiation by another factor of $\frac{H}{\sqrt{R^2+H^2}}$. Hence, the radiation pressure $p$ is:

$$\begin{aligned}
p &= (2-\varepsilon)\frac{I}{c}\frac{H^2}{R^2+H^2} \\
&= (2-\varepsilon)\frac{1}{c}\frac{H^2}{R^2+H^2}\frac{\sigma A T_0^4}{4\pi\left(R^2+H^2\right)} \\
&= (2-\varepsilon)\frac{\sigma A T_0^4}{4\pi c}\frac{H^2}{\left(R^2+H^2\right)^2}
\end{aligned}$$

Since the disc is in translational equilibrium, we equate the force due to radiation to the weight of the disc:

$$\begin{aligned} p\pi r^2 &= mg \\ m &= \frac{\pi r^2}{g}(2-\varepsilon)\frac{\sigma A T_0^4}{4\pi c}\frac{H^2}{\left(R^2+H^2\right)^2} \\ &= (2-\varepsilon)\frac{\sigma A r^2 T_0^4}{4cg}\frac{H^2}{\left(R^2+H^2\right)^2} \\ &\approx \boxed{0.64\text{ ng}} \end{aligned}$$

*Note:* 10000 K is extremely hot, and probably cannot be made out of any known physical material without the ring boiling. Despite the immense amount of power needed to maintain such a temperature, the mass that it can levitate is extremely tiny. Thus, using thermal radiation pressure to levitate something is very impractical.

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

**Problem 49: Bobbing Ball** (5 points)

A uniform solid spherical ball of radius $r$ floats in water. The ball is displaced vertically downwards by a distance $a \ll r$ and released from rest, after which it oscillates with period $T$. After a long time $t_1 \gg T$, its amplitude of oscillations drops below a fixed value $\delta \ll a$. If a ball of the same material and increased radius $2r$ is displaced by the same distance $a$, the time taken for its amplitude to drop below $\delta$ becomes $t_2$. Determine the ratio $t_2/t_1$.

Assume that the damping force on the ball is given by Stokes' Law as $F = -6\pi r\eta v$, where $\eta$ is the viscosity of the water and $v$ is the velocity of the ball. Assume also that the ball's oscillations are underdamped.

*Leave your answer to 2 significant figures.*

---

*Solution:* Let the $x$-axis be the vertical axis, with $x = 0$ denoting the equilibrium position of the ball in water. Let $\rho$ be the density of the ball and $\rho_w$ be the density of water. The mass of the ball is given by $\rho\frac{4}{3}\pi r^3$.

When the ball is displaced downwards by a small distance $x$ from equilibrium, the upthrust on the ball increases by $\rho_w \pi r_0^2 x$, where $r_0$ is the radius of the cross section of the ball at the water surface which remains approximately constant during the small oscillations.

Furthermore, according to Stokes' Law, the damping force on the ball is proportional to its radius and its velocity. We can then write an equation of motion for the ball:

$$\rho\frac{4}{3}\pi r^3 \ddot{x} = -\rho_w \pi r_0^2 x - 6\pi r\eta\dot{x}$$

Regardless of the radius of the ball, the same fraction of the ball is submerged at equilibrium, so $r_0$ is proportional to $r$ and the increase in upthrust is proportional to $r^2$. We can rewrite the equation as follows:

$$k_1 r^3 \ddot{x} = -k_2 r^2 x - k_3 r\dot{x}$$

with $k_1$, $k_2$ and $k_3$ as constants of proportionality independent of $r$. This is analogous to the equation for a linearly damped spring-mass system:

$$m\ddot{x} = -kx - b\dot{x}$$

The general solution to this equation for underdamped motion is:

$$x = Ae^{-\frac{t}{\tau}}\sin(\omega t - \phi)$$

where $A$ is the amplitude, $\phi$ is the phase constant, $\tau$ is the time constant and $\omega$ is the angular frequency of the oscillation. The expression for $\tau$ here is:

$$\tau = \frac{2m}{b}$$

For our ball's motion, the corresponding expression for $\tau$ is:

$$\tau = \frac{2k_1 r^3}{k_3 r} \propto r^2$$

The time constant is a measure of the time taken for the amplitude of the oscillation to decrease by a factor $\frac{1}{e}$ of the initial amplitude. Since the oscillations of both balls have the same initial amplitude, the time taken for the amplitude of the balls to decrease to the same value is proportional to the time constants of each oscillation:

$$\frac{t_2}{t_1} = \frac{\tau_2}{\tau_1} = \frac{r_2^2}{r_1^2} = 2^2 = \boxed{4.0}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

### Problem 50: Friendly Fire

The Death Star continuously fires a deadly, perfectly collimated laser with power $P = 9.0 \times 10^{14}$ W at an initially stationary rebel X-wing with mass $m = 100$ kg and positioned a distance $l$ away. Much to Darth Vader's anger, the X-wing has a perfectly reflecting surface, and the moment the first photon fired from the laser returns to him, his laser is destroyed. The X-wing and Death Star are both initially stationary in the lab frame, and the mass of the Death Star is large enough such that it remains effectively stationary throughout.

(a) If $l = 2.0 \times 10^3$ m, find the final velocity of the X-wing in the lab frame.

*Leave your answer to 2 significant figures in units of* m s$^{-1}$. (2 points)

(b) If $l = 2.0 \times 10^{12}$ m, the final velocity of the X-wing in the lab frame is given by $v = \beta c$, where $c$ is the speed of light. Find $\beta$.

*Leave your answer to 2 significant figures.* (5 points)

---

*Solution:*

(a) The laser beam travels at the speed of light, hits the reflective X-wing, and returns to the Death Star. The total time taken for the first photon to return is given by:

$$t = \frac{2l}{c}$$

If the momentum of each photon is $p_1$, its final momentum after reflection is $-p_1$, so it has imparted a momentum of $2p_1$ to the X-wing. The energy $E_1$ of each photon is then given by:

$$E_1 = p_1 c$$

The total energy of all the photons that are shot out is given by:

$$E = Pt = P\frac{2l}{c}$$

The total momentum imparted to the X-wing is therefore:

$$p = 2\frac{E}{c} = \frac{4Pl}{c^2}$$

We can then divide the momentum by the mass of the X-wing to obtain its final velocity $v$:

$$v \approx \boxed{0.80 \text{ m s}^{-1}}$$

(b) If we were to try applying the same equation obtained previously with the new length, we get:

$$v \approx 8 \times 10^8 \text{ m s}^{-1}$$

which should send alarm bells ringing, because this is faster than the speed of light! This suggests that the amount of momentum that has been imparted to the X-wing is so huge that we need to consider relativistic effects.

If we were to apply the same approach as part (a), we quickly find the situation to be very complex - the relativistic Doppler effect comes into play, for instance, as the photons are red-shifted in the frame of the rapidly moving X-wing. To overcome these complexities, we will make use of an important fact: regardless of the wavelength of a photon, we can always apply the equation:

$$E = pc$$

Motivated by this observation, we apply conservation of momentum and energy to the system of the photons and the X-wing. The total momentum of the system is equal to the initial momentum of the photons. Since the time taken for the first photon to return to the laser is still $t = \frac{2l}{c}$, and photons are constantly shot out over this period of time, the total momentum of the system is:

$$p = \frac{Pt}{c} = \frac{2Pl}{c^2}$$

The total energy of the system is the sum of the initial energy of the photons and the rest mass of the X-wing:

$$E = Pt + mc^2 = \frac{2Pl}{c} + mc^2$$

Now, consider the situation after a long time where all the photons have already collided with the X-wing. The photons have a negative momentum $-p_\gamma$, and the X-wing has a positive momentum $p_x$. Since total momentum is conserved, we have:

$$\frac{2Pl}{c^2} = -p_\gamma + p_x \tag{1}$$

The photons have energy $p_\gamma c$ and the X-wing has energy $E_x$. Since total energy is conserved, we have:

$$\frac{2Pl}{c} + mc^2 = p_\gamma c + E_x \tag{2}$$

We also apply the equations for relativistic momentum and energy:

$$p_x = \gamma m v \tag{3}$$

$$E_x = \gamma m c^2 \tag{4}$$

where $v$ is the velocity of the X-wing and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

We can directly solve equations (1) to (4) simultaneously to obtain:

$$v = \frac{4Plc(mc^3 + 2Pl)}{m^2c^6 + 4Plmc^3 + 8P^2l^2}$$

$$\beta = \frac{v}{c} \approx \boxed{0.86}$$

---

*Alternative solution:* A more practical way to solve these equations by hand is to use another equation for relativistic energy:

$$E_x = \sqrt{m^2c^4 + p_x^2c^2}$$

Upon substitution we can solve equations (1) and (2) only, to obtain:

$$p_x = \frac{\frac{4Pl}{c^2}mc + \frac{2Pl}{c^2}}{mc + \frac{2Pl}{c^2}}$$

Then, to obtain $v$, we can rearrange the equation $p_x = \gamma mv$ to obtain:

$$v = \frac{p_x c}{\sqrt{m^2c^2 + p_x^2}}$$

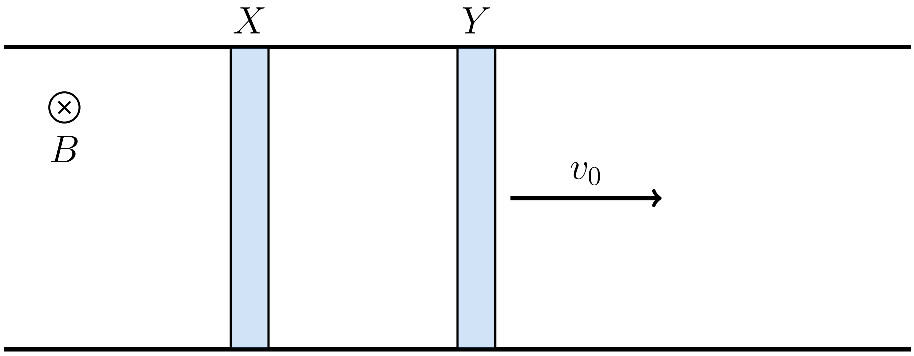
This will give us the same numerical result.

**Setter: Chen Guangyuan,** guangyuan.chen@sgphysicsleague.org

**Problem 51: Catch-up Capacitor** (6 points)

Two conducting rods, $X$ and $Y$, are placed perpendicular on top of smooth, parallel rails. Each rod can be considered a series combination of a capacitor and resistor. The capacitance of $X$ across its length is $C_x = 100\,\mu\text{F}$ and the capacitance of $Y$ across its length is $C_y = 2C_x$. Each of them has resistance $R = 10\ \Omega$ and mass $m = 0.10$ kg. A strong magnetic field $B = 10$ T is directed into the page. The rails are a distance $d = 0.50$ m apart and have zero resistance.

$Y$ moves at an initial speed $v_0 = 10\ \text{ms}^{-1}$ to the right. This induces $X$ to move to the right following $Y$. Sadly, $X$ never catches up with $Y$. Once steady state is reached, at what rate does the distance between them increase?




*Leave your answer to 2 significant figures in units of* m s$^{-1}$.

---

*Solution:* We start by writing out the equations of motion for $X$ and $Y$ respectively. Let the current flowing through the closed loop be $I$. For $X$, we have:

$$ma_X = BIl$$

For $Y$, the current flows in the opposite direction, hence:

$$ma_Y = -BIl$$

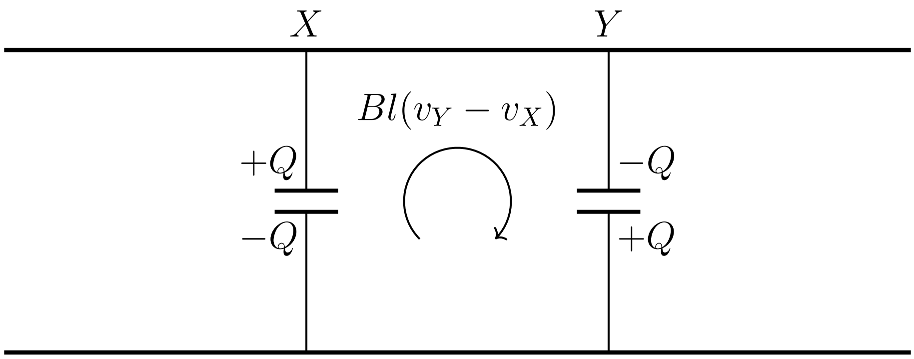
Since we are only interested in the steady state, we can integrate the above equations with respect to time (or directly use the impulse-momentum theorem). Letting the final velocities of $X$ and $Y$ be $v_X$ and $v_Y$ respectively and the final charge accumulated on $X$ and $Y$ be $Q$ (the charge on $X$ and $Y$ is the same due to conservation of charge), we have:

$$mv_X = BQl \tag{1}$$

$$m(v_Y - v_0) = -BQl \tag{2}$$

The rate of increase of the distance between $X$ and $Y$ is the difference in their velocities. Subtracting equation (2) from equation (1), we have:

$$v_Y - v_X = v_0 - \frac{2BQl}{m}$$




By Faraday's Law, the EMF induced due to the change in magnetic flux through the rectangular loop as rods $X$ and $Y$ move is given by $Bl(v_Y - v_X)$. To find $Q$, we use the fact that the current in the closed loop is zero in the steady state. Applying Kirchhoff's Voltage Law along the loop, we have:

$$\frac{Q}{C} + \frac{Q}{2C} + Bl(v_Y - v_X) = 0$$

Solving for $Q$, we obtain:

$$Q = \frac{2}{3}CBl(v_Y - v_X)$$

Substituting the expression for $Q$ into the expression for $v_Y - v_X$, we have:

$$v_Y - v_X = v_0 - \frac{4}{3}\frac{CB^2l^2(v_Y - v_X)}{m}$$

$$v_Y - v_X = \frac{3m}{3m + 4CB^2l^2}v_0$$

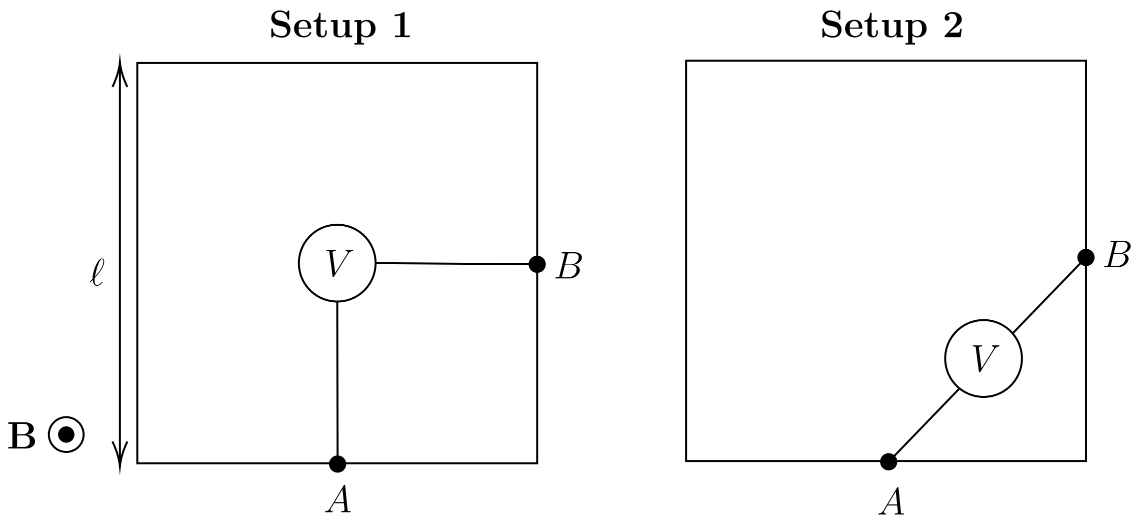
$$\approx \boxed{9.7 \text{ m s}^{-1}}$$

**Setter: Li Yichen, yichen.li@sgphysicsleague.org**

**Problem 52: Sussy Voltmeter** (5 points)

Bobby places a uniform, square-shaped wire with side length $\ell = 1.0$ m and total resistance $R = 10\ \Omega$ in a plane perpendicular to a magnetic field. The magnetic field points out of the page and increases in magnitude at a constant rate $\alpha = 10\ \mathrm{T\ s^{-1}}$.

He wants to measure the potential difference between points $A$ and $B$ as shown, where $A$ and $B$ are both the midpoints of the sides of the square wire. He connects the voltmeter in two different setups, and records two different readings $V_1$ and $V_2$, despite placing the voltmeter probes at the exact same two points on the wire. The voltmeter has an internal resistance $r = 1000\ \Omega$. Determine $|V_1 - V_2|$.

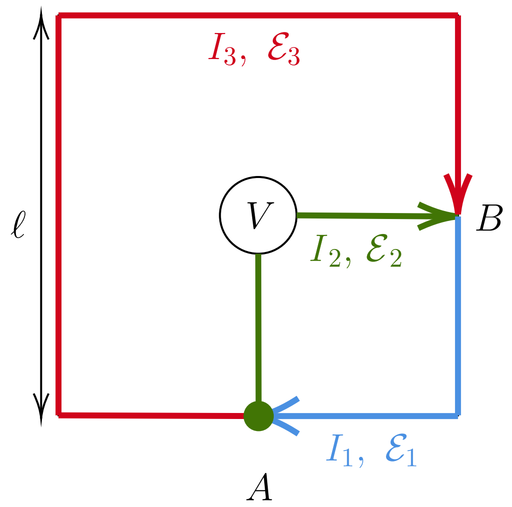



*Leave your answer to 3 significant figures in units of* V.

---

*Solution:* The key to solving this problem is to define the directions of currents and the signs of potentials systematically.

Setup 1: By considering the induced electromotive force (EMF) along the 3 sections, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ (defined to be positive if potential increases counter-clockwise), and current along the three sections as $I_1$, $I_2$ and $I_3$ (also defined positive counter-clockwise, meaning that it leads to a potential drop counter-clockwise), we can write down Kirchhoff's Voltage Law for the three wire loops:




$$V_B - V_A = \mathcal{E}_1 - I_1\frac{R}{4}$$
$$V_A - V_B = \mathcal{E}_2 - I_2 r$$
$$V_A - V_B = \mathcal{E}_3 - I_3\frac{3R}{4}$$

From Faraday's Law, we know further that the total EMF induced in each loop is proportional to the rate of change of magnetic flux linkage, obtaining the following equations:

$$\mathcal{E}_1 + \mathcal{E}_3 = \alpha\ell^2 \tag{1}$$
$$\mathcal{E}_1 + \mathcal{E}_2 = \alpha\frac{\ell^2}{4} \tag{2}$$

By Kirchhoff's Current Law:

$$I_1 = I_2 + I_3 \tag{3}$$

Solving equations (1), (2) and (3) simultaneously, we get $I_2 = 0$. Since the voltmeter reading $V_1 = I_2 r = 0$, the voltage Bobby sees on the voltmeter would be 0.

Setup 2: We have the same equations derived using Kirchhoff's Voltage and Current Laws, but the induction equations are slightly different. This is because the area enclosed by each loop is slightly different.

$$\mathcal{E}_1 + \mathcal{E}_3 = \alpha\ell^2$$
$$\mathcal{E}_1 + \mathcal{E}_2 = \alpha\frac{\ell^2}{8}$$

Solving for $I_2$, we get:

$$I_2 = -\frac{2\ell^2\alpha}{16r + 3R}.$$

Hence, $|V_1 - V_2| = |0 + I_2 r| = \frac{2\ell^2\alpha r}{16r+3R} = \boxed{1.25\text{ V}}$.

**Setter: Li Xinrui,** xinrui.li@sgphysicsleague.org

**Problem 53: Escaped Proton** (6 points)

An infinitely long charged wire with linear charge density $\lambda = 5 \times 10^{-9}$ C m$^{-1}$ and radius $R = 0.1$ m is placed inside a uniform magnetic field $B = 0.1$ T directed parallel to the wire. A proton leaves the surface of the wire at an initial speed $v_i$ in an arbitrary direction. What is the **minimum** magnitude of $v_i$ required such that the **maximum** distance from the surface of the wire that the proton can reach is also $R$?

*Leave your answer to 3 significant figures in units of* m s$^{-1}$.

---

*Solution:* From Gauss' Law, we can find the electric field and potential at a distance $r$ from the wire ($r > R$):

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$
$$V(r) = -\int_{r_0}^{r} E(r)\,\mathrm{d}r$$
$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r}$$

where $r_0$ is the reference point with zero potential. By conservation of energy, the velocity of the particle must hence change as it travels. We consider each of its perpendicular components separately.

We notice that the axial component of the proton's velocity is parallel to the magnetic field and perpendicular to the electric field. Hence, it does not experience any forces due to this component. This component thus does not affect the maximum distance the proton can reach. To minimise the proton's initial velocity, we may set this component to 0 and only consider the proton's motion in the plane perpendicular to the wire.

We separate the proton's velocity into its radial component $v_r$ and tangential component $v_t$. The angular momentum of the proton is then $L = mv_t r$, and the torque about the wire by the magnetic field is $\tau = F_B r = ev_r Br$. Since the electric field is radial, it contributes no torque. Hence, we have:

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t}$$
$$eBr\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}t}$$

Integrating both sides with respect to time, we have:

$$\frac{1}{2}eBr^2 - \frac{1}{2}eBr_0^2 = L - L_0$$
$$= m_p v_t r - L_0$$
$$m_p v_t r - \frac{1}{2}eBr^2 = L_0 - \frac{1}{2}eBr_0^2$$

which is constant. This also implies that it is impossible for the proton to reach infinity. When the proton is farthest away from the wire, we have $v_{r,f} = \frac{\mathrm{d}r}{\mathrm{d}t} = 0$, hence the final velocity is purely tangential, i.e. $v_f = v_{t,f}$.

Let the farthest distance the proton can reach be $r_f = 0.2\,\mathrm{m}$. From conservation of energy, we have:

$$\frac{1}{2}m_p v_i^2 + e\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{R} = \frac{1}{2}m_p v_f^2 + e\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r_f}$$
$$\implies \frac{1}{2}m_p v_f^2 - \frac{1}{2}m_p(v_{t,i}^2 + v_{r,i}^2) = e\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_f}{R} \tag{1}$$

Using the conserved quantity $m_p v_t r - \frac{1}{2}qBr^2$ we derived earlier:

$$m_p v_{t,i}R - \frac{1}{2}eBR^2 = m_p v_f r_f - \frac{1}{2}eBr_f^2 \tag{2}$$

From (1), to minimise the initial velocity $v_i = \sqrt{v_{t,i}^2 + v_{r,i}^2}$, we need to minimise $v_f$. If we assume that $v_f$ is positive, then from (2), to minimise $v_f$, we need to minimise $v_{t,i}$. This is achieved when $v_i = v_{t,i}$ and when $v_{t,i}$ is negative. Letting $v_{r,i} = 0$ and solving equations (1) and (2) simultaneously, we have:

$$v_f \approx 483215\,\mathrm{m\ s^{-1}}$$
$$v_{t,i} \approx -470695\,\mathrm{m\ s^{-1}}$$

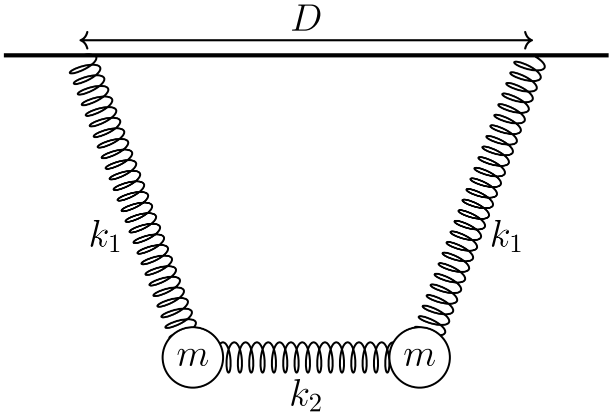
Our assumption that $v_f$ is positive is satisfied, hence $v_i$ has minimum magnitude $\boxed{471000\ \mathrm{m\ s^{-1}}}$ and is purely tangential.

A common mistake is to assume that the proton's velocity at the maximum distance from the wire is 0 and use conservation of energy, as the work is only done by the electric field and not the magnetic field. However, this neglects the torque due to the magnetic field, which will affect the maximum radial distance reached.

**Setter: Li Yichen, yichen.li@sgphysicsleague.org**

## Problem 54: Spring Swing

A "swing" is made of 3 springs and 2 masses. Two identical springs of spring constant $k_1 = 100$ N m$^{-1}$ are attached to a ceiling at points a distance $D = 0.50$ m apart. At their other ends, they are each attached to identical point masses $m = 20$ kg, which are connected by a spring of spring constant $k_2 = 200$ N m$^{-1}$. Take all springs to have zero natural length.




(a) Find the distance $d$ between the two masses at equilibrium.

*Leave your answer to 2 significant figures in units of* m. (2 points)

(b) Find the sum of all distinct resonant frequencies, $\sum \omega_i$, of the system.

*Leave your answer to 3 significant figures in units of* rad s$^{-1}$. (5 points)

---

*Solution:* The crux to simplifying this problem is to notice that for a spring with zero natural length, the spring force $\vec{F}$ on a mass at its end can be expressed in terms of the displacement $\vec{r}$ of the mass:

$$\vec{F} = -k\vec{r}$$

This implies that we can resolve components of $\vec{F}$ identically to $\vec{r}$ (e.g. $\vec{F_x} = -k\vec{x}$).

(a) Observe that we may solve for just one mass due to the symmetry of the system.

Using only the $x$ component of each spring's extension, the mass is in horizontal equilibrium when:

$$k_1 x_1 = k_2 x_2 \tag{1}$$

Additionally, since $D$ is a fixed value, we also have:

$$x_1 + x_2 + x_1 = D \tag{2}$$

Solving equations (1) and (2) simultaneously, we obtain $d = x_2 = \boxed{0.10 \text{ m}}$.

(b) Let the initial displacements of the left and right masses be $\vec{r}_a$ and $\vec{r}_b$ respectively. At equilibrium, each mass must experience zero net force:

$$-k_1\vec{r}_a - k_2\left(\vec{r}_a - \vec{r}_b\right) + m\vec{g} = 0$$

If the left mass were displaced by $\vec{r}_a'$ while the right mass remained stationary, then the net force on the left mass would be:

$$\begin{aligned}\vec{F}_{\text{net}} &= -k_1\left(\vec{r}_a + \vec{r}_a'\right) - k_2\left(\vec{r}_a + \vec{r}_a' - \vec{r}_b\right) + m\vec{g} \\ &= -k_1\vec{r}_a' - k_2\vec{r}_a'\end{aligned}$$

In other words, we may treat the equilibrium positions as our reference points, and we may find the spring force directly using the relative displacements to the equilibrium positions.

Letting the displacements from each mass's equilibrium position be $\vec{r}_a'$ and $\vec{r}_b'$ respectively, we apply Newton's Second Law to obtain:

$$\begin{aligned}m\ddot{\vec{r}}_a' &= -k_1\vec{r}_a' - k_2\left(\vec{r}_a' - \vec{r}_b'\right) \\ m\ddot{\vec{r}}_b' &= -k_1\vec{r}_b' - k_2\left(\vec{r}_b' - \vec{r}_a'\right)\end{aligned}$$

Noticing that the equations of motion are identical regardless of the axis chosen, let us consider only the $x$-axis for simplicity:

$$\begin{aligned}m\ddot{x}_a' &= -(k_1 + k_2)x_a' + k_2x_b' \\ m\ddot{x}_b' &= k_2x_a' - (k_1 + k_2)x_b'\end{aligned}$$

In simple harmonic motion, we have $\ddot{x} = -\omega^2 x$. This system of equations becomes an eigenvalue problem (one of the form $\mathbf{Ax} = \lambda\mathbf{x}$):

$$m\omega^2\begin{pmatrix}x_a \\ x_b\end{pmatrix} = \begin{pmatrix}k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2\end{pmatrix}\begin{pmatrix}x_a \\ x_b\end{pmatrix}$$

To solve for $\omega$, we hence equate the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ to 0:

$$\begin{vmatrix}k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_1 + k_2 - m\omega^2\end{vmatrix} = 0$$

$$\left(k_1 + k_2 - m\omega^2\right)^2 - k_2^2 = 0$$

This is a quadratic equation in $\omega^2$, yielding two positive roots. Hence:

$$\sum\omega_i = \sqrt{\frac{k_1}{m}} + \sqrt{\frac{k_1 + 2k_2}{m}} \approx \boxed{7.24 \text{ rad s}^{-1}}$$

---

*Alternative solution:* Recalling our system of differential equations:

$$m\ddot{x}'_a = -(k_1 + k_2)x'_a + k_2 x'_b \tag{3}$$
$$m\ddot{x}'_b = k_2 x'_a - (k_1 + k_2)x'_b \tag{4}$$

We may take the sum and difference of equations (3) and (4) to obtain:

$$m(\ddot{x}'_a + \ddot{x}'_b) = -k_1(x'_a + x'_b) \tag{5}$$
$$m(\ddot{x}'_a - \ddot{x}'_b) = -(k_1 + 2k_2)(x'_a - x'_b) \tag{6}$$

Equations (5) and (6) are equations of simple harmonic motion, with angular frequencies $\sqrt{\frac{k_1}{m}}$ and $\sqrt{\frac{k_1+2k_2}{m}}$ respectively. This yields the same result as the previous solution.

**Setter: Paul Seow, paul.seow@sgphysicsleague.org**

**Problem 55: Relativistic Submarines** (5 points)

There are two submarines underwater, Submarine A and Submarine B, operated by Operator A and Operator B respectively, moving along the same axis towards each other. Relative to the water, Submarine A is moving at a speed of $v_A$ while Submarine B is moving at a speed of $v_B = 5v_A$.

Operator A shines a monochromatic red light, which Operator A measures to have a wavelength of $\lambda_i = 666$ nm in air, at Submarine B. The light bounces off the reflective window of Submarine B and returns to Submarine A, where Operator A sees it as dark blue light and measures it to have a wavelength of $\lambda_f = 420$ nm in air. Some of the light passes through the reflective window of Submarine B and is seen by Operator B, who measures the light to have a wavelength of $\lambda_m$ in air. Find $\lambda_m$.

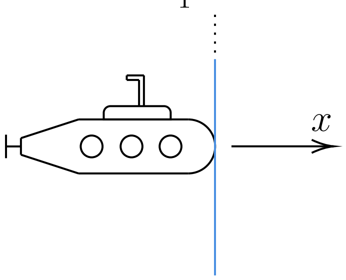
Take the refractive index of water to be exactly $n = \frac{4}{3}$. Take the refractive index of air to be 1.

*Leave your answer to 3 significant figures in units of* nm.

---

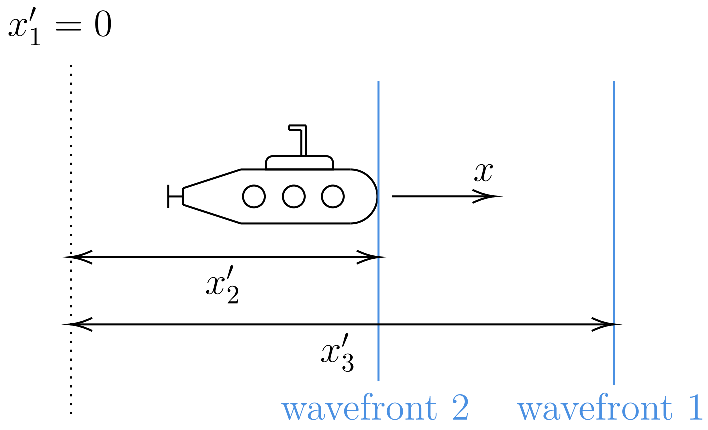
*Solution:* We define $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. $\beta_A$ and $\gamma_A$ are calculated with $v_A$, while $\beta_B$ and $\gamma_B$ are calculated with $v_B$. We define the unprimed frame as the frame of Submarine A, and the primed frame as the frame of the water.

Consider two adjacent wavefronts. The first wavefront is emitted at $t_1 = 0, x_1 = 0$ in A's frame, which we also define as $t'_1 = 0, x'_1 = 0$ in the water's frame.




Then, in A's frame, the second wavefront is emitted at $t_2 = \frac{\lambda_i}{c}, x_2 = 0$. Using relativistic time dilation[5], the second wavefront is emitted at $t'_2 = \frac{\gamma_A \lambda_i}{c}$ in the water's frame.

[5] Since the two wavefronts are emitted in the same position in A's frame, $t_2 - t_1$ is the proper time and $t'_2 - t'_1$ is multiplied by $\gamma_A$




At this point, in the water's frame, A is at $x'_2 = x'_1 + v_A t'_2 = \beta_A \gamma_A \lambda_i$, and the first wavefront is now at $x'_3 = x'_1 + \frac{c}{n} t'_2 = \frac{\gamma_A \lambda_i}{n}$. Therefore, the wavelength in the water is $x'_3 - x'_2 = \gamma_A \lambda_i \left(\frac{1}{n} - \beta_A\right)$.

Now we consider the wavelength $\lambda_m$ observed by Observer B. In the frame of the water, the time gap observed between two wavefronts reflecting off Submarine B is $\Delta t = \frac{x'_3 - x'_2}{\frac{c}{n} + v_B} = \frac{\gamma_A \lambda_i}{c} \frac{\frac{1}{n} - \beta_A}{\frac{1}{n} + \beta_B}$.

Now we transform $\Delta t$ into the frame of Submarine B, by dividing $\Delta t$ by $\gamma_B$. The wavelength seen by Operator B is thus $\lambda_m = \frac{\Delta t}{\gamma_B} c = \frac{\gamma_A (1 - n\beta_A)}{\gamma_B (1 + n\beta_B)} \lambda_i$.

We see that having the light reflect off the window of the submarine is actually equivalent to having the light pass through, reflect off a reflective surface inside the submarine, and then pass through the window again back into the water. Since reflection does not change the wavelength of light in the frame of Submarine B, the behaviour of the reflected light is equivalent to the behaviour of light of wavelength $\lambda_m$ that Observer B emits, which Observer A measures to have a wavelength of $\lambda_f$. By symmetry between the two observers, $\lambda_f = \frac{\gamma_B (1 - n\beta_B)}{\gamma_A (1 + n\beta_A)} \lambda_m$. Substituting the expression for $\lambda_m$, $\lambda_f = \frac{(1 - n\beta_A)(1 - n\beta_B)}{(1 + n\beta_A)(1 + n\beta_B)} \lambda_i$. Solving this expression, we get $v_A \approx 0.0285c$.

Now that we know $v_A$ we can solve the problem trivially by simply substituting in the relevant quantities to the expression for $\lambda_m$ that we found previously.

$$\lambda_m = \frac{\gamma_A (1 - n\beta_A)}{\gamma_B (1 + n\beta_B)} \lambda_i \approx \boxed{533 \text{ nm}}$$

*Note:* Previously, partial credit was provided for the correct value of $v_A$. However, finding $\lambda_m$ after getting the expression for $v_A$ is trivial, and various physically wrong solutions would get the correct value for $v_A$ from cancelling terms as there was too much symmetry in the setup. These wrong solutions seem to universally find $\lambda_m$ to be the geometric mean of $\lambda_i$ and $\lambda_f$, which is 529 nm. Thus, in this version of the question, credit is only provided for finding the correct value of $\lambda_m$, and 529 nm is too far off the correct answer to be accepted.

**Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org**

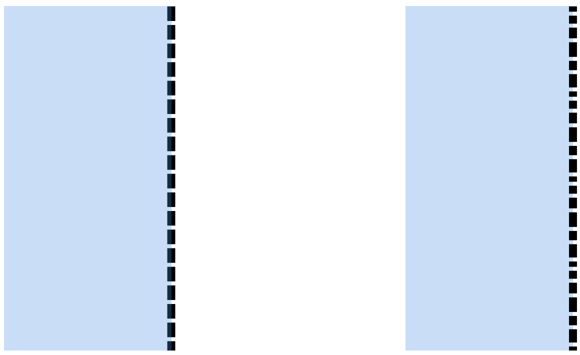
**Problem 56: Drunk Grating** (5 points)

Jennifer uses a 3D printer to create diffraction gratings. Each diffraction grating has $N = 1 \times 10^6$ slits, with adjacent slits separated by distance $d = 1200$ nm. She tests the grating by passing a laser of wavelength $\lambda = 600$ nm through the grating onto a screen placed a distance $D \gg d$ away. She records the intensity $I_0$ of light at an angular position $\theta_0 = 0°$.

One day, while drinking on the job, she accidentally sets the slit separation between each adjacent slit to be a random value between 0 and $d$ with uniform probability, while keeping the total number of slits constant at $N$. Now, she conducts the test again with the same laser and screen, but records the intensity $I_1$ of light at an angular position $\theta_1 = 30°$ instead. Determine the expected value of the ratio $\frac{I_0}{I_1}$.

Assume that the width of each slit is constant and much smaller than slit separations in both gratings. Assume further that the beam illuminates the whole grating.

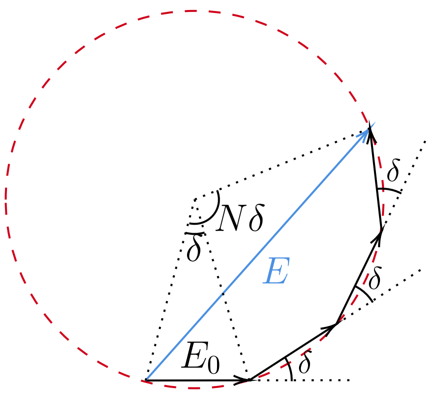
*Leave your answer to 2 significant figures.*



regular grating    drunk grating

---

*Solution:* Firstly, for a regular diffraction grating, let the electric field amplitude of light passing through each slit be $E_0$. The phase difference between light passing through each slit is given by $\delta = \frac{2\pi d \sin\theta}{\lambda}$. We can draw a phasor diagram as shown:



Then, the net electric field $E$ is given by:

$$E = E_0 \frac{\sin \frac{N\delta}{2}}{\sin \frac{\delta}{2}}$$

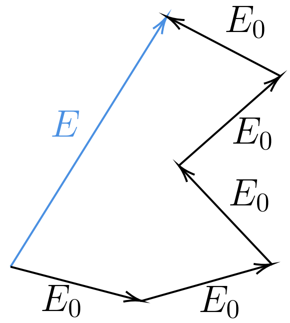
At $\theta_0 = 0°$, $\delta = 0$. This makes the fraction in the previous expression undefined. However, we observe that when $\delta = 0$, there is no phase difference between the electric fields passing through each slit, and thus $E = NE_0$.

Since $I = kE^2$, where $k$ is a constant of proportionality, we have:

$$I_0 = kE_0^2 N^2$$

Now, we consider the randomly constructed grating. The electric field amplitude passing through each slit is still $E_0$, since the slit width has not changed. The phase separation between light passing through the $i^{\text{th}}$ slit and the $(i+1)^{\text{th}}$ slit is $\delta_i = \frac{2\pi x_i \sin\theta}{\lambda}$. Coincidentally, we notice that $d \sin\theta_1 = \lambda$ for the values given in the problem.

Since $x_i$ has an equal probability to take any value between 0 and $d$, the phase difference has an equal probability to take any value between 0 and $2\pi$. We can draw a new phasor diagram as shown:




This is analogous to a "random walk" in two dimensions, where each step has length $E_0$ and arbitrary direction. The displacement in the $i$-th step is $E_0 \cos\delta_i$ in the horizontal direction and $E_0 \sin\delta_i$ in the vertical direction. The total displacement $E$ due to the $N$ slits is then given by Pythagoras' Theorem:

$$\begin{aligned}
E^2 &= \left(\sum_{i=1}^{N} E_0 \cos\delta_i\right)^2 + \left(\sum_{i=1}^{N} E_0 \sin\delta_i\right)^2 \\
&= E_0^2 \left(\sum_{i=1}^{N} \cos^2\delta_i + \sum_{i=1}^{N} \sin^2\delta_i + 2\sum_{i,\,j=1,\,i\neq j}^{N} \cos\delta_i \cos\delta_j + 2\sum_{i,\,j=1,\,i\neq j}^{N} \sin\delta_i \sin\delta_j\right) \\
&= E_0^2 \left(\sum_{i=1}^{N} 1 + 2\sum_{i,\,j=1,\,i\neq j}^{N} \cos\delta_i \cos\delta_j + 2\sum_{i,\,j=1,\,i\neq j}^{N} \sin\delta_i \sin\delta_j\right)
\end{aligned}$$

Since $\delta_i$ and $\delta_j$ are uncorrelated for $i \neq j$, and the sin and cos functions range from $-1$ to 1, the expected values of the latter two summation terms are both 0. This leaves only the first summation term which adds up to $N$. Thus:

$$
\begin{aligned}
E^2 &= E_0^2 N \\
I_1 &= k E_0^2 N \\
\frac{I_0}{I_1} &= N \\
&= \boxed{1000000}
\end{aligned}
$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Problem 57: Toilet Bowl**

Why do some toilet bowl lids close so slowly?

Let us consider a model of a hinged lid falling towards the ground. The lid is of a square shape with side length $\ell = 40$ cm and has uniform mass $m = 0.50$ kg. Assume that there are vertical walls blocking the red cross-section at the sides (as shown in the figure), such that air between the lid and ground can only escape from the blue cross-section at the edge furthest from the hinge.

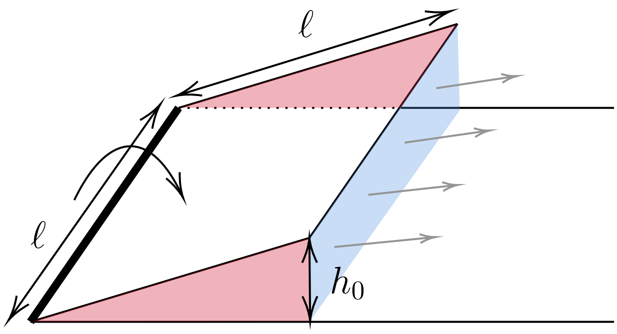
The air has density $\rho = 1.3$ kg m$^{-3}$ and the airflow is laminar and incompressible. The slowing of air near surfaces can be ignored, and the air quickly becomes stationary once it escapes and diffuses into the surroundings.

(a) When the angular velocity of the lid is $\omega$, the velocity of the air flow $v$ at a radial distance $x$ from the hinge is given by $v = \alpha\omega x$, where $\alpha$ is a dimensionless constant. Determine $\alpha$ when the height of the lid $h = 1.0$ cm (measured from the highest side to the ground).

*Leave your answer to 2 significant figures.* (3 points)

(b) The lid is released from rest at a small initial height $h_0 = 3.0$ cm. Determine the angular velocity $\omega_1$ of the lid when its height above ground reaches $h_1 = 1.0$ cm.

*Leave your answer to 3 significant figures in units of* rad s$^{-1}$. (3 points)




---

*Solution:*

(a) As the lid falls, the volume of the gap between the lid and ground decreases. The air is assumed to be incompressible, so continuity applies and air exits the gap at the same rate. Noting that the cross-sectional area of the portion of the gap within a radial distance $x$ from the hinge is a circular sector with area $x^2\theta/2$, the rate of change of the volume of this portion of the gap can be expressed as follows:

$$-\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \frac{x^2\omega\ell}{2}$$

The radial volume flow rate at a radial position $x$ is given by $x\theta\ell v$, where $v$ is the speed of the air flowing out. Equating these two expressions, we have:

$$\frac{x^2\omega\ell}{2} = x\theta\ell v$$

Using the small angle approximation ($h \ll \ell$), we can approximate $\theta \approx h/\ell$. With that, we find:

$$v = \frac{\ell}{2h}\omega x \Longrightarrow \alpha = \frac{\ell}{2h} = \boxed{20}$$

(b) It is important to realise that energy is conserved in laminar air flow. The rotational kinetic energy of the lid is given by $I\omega^2/2$, with $I = m\ell^2/3$ being the moment of inertia of the square lid about its side edge, while the kinetic energy of the air underneath the lid can be calculated using an integral:

$$\text{K.E.} = \int_0^\ell \frac{1}{2}\rho v^2 \, \mathrm{d}V = \int_0^\ell \frac{1}{2}\rho \left(\frac{\ell\omega x}{2h}\right)^2 \left(\frac{xh}{\ell}\,\mathrm{d}x\right) = \frac{\rho\omega^2\ell^5}{32h}$$

Since the air that flows out diffuses quickly and becomes stationary, the majority of kinetic energy will be concentrated in the air that is trapped underneath the lid and we can neglect kinetic energy of the air outside the lid. The conservation of energy equation is then:

$$mg\frac{h_0}{2} = \frac{\rho\omega_1^2\ell^5}{32h_1} + \frac{1}{2}\frac{m\ell^2}{3}\omega_1^2 + mg\frac{h_1}{2}$$

Solving for $\omega$, we have:

$$\omega_1 = \sqrt{\frac{mg(h_0 - h_1)}{\frac{\rho\ell^5}{16h_1} + \frac{m\ell^2}{3}}} \approx \boxed{0.945 \text{ rad s}^{-1}}$$

**Setter: Li Xinrui, xinrui.li@sgphysicsleague.org**

### Problem 58: Pulling an Inductor

In the following circuit, a very long solenoidal inductor with $n_0 = 5000 \text{ m}^{-1}$ turns per unit length is connected to a capacitor in series. The length $\ell$ and radius $r$ of the inductor satisfy $\ell \gg r$, so only the magnetic field inside the inductor needs to be considered. The initial maximum current in the circuit is $I_0 = 1.00$ A, and the initial period of oscillations of the current is $T_0$.

At time $t = 0$, there is no current flowing through the wires. At this instant, the inductor is pulled on and stretched out, such that the turns per unit length decreases to $n_1 = 2000 \text{ m}^{-1}$. This process is done in time $t_p$. Find the new maximum electric current $I_1$ in the circuit if:

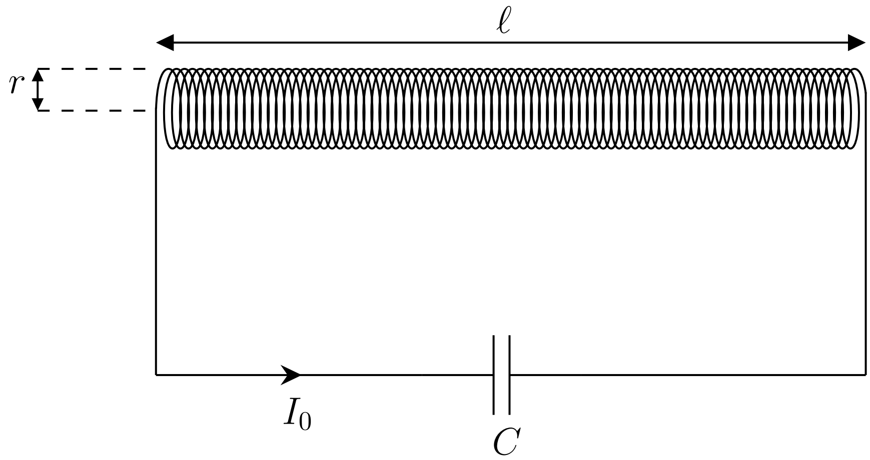
(a) $t_p \ll T_0$.

*Leave your answer to 3 significant figures in units of* A. (3 points)

(b) $t_p \gg T_0$.

*Leave your answer to 3 significant figures in units of* A. (4 points)

You may assume the turns remain equally spaced apart throughout this process.




---

*Solution:*

(a) The magnetic field inside a solenoid with $n$ turns is $B = \mu_0 n I$ where $I$ is the current passing through the solenoid. If we let $N$ be the total number of turns of the solenoid, the electromotive force (EMF) $\mathcal{E}$ induced as the flux $\Phi$ through the solenoid varies over time is given by:

$$\begin{aligned}\mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= -N(\pi r^2)\frac{\mathrm{d}B}{\mathrm{d}t} \\ &= -\pi\mu_0 n N r^2 \frac{\mathrm{d}I}{\mathrm{d}t}\end{aligned}$$

From the definition of inductance $L = \mathcal{E}\frac{\mathrm{d}I}{\mathrm{d}t}$, the inductance of the solenoid is hence $L = \pi\mu_0 n N r^2$.

The energy stored in an inductor is given by $E = \frac{1}{2}LI^2$. At time $t = 0$, there is no current flowing through the circuit. Since the time taken to stretch out the solenoid $t_p \ll T_0$, the current remains at approximately zero throughout the process. The instantaneous work down on the inductor is given by $\mathrm{d}W = LI\,\mathrm{d}I$, and $\mathrm{d}I \approx 0$ for this process, so there is effectively no work done on the inductor. We can then apply conservation of the total energy of the LC circuit to obtain:

$$\begin{aligned}\frac{1}{2}L_1 I_1^2 &= \frac{1}{2}L_0 I_0^2 \\ I_1 &= \sqrt{\frac{L_0}{L_1}}I_0 \\ &= \sqrt{\frac{\pi\mu_0 n_0 N r^2}{\pi\mu_0 n_1 N r^2}}I_0 \\ &= \sqrt{\frac{n_0}{n_1}}I_0 \\ &\approx \boxed{1.58\ \mathrm{A}}\end{aligned}$$

Here, we avoid using the variable $\ell$ (the length of the solenoid) since it changes during the pulling process, which makes calculations more annoying. In contrast, the variable $N$ stays fixed.

(b) As the inductance varies slowly, we can no longer approximate that no work is done on the inductor and apply conservation of energy. Instead, we will conserve another quantity, the adiabatic invariant of the system.

When oscillations have periods much less than the time scale at which the parameters of the system are changing, the adiabatic invariant can be regarded as constant[6]. The adiabatic invariant $I$ is the area enclosed by the closed contour traced by the trajectory of the system on the phase diagram. For a mechanical harmonic oscillator, the axes of the phase diagram are the momentum $p_x = m\frac{\mathrm{d}x}{\mathrm{d}t}$

[6] For a brief proof, refer to this article.

and spatial coordinate $x$, and the area traced out is an ellipse. For an $LC$ circuit, the analogous axes are $LI$ and $Q$ respectively ($Q$ is the charge stored in the capacitor). The semi-major and semi-minor axes lengths are the amplitude of oscillation in these two axes, $LI_m$ and $Q_m$. The area of the ellipse $I$ is given by:

$$\begin{aligned} I &= \pi Q_m L I_m \\ &= \pi \frac{I_m}{\omega} L I_m \\ &= \pi \sqrt{C} L^{\frac{3}{2}} I_m^2 \end{aligned}$$

Since this quantity is conserved, we have:

$$\begin{aligned} L_0^{\frac{3}{2}} I_0^2 &= L_1^{\frac{3}{2}} I_1^2 \\ I_1 &= I_0 \left(\frac{L_0}{L_1}\right)^{\frac{3}{4}} \\ &= I_0 \left(\frac{n_0}{n_1}\right)^{\frac{3}{4}} \\ &\approx \boxed{1.99 \text{ A}} \end{aligned}$$

---

*Alternative solution 1:* Let the current in the system be $I = I_m \sin(\omega t)$ at time $t$, where $I_m$ is the maximum current and $\omega$ is the angular frequency of oscillations at that point in time. The instantaneous energy stored in the inductor is $\frac{1}{2}LI^2$, so the average energy stored in the inductor over a period $T = \frac{2\pi}{\omega}$ is $\langle E \rangle = \frac{1}{2}L\langle I \rangle^2$ where $\langle I^2 \rangle$ is the average value of $I^2$ over a period. As this energy changes, we have:

$$\langle \mathrm{d}E \rangle = \left\langle LI \,\mathrm{d}I + \frac{1}{2} I^2 \,\mathrm{d}L \right\rangle \tag{1}$$

We can then write Kirchoff's Voltage Law for the circuit, taking into account that the inductance is varying in time:

$$\begin{aligned} \frac{Q}{C} + \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= 0 \\ \frac{Q}{C} + L\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\mathrm{d}L}{\mathrm{d}t} &= 0 \\ \mathrm{d}I &= -\frac{I}{L}\,\mathrm{d}L - \frac{Q}{LC}\,\mathrm{d}t \end{aligned}$$

Substituting this into equation (1):

$$\langle \mathrm{d}E \rangle = \left\langle -I^2 \,\mathrm{d}L - \frac{QI}{C}\,\mathrm{d}t + \frac{1}{2} I^2 \,\mathrm{d}L \right\rangle$$

$$= -\frac{1}{2}\langle I^2 \rangle \,\mathrm{d}L - \frac{1}{C}\langle QI \rangle \,\mathrm{d}t$$

Since $T \ll t_p$ here, we can take a time average of $I^2$ over a period to obtain:

$$\left\langle I^2 \right\rangle = I_m^2 \left\langle \sin^2(\omega t) \right\rangle = \frac{1}{2} I_m^2$$

Similarly, since $Q$ is of the form $Q = Q_m \cos(\omega t)$, we have:

$$\langle QI \rangle = Q_m I_m \langle \sin(\omega t) \cos(\omega t) \rangle = 0$$

Therefore, we have:

$$\langle \mathrm{d}E \rangle = -\frac{1}{4} I_m^2 \,\mathrm{d}L$$

Dividing this by the total energy of the system $E = \frac{1}{2} L I_m^2$, we have:

$$\frac{\langle \mathrm{d}E \rangle}{E} = -\frac{1}{2}\frac{\mathrm{d}L}{L}$$

At this point we may remove the angled brackets as our time scale shifts from $T_0$ to $t_p$. Integrating on both sides, we obtain a conserved quantity:

$$L^{\frac{1}{2}} E = \text{const.}$$
$$L^{\frac{3}{2}} I^2 = \text{const.}$$

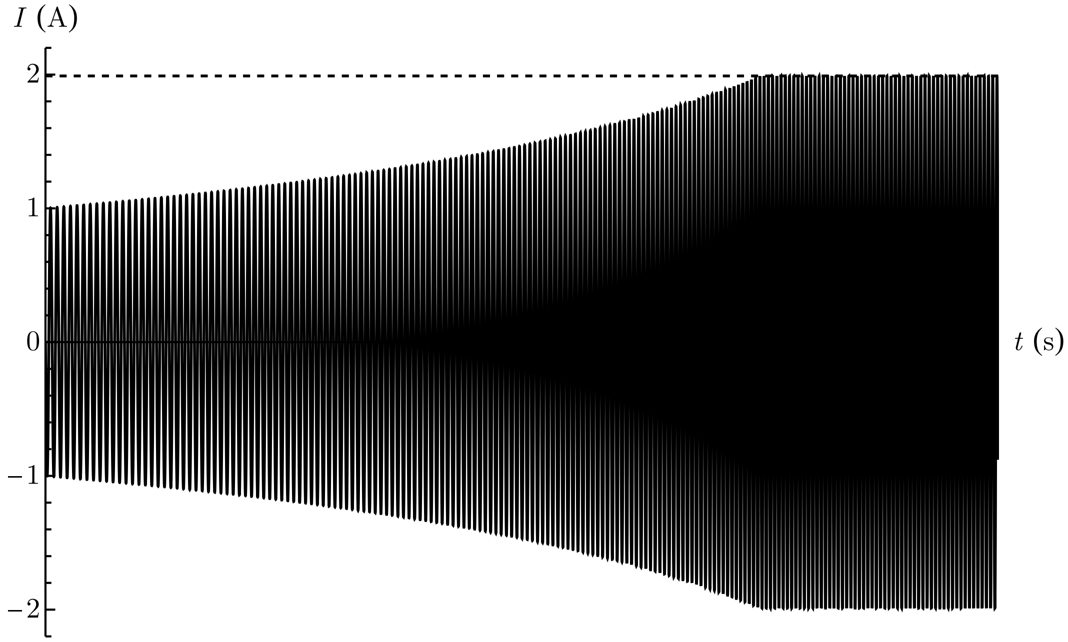
This is the same result obtained in our previous solution.

---

*Alternative solution 2:* We can also proceed via numerical integration. The exact form of the current increase does not matter, as long as the rate of increase is sufficiently slow. We assume a linear increase in $n$ from $n_0$ to $n_1$, from time $t = 0$ to $t = t_p$, in other words $L = L_0 + (L_1 - L_0)\frac{t}{t_p}$ when $t \leq t_p$. The required equation we have to solve is given by Kirchoff's Voltage Law:

$$\frac{Q}{C} + L\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\mathrm{d}L}{\mathrm{d}t} = 0$$
$$\frac{Q}{C} + L\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + \frac{\mathrm{d}Q}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}t} = 0$$

We set the initial conditions $Q = I_0\sqrt{L_0 C}$ and $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$ at $t = 0$. We also choose the constant values $C = 1$ F, $L_0 = 2$ H, $L_1 = 5$ H and $t_p = 1500$ s, such that the maximum period of oscillations $T_0 = 2\pi\sqrt{\frac{1}{L_0 C}} \approx 4.4$ s $\ll t_p$. Solving numerically, we obtain the following graph.




The dotted line indicates the current $I = \boxed{1.99 \text{ A}}$, which is the final amplitude of current oscillations.

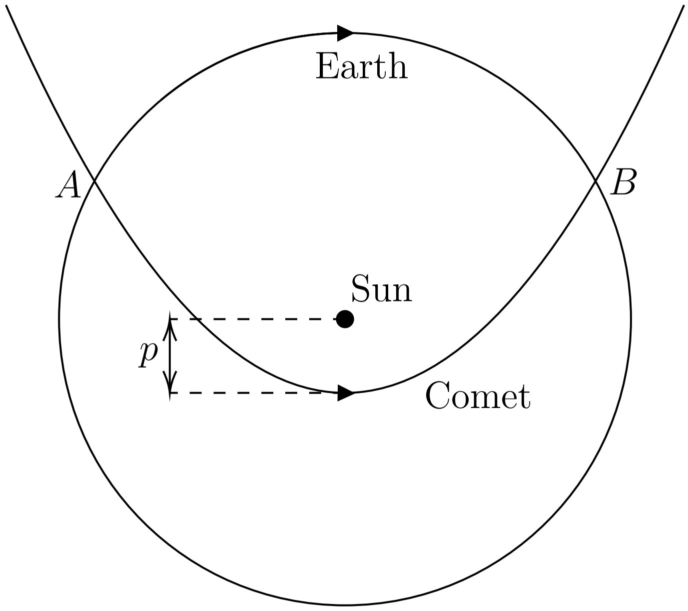
**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

**Problem 59: Viewing a Comet** (6 points)

A comet travels in a parabolic path around the Sun. Its closest distance to the Sun is $p$. The comet's position first coincides with the Earth's position at point A, and again at point B. You may assume that:

- Although the Earth and comet are close to each other at points A and B, they do not exert any gravitational influence on each other. Only the gravitational force from the Sun acts on the comet.
- The Earth's orbit is circular with radius $R$.
- The Earth travels along the minor arc $AB$ to pass by the comet again.

Find the value of $\frac{p}{R}$.

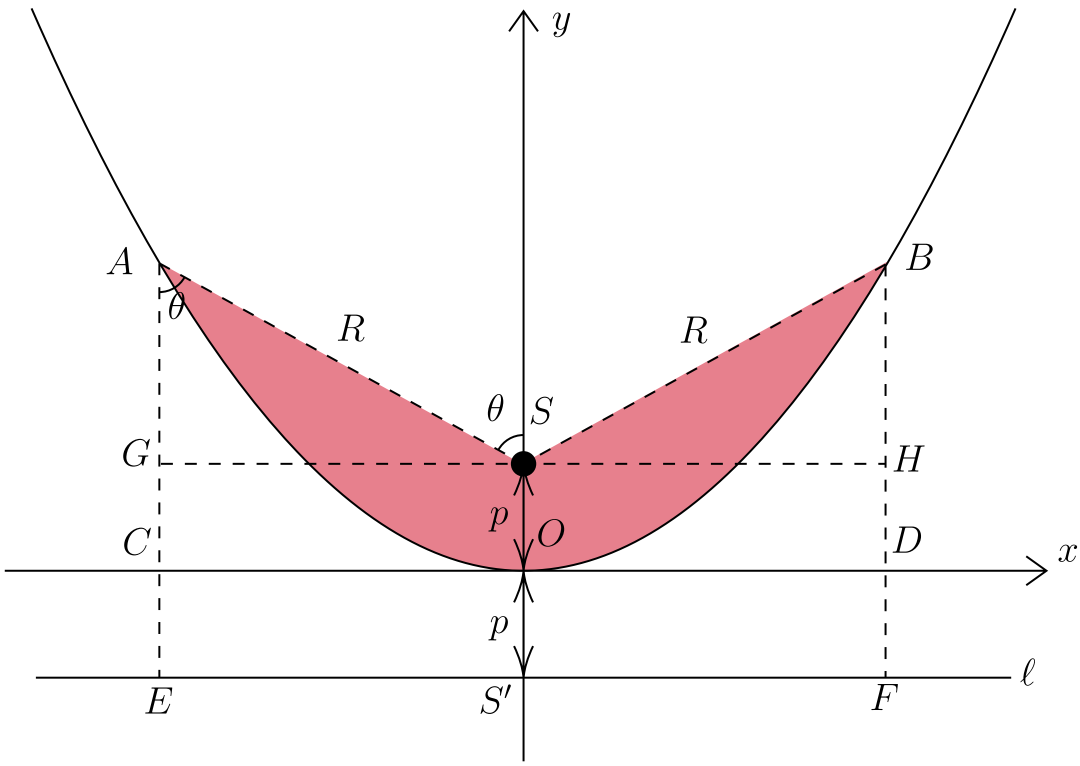



*Leave your answer to 2 significant figures.*

---

*Solution:* We start by setting up a coordinate system with the origin at the point of closest approach of the comet to the Sun, and have the $y$-axis point in the direction of the Sun. Let the Sun be at point $S$.

Point $S$ is the focus of the parabola. Parabolas have the geometrical property that every point along the parabola is equidistant from the focus and a line known as the directrix. As such, since the origin has to be equidistant from the directrix and point $S$, the directrix $\ell$ has the equation $y = -p$.




Hence, we know that $AS = AE$ from the geometrical property of parabolas. Since $AS = R$, we have $AE = R$, $AC = R - p$ and $AG = R - 2p$. From the Pythagorean theorem, $OC = GS = \sqrt{R^2 - (R-2p)^2} = \sqrt{4pR - 4p^2} = 2\sqrt{p(R-p)}$.

Let the parabola have equation $y = cx^2$ for some constant $c$. Since point $B$ lies on the parabola and has coordinates $(2\sqrt{p(R-p)}, R-p)$, we can substitute its coordinates into the equation of the parabola to find $c$:

$$\begin{aligned} c &= \frac{y}{x^2} \\ &= \frac{R-p}{4p(R-p)} \\ &= \frac{1}{4p} \end{aligned}$$

Hence, the parabola has equation $y = \frac{1}{4p}x^2$.

Suppose the comet has speed $v_0$ at point $O$. In a parabolic trajectory, the comet has an infinitesimally small velocity at distances far from the Sun. Hence, the total energy of the comet is 0. By conservation of energy:

$$\begin{aligned} \frac{1}{2}mv_0^2 - \frac{GMm}{p} &= 0 \\ v_0^2 &= \frac{2GM}{p} \\ v_0 &= \sqrt{\frac{2GM}{p}} \end{aligned}$$

To continue, we note that from Kepler's Second Law, the area swept out by an arc of the comet's orbit is proportional to the time taken for it to travel through that arc. At point $O$, in time $\mathrm{d}t$, the comet sweeps out a triangle with base $v_0\,\mathrm{d}t$ and height $p$. Hence, $\mathrm{d}A = \frac{1}{2}pv_0\,\mathrm{d}t$. Therefore, if the comet sweeps out an area $A$ in time $t$, then $A = \frac{1}{2}pv_0 t$.

Let the time taken for the comet to travel from point $A$ to point $B$ be $T_c$, and the area $S_R$ swept out during this time be the Red Area shaded in the diagram above. Furthermore, let the area of trapezium ASOC be $S_A$. We have:

$$\begin{aligned}
T_c &= \frac{S_R}{\frac{1}{2}pv_0} \\
&= \frac{2}{pv_0}\left(2S_A - 2\int_0^{2\sqrt{p(R-p)}} \frac{1}{4p}x^2\,\mathrm{d}x\right) \\
&= \frac{4}{pv_0}\left(\frac{1}{2}(p+R-p)\left(2\sqrt{p(R-p)}\right) - \frac{1}{4p}\left[\frac{x^3}{3}\right]_0^{2\sqrt{p(R-p)}}\right) \\
&= \frac{4}{pv_0}\left(R\sqrt{p(R-p)} - \frac{1}{12p}\left(2\sqrt{p(R-p)}\right)^3\right) \\
&= \frac{4}{p}\sqrt{\frac{p}{2GM}}\left(R\sqrt{p(R-p)} - \frac{2}{3p}\left(\sqrt{p(R-p)}\right)^3\right) \\
&= 4\sqrt{\frac{1}{2GMp}}\left(R\sqrt{p(R-p)} - \frac{2}{3}(R-p)\sqrt{p(R-p)}\right) \\
&= 4\sqrt{\frac{1}{2GMp}}\sqrt{p(R-p)}\left(R - \frac{2}{3}(R-p)\right) \\
&= \frac{4}{3}(R+2p)\sqrt{\frac{R-p}{2GM}}
\end{aligned}$$

Next, let the time taken for the Earth to move from $A$ to $B$ be $T_e$. From Kepler's Third Law, the period of the Earth's orbit $T$ satisfies $T^2 = \frac{4\pi^2}{GM}R^3$. Since $T_e = \frac{2\theta}{2\pi}T$, we have:

$$\begin{aligned}
T_e &= \frac{\theta}{\pi}T \\
&= \frac{\cos^{-1}\left(\frac{AG}{AS}\right)}{\pi}\sqrt{\frac{4\pi^2R^3}{GM}} \\
&= 2R\cos^{-1}\left(\frac{R-2p}{R}\right)\sqrt{\frac{R}{GM}}
\end{aligned}$$

Equating $T_c$ and $T_e$:

$$\frac{4}{3}(R+2p)\sqrt{\frac{R-p}{2GM}} = 2R\cos^{-1}\left(\frac{R-2p}{R}\right)\sqrt{\frac{R}{GM}}$$

$$\frac{2}{3}\left(1+2\frac{p}{R}\right)\sqrt{\frac{1}{2}\left(1-\frac{p}{R}\right)}=\cos^{-1}\left(1-2\frac{p}{R}\right)$$

If we let $\frac{p}{R}=x$, we get:

$$\frac{2}{3}(1+2x)\sqrt{\frac{1}{2}(1-x)}=\cos^{-1}(1-2x)$$

Solving numerically, we obtain:

$$x \approx \boxed{0.065}$$

**Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org**

## Problem 60: Bouncing in a Valley

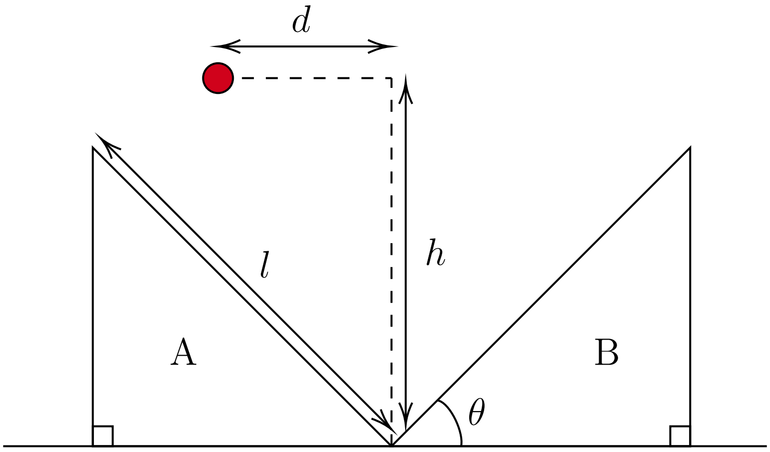
Slope A and Slope B, two smooth right triangular prisms, are fixed to the ground and joined at their bottom corners, forming a V-shaped valley. Each slope has slope length $l$ and is inclined at $\theta = 45°$ above the horizontal. A ball is dropped above Slope A, at a height $h$ above the ground, and a horizontal distance $d$ from the centre of the valley. Model the ball as a point mass and assume that all collisions are perfectly elastic and frictionless.

(a) Given that $l = 2.00$ m, $h = 1.25$ m, $d = 0.35$ m, find the time taken $t_1$ for the ball to first return to its starting point. If the ball never returns to its starting point, enter $-999$ as your answer.

*Leave your answer to 3 significant figures in units of* s. (4 points)

(b) Given instead that $h = 2.50$ m while $l$ and $d$ take the same values as in part (a), the ball will eventually escape over the edge of the valley. Find the duration $t_2$ between the ball's initial release and its final contact with the prisms.

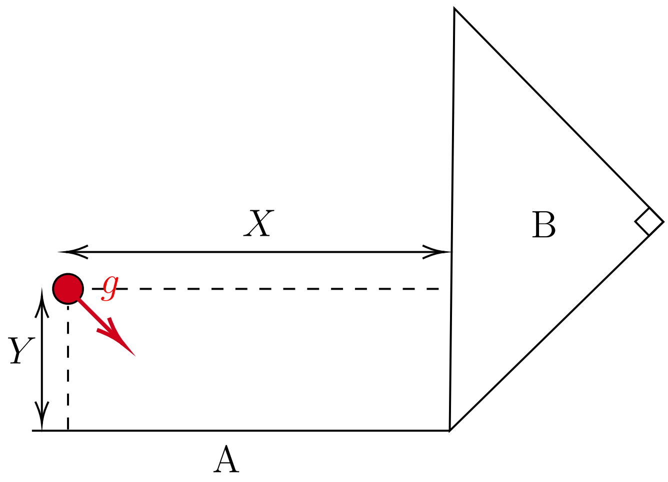
*Leave your answer to 3 significant figures in units of* s. (4 points)




---

*Solution:*

(a) Let us first rotate the system 45° anti-clockwise. Define the $x$-axis to be parallel to Slope A and the $y$-axis to be parallel to Slope B.




Let the initial distance of the ball from Slope B be $X$ and Slope A be $Y$. Using trigonometry, we find that $Y = \frac{h-d}{\sqrt{2}}$ and $X = \frac{h+d}{\sqrt{2}}$. We can also resolve the gravitational acceleration $g$ on the ball into its $x$ and $y$ components. Both components have magnitude $\frac{g}{\sqrt{2}}$ and are directed towards the slopes.

Now, we can calculate the time taken for the ball to first contact Slope B, $t_B$:

$$X = \frac{1}{2}\frac{g}{\sqrt{2}}t_B^2 \Rightarrow t_B = \sqrt{\frac{2\sqrt{2}X}{g}}$$

Likewise, we calculate the time taken to first contact Slope A, $t_A$:

$$t_A = \sqrt{\frac{2\sqrt{2}Y}{g}}$$

Notice that the horizontal displacement $x$ and vertical displacement $y$ of the ball can be considered independently. In order to return to its starting $x$ position, the ball must contact Slope B and then rebound elastically to its starting $x$ position. This happens periodically with a period of $2t_B$. Likewise, the ball reaches its original $y$ position with a period of $2t_A$.

For the ball to reach its original position, these two events must coincide:

$$\exists\ q, r \in \mathbb{N} \text{ such that } q \cdot 2t_A = r \cdot 2t_B$$

with the added condition of $\gcd(q, r) = 1$ for the first time this occurs. Thus:

$$\frac{q}{r} = \frac{t_B}{t_A} = \sqrt{\frac{2\sqrt{2}X}{g}}\Bigg/\sqrt{\frac{2\sqrt{2}Y}{g}} = \sqrt{\frac{X}{Y}}$$

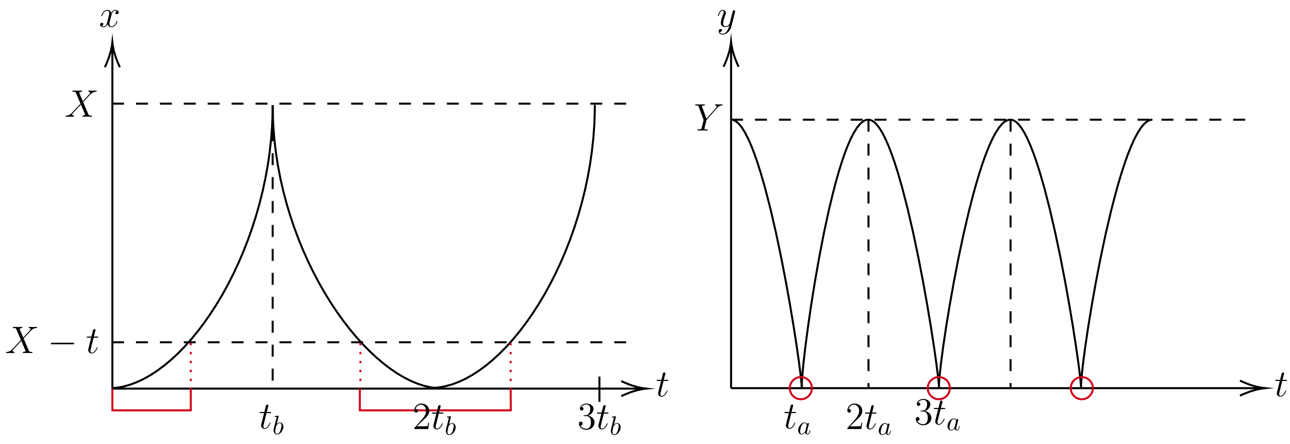
Substituting our numerical values, we get $\frac{q}{r} = \frac{4}{3}$, so $q = 4, r = 3$. Thus:

$$t_1 = q \cdot 2t_A \approx \boxed{3.43 \text{ s}}$$

*Note:* The value of $l$ is provided for you to verify that the ball lands within the structure in the first place.

(b) For the ball to escape over the edge of the slope, either of its initial horizontal or vertical displacement must be greater than the slope length. This is because its initial displacements bound its displacement along that axis. Using $X = \frac{h+d}{\sqrt{2}}$, we find that $X \approx 2.02$ m $> l$, so the ball will end up (just) escaping over the edge of Slope A.

To visualise what it means for the ball to escape over the edge of the slope, we will graph out $x$ against $t$ and $y$ against $t$.




At the point of escape, the ball must satisfy $0 \leq x < X - l$, i.e. it is past the edge of Slope A. Additionally, at the point of no return, $y = 0$. We consider this the point of no return because for $y > 0$, it is still possible for the ball to return to within the slopes. Graphically, we need one of the points circled in red to lie within one of the red intervals. These intervals are defined by the general formula:

$$(2qt_B - t_{\text{pass}}, 2qt_B + t_{\text{pass}}),\ q \in \mathbb{N}$$

where $t_{\text{pass}}$ is equivalent to the time where the ball first reaches $x = X - l$:

$$X - l = \frac{1}{2}\frac{g}{\sqrt{2}}t_{\text{pass}}^2 \Rightarrow t_{\text{pass}} = \sqrt{\frac{2\sqrt{2}(X-l)}{g}}$$

Next, for the condition $y = 0$:

$$t = (2r+1)t_A,\ r \in \mathbb{N}$$

Piecing this together, we want to find $q, r$ that satisfy:

$$2qt_B - t_{\text{pass}} < (2r+1)t_A < 2qt_B + t_{\text{pass}}$$

$$\Rightarrow \frac{2q\left(\sqrt{X} - \sqrt{X-l}\right)}{\sqrt{Y}} < 2r + 1 < \frac{2q\left(\sqrt{X} + \sqrt{X-l}\right)}{\sqrt{Y}}$$

We can use Desmos to solve this, giving us $q = 3,\ r = 3$.

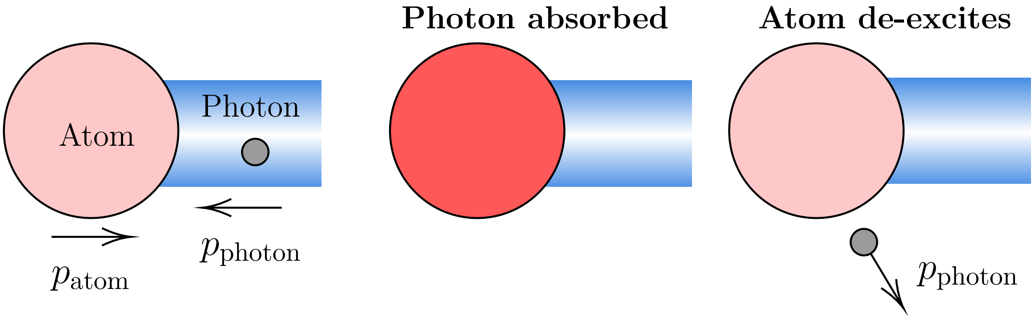
From the graphs, we can see that this means the final contact on Slope B (the last time $x = X$) is $(2q-1)t_B = 3.81$ s and the final contact on Slope A (the last time $y = 0$) is $(2r+1-2)t_A = 3.31$ s. Therefore, $t_2 \approx \boxed{3.81 \text{ s}}$

**Setter: Gerrard Tai, gerrard.tai@sgphysicsleague.org**

**Problem 61: Why So Cool?** (5 points)

It was recently discovered that one can cool an atom (i.e. slow it down) by directing a laser at it.

Consider the simplest case of 1D motion. A photon is absorbed when its frequency **in the atom's frame** coincides with the atom's "resonant frequency". The excited atom will then quickly de-excite by emitting a photon spontaneously but in a random direction. The net result is that the atom's speed decreases.




Consider a $Ca^{2+}$ ion of mass $6.6 \times 10^{-26}$ kg and initial velocity $v_0 = 0.10c$ at $t = 0$ in the lab frame, where $c$ denotes the speed of light. The ion can absorb photons with a frequency range of $5.3 \times 10^{14}$ Hz $< f <$ $5.5 \times 10^{14}$ Hz (in the ion's frame). The laser is directed against the atom's initial velocity, and has a frequency $f_0 = 4.9 \times 10^{14}$ Hz with power $P_0 = 1.0 \times 10^{-8}$ W as observed in the lab frame. Assume that only **half** the photons from the laser reach the atom and get absorbed. Find the time taken $t$ for the atom to stop absorbing photons.

*Leave your answer to 2 significant figures in units of* ms.

---

*Solution:* Let $\beta = v/c$, where $v$ is the speed of the atom. The net force exerted by the photon on the atom in the lab frame is given by:

$$F_{\text{photon,lab}} = \frac{\Delta p_{\text{photon}}}{\Delta t} = \frac{1}{2}\frac{N}{t}\frac{hf}{c}\left(1 + \frac{v}{c}\right) = \frac{P_0}{2c}(1 + \beta)$$

where $N$ is the number of photons emitted in time $t$ in the lab frame. The additional factor of $(1+\beta)$ arises because the atom is moving against the direction of the photons. This motion causes more photons to collide with the atom per unit time compared to when the atom is stationary.

The frequency of the light observed in the frame of the atom is given by the relativistic Doppler equation:

$$f = f_0\sqrt{\frac{1 + \beta}{1 - \beta}}$$

Substituting in the values, $f = \sqrt{11/9}\ f_0 \approx 5.42 \times 10^{14}$ Hz. This is within the absorption frequency range of the atom.

Now, we find the critical velocity $v_{\text{crit}}$ below which the atom will no longer absorb any photon. Taking the lower bound of the absorption frequency range, we get:

$$v_{\text{crit}} = \frac{f^2 - f_0^2}{f^2 + f_0^2}\, c$$

Going back to the lab frame, we know that the atom starts with an initial velocity $v_0 = 0.1c$ and ends with a final velocity $v_{\text{crit}}$ before it stops absorbing any photons.

Writing the relativistic form of Newton's Second Law:

$$\frac{\mathrm{d}(\gamma m v)}{\mathrm{d}t} = \gamma^3 m a = \frac{m}{\sqrt{(1-\beta^2)^3}} \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{P_0}{2c}(1+\beta).$$

We can then solve this differential equation for $v(t)$ with initial conditions $v|_{t=0} = v_0$. Solving for the time $t$ when $v(t) = v_{\text{crit}}$, we get $t \approx \boxed{24\text{ ms}}$.

---

*Alternative solution*: In the atom's frame, the laser becomes Doppler shifted, and the time taken $t$ for $N$ photons to be emitted will be dilated by a factor of $\gamma$. The force experienced in the atom's frame is then given by:

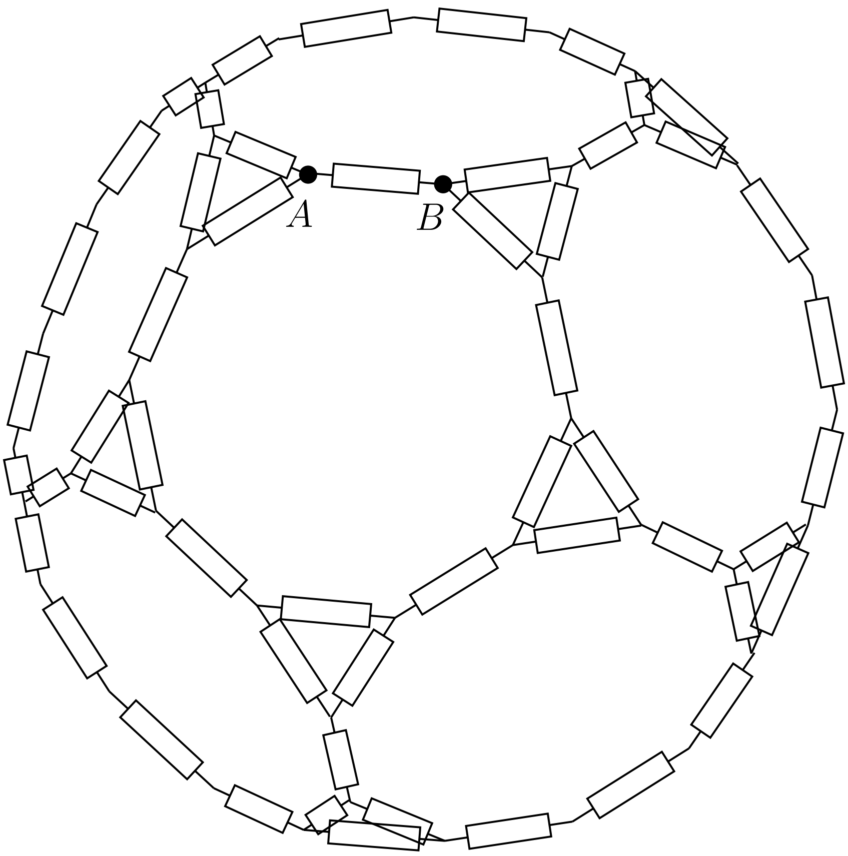
$$F_{\text{photon, atom}} = \frac{1}{2}\frac{N}{\gamma t}\frac{h f_0}{c}\sqrt{\frac{1+\beta}{1-\beta}} = \frac{P_0}{2c}(1+\beta).$$

As longitudinal forces do not change under Lorentz transformation, we have $F_{\text{photon, lab}} = F_{\text{photon, atom}}$, recovering the same result as above.

**Setter: Li Xinrui, xinrui.li@sgphysicsleague.org**

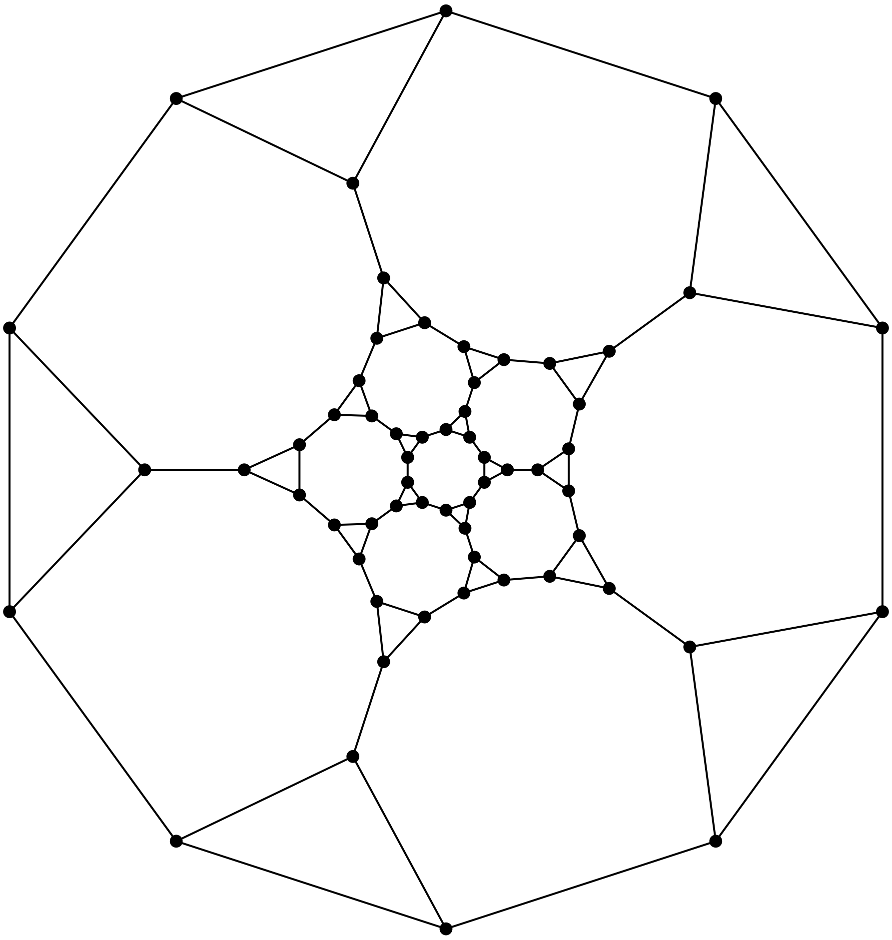
**Problem 62: Truncated Dodecahedron** (6 points)

Identical resistors of resistance $R = 1.0\ \Omega$ are connected by wires of zero resistance into the shape of a truncated dodecahedron, which is a polyhedron formed by cutting off the corners of a regular dodecahedron. The resistors thus form 12 decagonal faces and 20 triangular faces. Determine the resistance $R_{AB}$ between points $A$ and $B$.

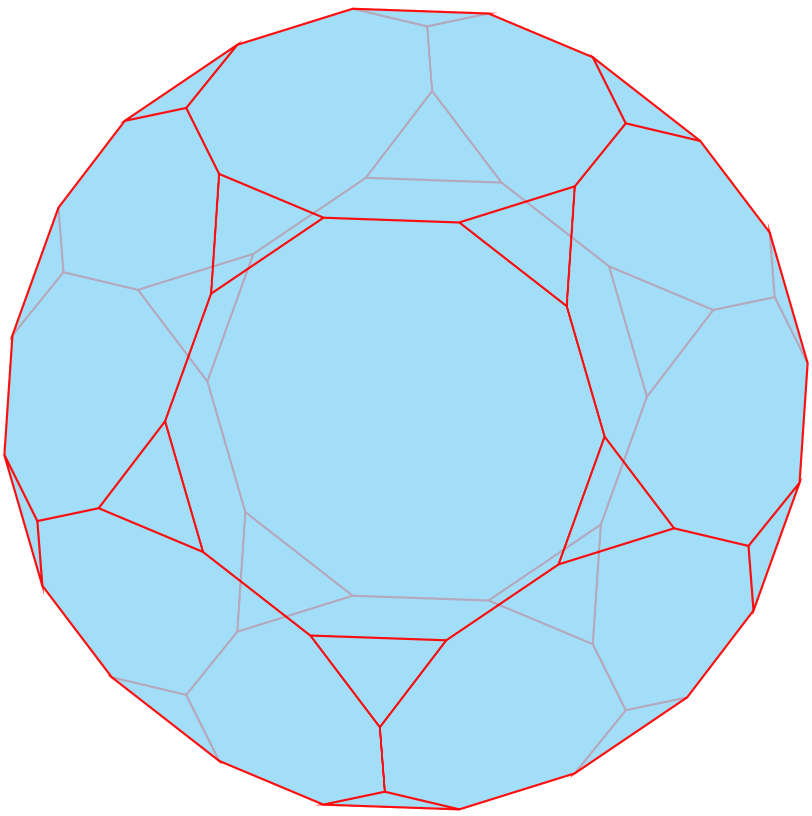



Note that in the circuit diagram, only the resistors in the front are shown to avoid clutter. To see more diagrams of the truncated dodecahedron, see the next page.

*Leave your answer to 2 significant figures in units of* $\Omega$.



A truncated dodecahedron, as a planar graph

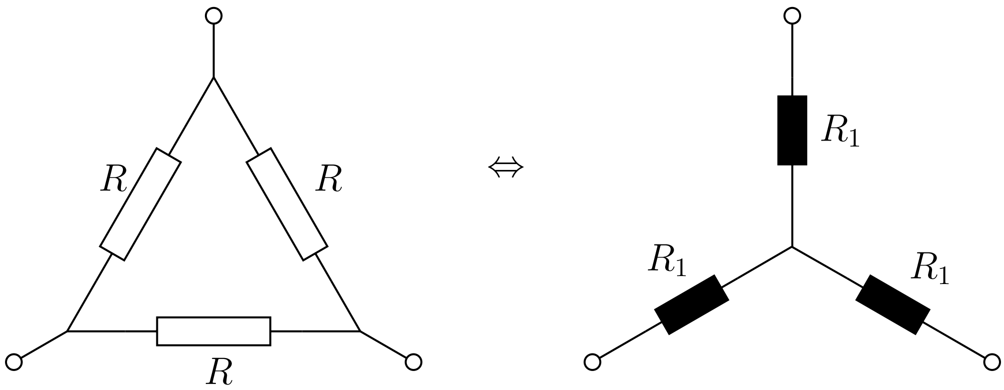


A truncated dodecahedron

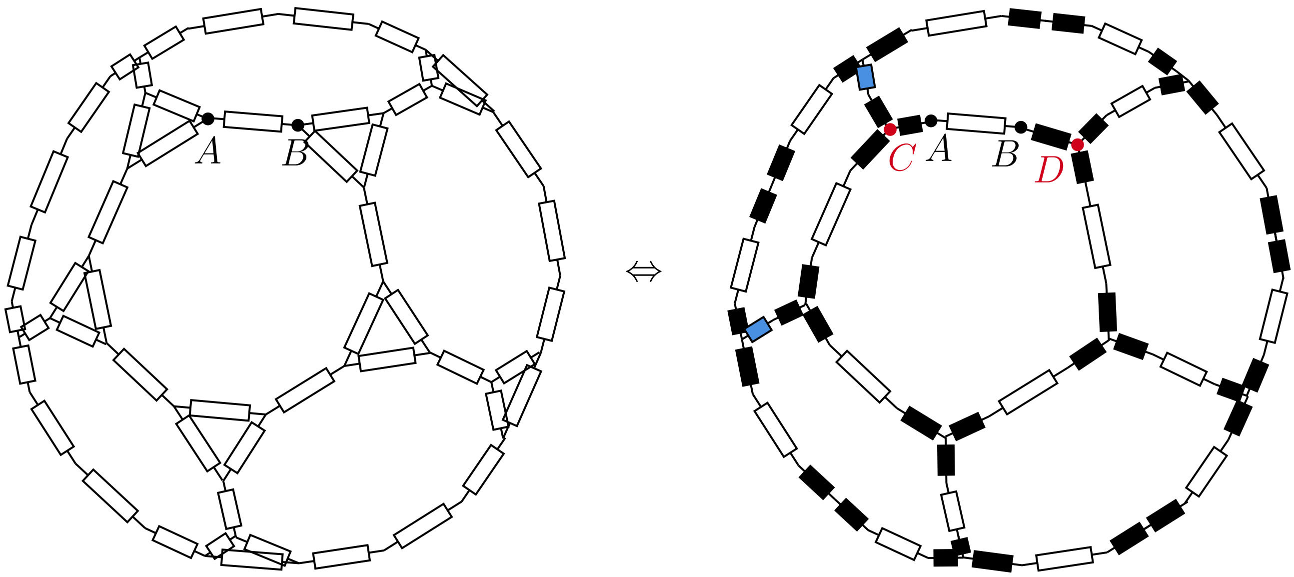
---

*Solution:* We first apply the Δ-Y transformation to the triangles of the truncated dodecahedron. We can replace every triangle with a 'Y'-shaped configuration where:

$$R_1 = \frac{R^2}{R+R+R} = \frac{1}{3}R$$




After replacing every triangle with this 'Y' shaped configuration, the truncated dodecahedron now loses all its triangular faces and becomes a dodecahedron:



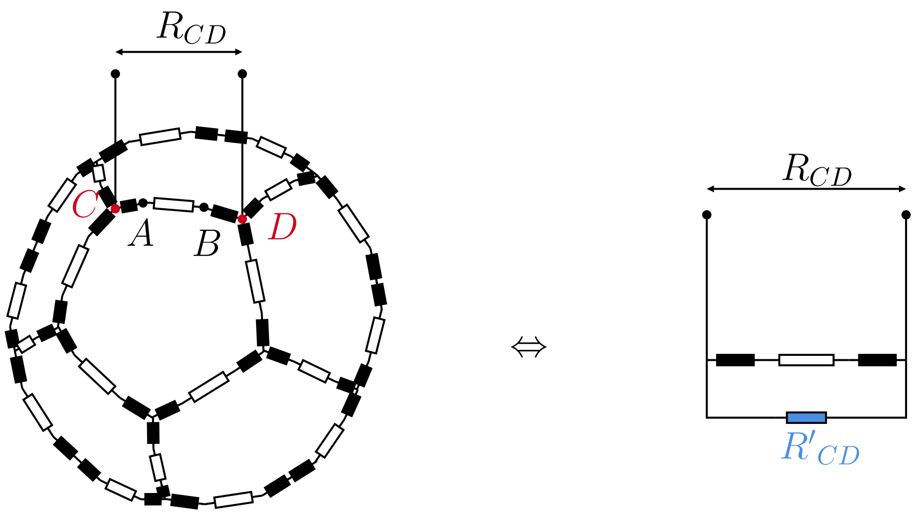

Note that each edge of the new dodecahedron has resistance $R_2 = \frac{1}{3}R + R + \frac{1}{3}R = \frac{5}{3}R$.

Next, we consider the resistance between points $C$ and $D$, $R_{CD}$. Consider injecting current $I$ into point $C$, and extracting equal currents out from every vertex of the dodecahedron. Since there are 20 vertices, the extracted current out from every vertex is $\frac{I}{20}$. Therefore, the net current entering the system at point $C$ is $I - \frac{I}{20} = \frac{19}{20}I$. By the symmetry in the system, the current flowing from $C$ to $D$ in this setup is $\frac{1}{3}\left(\frac{19}{20}I\right) = \frac{19}{60}I$.

In a similar manner, consider injecting $\frac{I}{20}$ of current into every vertex of the dodecahedron, and extracting current $I$ out from point $D$. Using similar reasoning, the current flowing from $C$ to $D$ in this setup is also $\frac{19}{60}I$.

We now superimpose the two setups. Each vertex other than $C$ and $D$ will have current $\frac{I}{20}$ injected and extracted simultaneously, no net current enters/exits the resistor network at these vertices. Point $C$ has current $I$ entering it and point $D$ has current $I$ exiting it. The net current flowing from $C$ to $D$ is $\frac{19}{60}I + \frac{19}{60}I = \frac{19}{30}I$. Since edge $CD$ has resistance $R_2 = \frac{5}{3}R$, the potential difference between points $C$ and $D$ is $\left(\frac{19}{30}I\right)\left(\frac{5}{3}R\right) = \frac{19}{18}IR$. Dividing this by the net current $I$ between point C and D, we have $R_{CD} = \frac{\frac{19}{18}IR}{I} = \frac{19}{18}R$.

However, we still need $R_{AB}$, not $R_{CD}$. Define $R'_{CD}$ to be the resistance of the rest of the dodecahedron between $C$ and $D$, excluding edge $CD$. Redrawing the circuit:



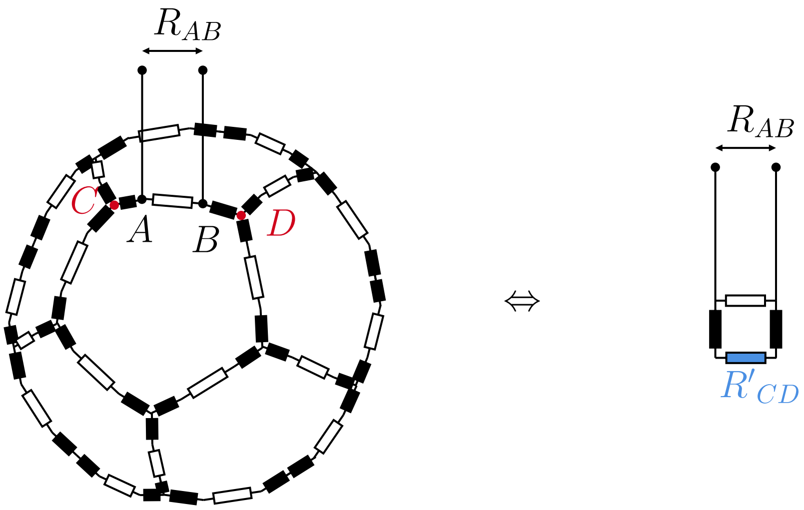

$$\frac{1}{R'_{CD}} + \frac{1}{R_2} = \frac{1}{R_{CD}}$$

$$R'_{CD} = \left(\frac{1}{R_{CD}} - \frac{1}{R_2}\right)^{-1}$$

$$= \left(\frac{1}{\frac{19}{18}R} - \frac{1}{\frac{5}{3}R}\right)^{-1}$$

$$= \frac{95}{33}R$$

We can now compute $R_{AB}$ by redrawing the circuit again:




We have:

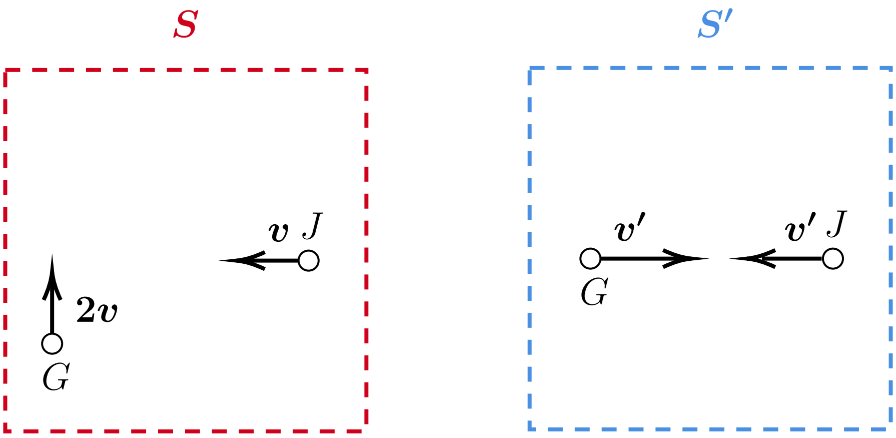
$$
\begin{aligned}
R_{AB} &= \left(\frac{1}{R} + \frac{1}{R'_{CD} + 2R_1}\right)^{-1} \\
&= \left(\frac{1}{R} + \frac{1}{\frac{95}{33}R + \frac{2}{3}R}\right)^{-1} \\
&= \frac{39}{50}R \\
&= \boxed{0.78\ \Omega}
\end{aligned}
$$

**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

### Problem 63: Delulu is the Solulu (5 points)

Guangyuan is a Twice stan. At a Twice concert, Jihyo is moving at velocity $v = \frac{2}{5}c$ in an inertial frame $S$. Unfortunately for Guangyuan, he can only move at velocity $2v$ in a direction perpendicular to her, as viewed in frame $S$. In his delusion, he argues that they are actually moving directly towards each other, as there exists an inertial frame $S'$ in which his velocity $v' = \beta' c$ has the same magnitude and opposite direction to Jihyo's (though its exact direction is not necessarily as shown in the diagram). Determine $\beta'$.

*Leave your answer to 3 significant figures.*




---

*Solution:* Our approach makes use of 4-vectors and the invariance of their inner products under Lorentz transformations[7]. We let Jihyo move along the $x$-axis and Guangyuan move along the $y$-axis in frame $S$, with $\gamma_v$ referring to the Lorentz factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. We can find the 4-velocities of Jihyo and Guangyuan in frame $S$, $V_J$ and $V_G$ respectively:

$$V_J = (\gamma_v c, \gamma_v v, 0, 0)$$
$$V_G = (\gamma_{2v} c, 0, 2\gamma_{2v} v, 0)$$

Now consider the frame $S'$. We choose the direction of the axes such that both Jihyo and Guangyuan have velocities directed along the $x$-axis. Their 4-velocities $V'_J$ and $V'_G$ can be written as follows:

$$V'_J = (\gamma_{v'} c, \gamma_{v'} v', 0, 0)$$
$$V'_G = (\gamma_{v'} c, -\gamma_{v'} v', 0, 0)$$

We make use of the fact that the inner product of two 4-vectors is invariant under Lorentz transformations. Furthermore, since the inner product is also invariant under

[7] For a more detailed discussion on 4-vectors and their interesting properties, see Chapter 13 of David Morin's *Introduction to Classical Mechanics*.

rotation, we are free to choose the axes directions in $S'$ arbitrarily. Taking the inner products of the two 4-velocities in $S$ and $S'$ and equating them, we have:

$$\begin{aligned}
V_J \cdot V_G &= V'_J \cdot V'_G \\
\gamma_v \gamma_{2v} c^2 &= \gamma_{v'}^2 (c^2 + v'^2) \\
\gamma_v \gamma_{2v} &= \frac{c^2 + v'^2}{c^2 - v^2} \\
v'^2 &= \frac{\gamma_v \gamma_{2v} - 1}{\gamma_v \gamma_{2v} + 1} c^2 \\
\beta' &= \sqrt{\frac{\gamma_v \gamma_{2v} - 1}{\gamma_v \gamma_{2v} + 1}} \approx \boxed{0.539}
\end{aligned}$$

---

*Alternative solution:* A more tedious approach uses the relativistic velocity-addition formulas. Let the frame $S'$ move at velocity $u$ along the $x$-axis with respect to the lab frame. Then, in $S'$, Jihyo and Guangyuan have velocities $v'_x$, $-v'_x$, $v'_y$ and $-v'_y$ in the $x$ and $y$ directions respectively. Let Jihyo move at angle $-\theta$ with respect to the $x$-axis and let Guangyuan move at angle $\frac{\pi}{2} - \theta$ in $S'$. We have:

$$\begin{aligned}
v'_x &= \frac{2v \sin\theta - u}{1 - \frac{u}{c^2} 2v \sin\theta} = -\frac{v \cos\theta - u}{1 - \frac{u}{c^2} v \cos\theta} \\
v'_y &= \frac{2v \cos\theta \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u}{c^2} 2v \sin\theta} = -\frac{-v \sin\theta \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u}{c^2} v \cos\theta}
\end{aligned}$$

Solving these equations simultaneously for $u$ and $\theta$ yields:

$$\begin{aligned}
u &\approx 1.5262 \times 10^8 \text{ m s}^{-1} \\
\theta &\approx 1.2545
\end{aligned}$$

Upon substituting these values back into the equations for $v'_x$ and $v'_y$, we can obtain:

$$\beta' = \frac{\sqrt{v_x'^2 + v_y'^2}}{c} \approx \boxed{0.539}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush M1: Weightlifting** (3 points)

In a simplified model of weightlifting, an athlete applies an initial upwards impulse to the weight, then allows it to move freely under the influence of only gravity. The maximum height of the weight can then be taken as the point where it reaches zero velocity.

The athlete can lift the weight in two ways:

1. **Clean and Jerk**: the weight is lifted in two stages, first to a momentary rest at a height $l = 0.90$ m, then lifted once again to height $2l$.
2. **Snatch**: the weight is lifted to height $2l$ immediately.

If a weightlifter lifts a maximum mass $m_c = 200$ kg for the clean and jerk, determine the maximum mass $m_s$ he can lift for the snatch. Assume that the maximum impulse that the weightlifter can impart remains the same.

*Leave your answer to 2 significant figures in units of* kg.

---

*Solution:* Let the initial velocity of the clean and jerk be $v_c$, and that of the snatch be $v_s$. Since the weightlifter imparts the same maximum impulse for both lifts, they have the same initial momentum:

$$m_c v_c = m_s v_s$$

Furthermore, by conservation of energy, $mgl = \frac{1}{2}mv^2 \implies v^2 \propto l$. We have:

$$\frac{v_c^2}{l} = \frac{v_s^2}{2l}$$
$$\frac{v_s}{v_c} = \sqrt{2}$$

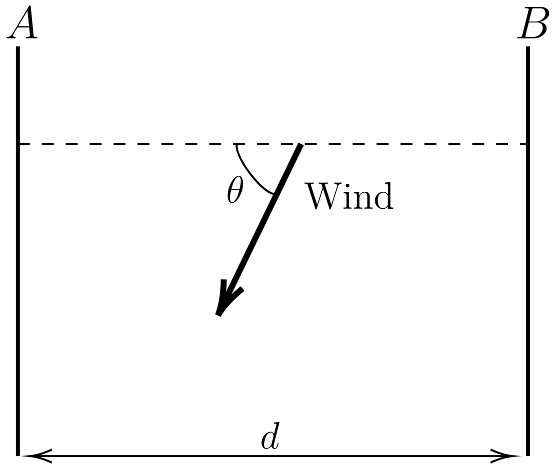
Hence we can solve for $m_s$ to obtain:

$$m_s = \frac{m_c v_c}{v_s} = \frac{m_c}{\sqrt{2}} \approx \boxed{140 \text{ kg}}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush M2: Smooth Sailing** (3 points)

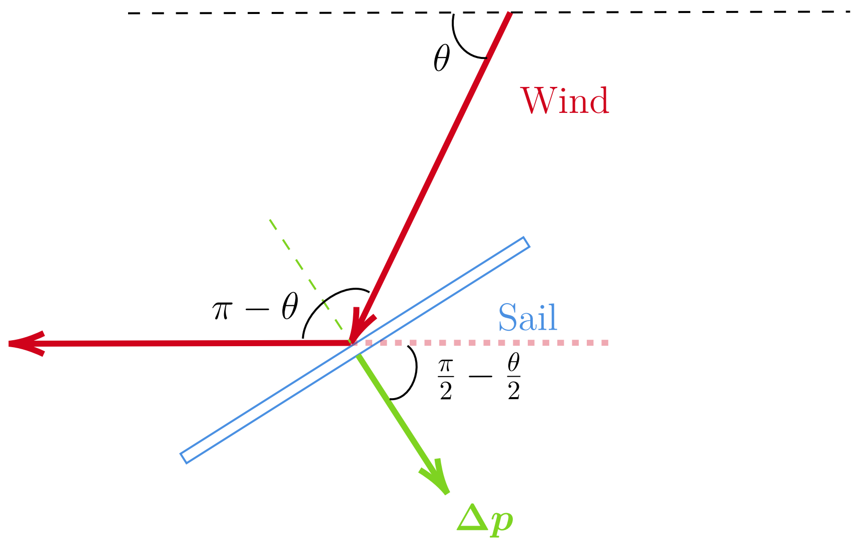
Roger is an Olympic sailor. He needs to travel from Shore A to Shore B, separated by a perpendicular distance $d = 2000$ m. However, the wind blows against him at an angle $\theta = 70^\circ$ to the axis perpendicular to the shores. His sail reflects the wind perfectly and can be oriented in any direction. He orientates it in a constant direction such that he reaches the other shore in the shortest possible time. Determine the distance $l$ that he travels from Shore A to Shore B.




*Leave your answer to 2 significant figures in units of* m.

---

*Solution:* The sail reflects the direction of motion of the air particles in the wind, changing their direction of momentum. If the sail is oriented such that the wind's horizontal momentum towards Shore A increases upon reflection, the boat's horizontal momentum towards Shore B will increase due to conservation of momentum. To maximise the boat's horizontal momentum towards Shore B, the final momentum of the wind should be perfectly horizontal and directed towards Shore A.



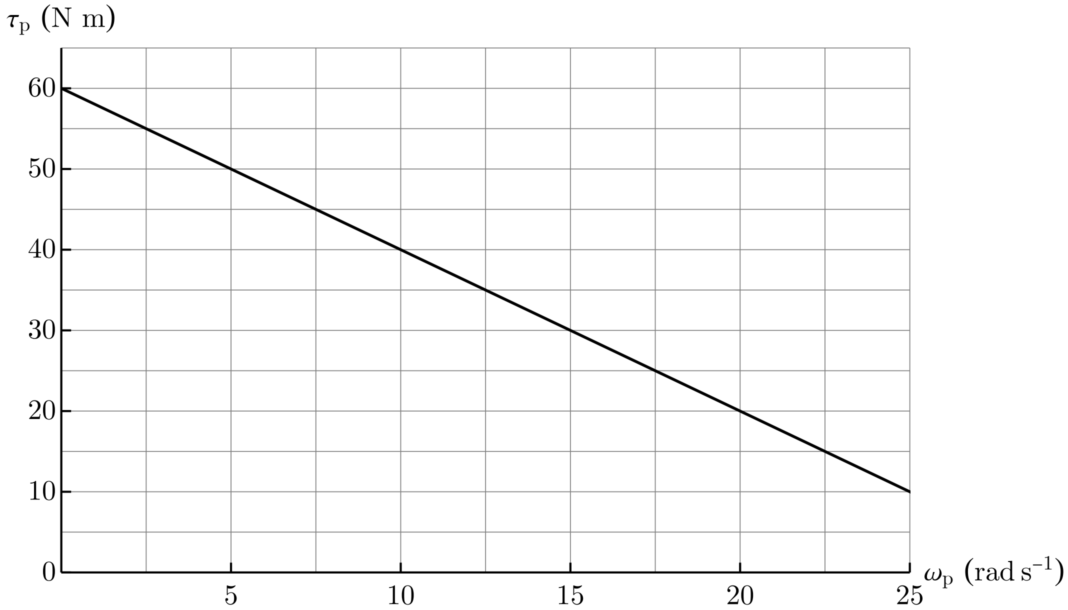
The boat's velocity will be in the direction of the change in momentum of the wind, which is perpendicular to the sail. Upon some angle chasing, we conclude that:

$$l = \frac{d}{\sin\frac{\theta}{2}} \approx \boxed{3500 \text{ m}}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush M3: Tour de France** (4 points)

Paul is an Olympic cyclist. The maximum torque $\tau_p$ that he can supply to the pedals on his bicycle is dependent on the angular velocity $\omega_p$ at which he pedals. The graph below shows the relation between $\tau_p$ and $\omega_p$.




Paul starts from rest pedalling with a 5 : 1 gear. At any time while cycling, he can choose to switch to a 10 : 1 gear. The gear ratio $a : b$ represents the ratio of the number of rotations of the wheels, $a$, to the number of rotations of the pedals, $b$.

Given that the radius of his wheels is $r = 0.20$ m, at what velocity $v$ of his bicycle should he switch gears to reach his top speed within the shortest time?

*Leave your answer to 2 significant figures in units of* m s$^{-1}$.

---

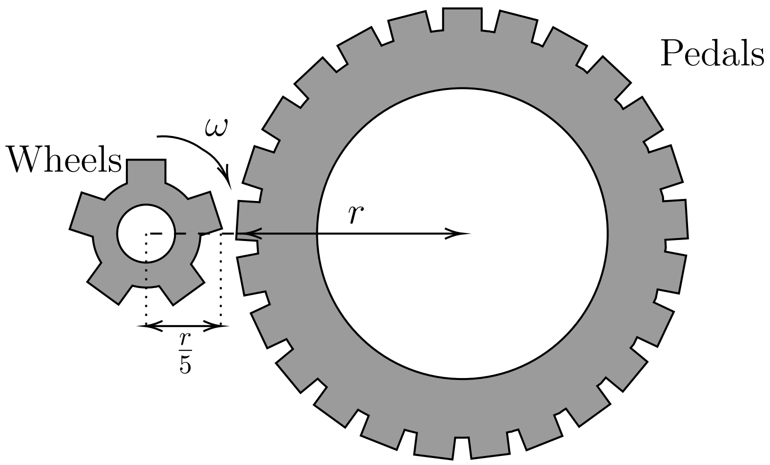
*Solution:* From the graph, we can obtain the equation for the linear relationship between $\tau_p$ and $\omega_p$:

$$\tau_p = \tau_0 - k\omega_p$$

where $\tau_0 = 60$ N m and $k = 2$ kg m$^2$ s$^{-1}$.

Let $i = a/b$ be the gear ratio (of 5 or 10 in this case), which is the ratio of the angular velocities of the gears. Since both gears share the same linear displacement as they rotate, the ratio of the wheel gear $r_w$ to the ratio of the pedal gear $r_p$ is $r_w/r_p = 1/i$.

To reach his top speed within the shortest time, Paul must maximise his average acceleration. This is done by maximising the torque supplied to the wheels $\tau_w$. Since the force between each pair of gears is equal by Newton's Third Law, the torque exerted on each gear is proportional to the radius of the gear, in other words $\tau_w/\tau_p = 1/i$.




Furthermore, the linear velocities of the gears are also equal, so the angular velocity of the gear will be inversely proportional to its radius. We thus have $\omega_w/\omega_p = i$. The actual velocity $v$ of his bicycle is then given by $v = r\omega_w = ir\omega_p$.

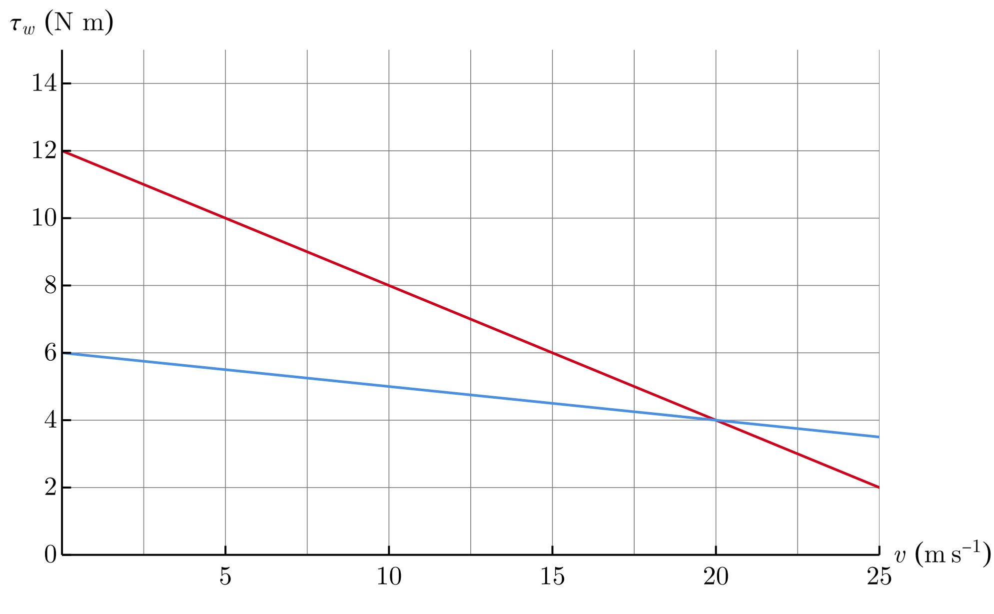
In summary, we have:

$$\tau_w = \frac{\tau_p}{i}$$
$$v = ir\omega_p$$

The equation for the graph of $\tau_w$ against $v$ can then be written as follows:

$$i\tau_w = \tau_0 - \frac{kv}{ir}$$
$$\tau_w = \frac{1}{i}\left(\tau_0 - \frac{kv}{ir}\right)$$

We can plot $\tau_w$ against $v$ for $i = 5$ and $i = 10$, shown in red and blue respectively:

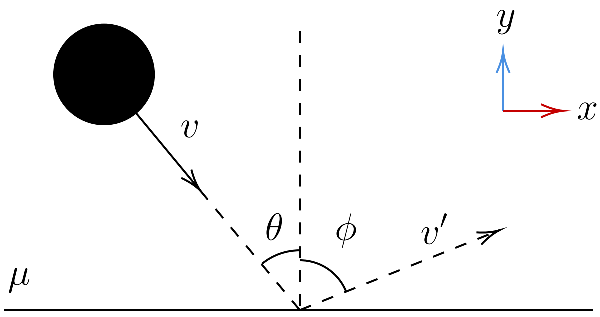



These two lines intersect at $v = \boxed{20\ \text{m s}^{-1}}$. Below this velocity, the torque provided by the first gear is larger, while above this velocity, the torque provided by the second gear is larger. We should thus switch gears at this velocity.

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

## Half Hour Rush M4: Tennis Ball (5 points)

Paul, an Olympic tennis player, throws a spherical tennis ball at a wall with coefficient of kinetic friction $\mu = 0.5$. The ball travels in a straight line with initial speed $v = 6.9$ m s$^{-1}$, at an angle $\theta = 50°$ with respect to the normal.




*(Top-down View)*

The coefficient of restitution $e$ of the ball-wall collision is a constant, defined as the ratio of the magnitude of normal velocities of the ball after and before the collision. Find $e$ such that the speed of the ball after the collision $v'$ is minimised.

Assume the ball is always slipping against the wall. Ignore gravity and air resistance and assume all energy loss comes from the reaction forces (i.e. friction and normal forces).

*Leave your answer to 2 significant figures.*

---

*Solution:* Throughout the collision, the only forces on the ball are the normal and frictional forces with the wall. Since our kinetic friction is always $f_k = \mu N$, the direction of the resultant of the normal force and friction, and therefore the direction of the impulse $\vec{J}$ from the wall, is always angled at $\tan^{-1}\mu$ relative to the normal, in the direction opposing the initial motion. Thus, the change in velocity $\Delta\vec{v}$ points in this direction as well.

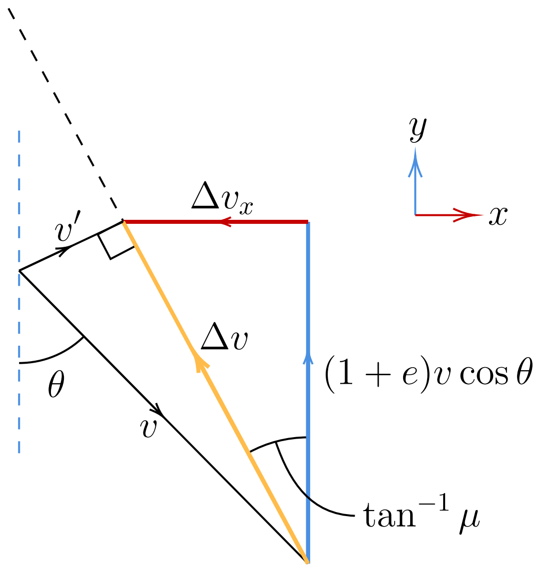
The change in normal velocity $\Delta v_y$ is given by:

$$\begin{aligned}\Delta v_y &= ev\cos\theta - (-v\cos\theta)\\ &= (1+e)v\cos\theta\end{aligned}$$

Now, there are two ways to represent $\Delta\vec{v}$:

1. the vector sum of $\Delta\vec{v}_x$ and $\Delta\vec{v}_y$
2. the vector sum of $\vec{v}'$ and $\vec{v}$

Since $\theta > \tan^{-1}\mu$, we can superimpose these two representations into the following vector diagram:




where $\vec{v}\,'$ is the final velocity of Paul's ball.

We notice that varying $e$ only varies the length of $\Delta\vec{v}$ with its direction staying constant. As such, the minimum $\vec{v}\,'$ is obtained when $\Delta v$ is oriented such that $\vec{v}\,' \perp \Delta\vec{v}$.

From the diagram, equating the length of the common side gives us:

$$v\cos\left(\theta - \tan^{-1}\mu\right) = \frac{(1+e)v\cos\theta}{\sin(\tan^{-1}\mu)}$$
$$\Rightarrow \frac{\cos\theta + \mu\sin\theta}{\sqrt{1+\mu^2}} = \sqrt{1+\mu^2}(1+e)\cos\theta$$

Rearranging, we get:

$$e = \frac{\mu(\tan\theta - \mu)}{1+\mu^2}$$
$$\approx \boxed{0.28}$$

---

*Alternative solution:* The normal impulse is given by:

$$J = mv_y + mv_y'$$
$$= mv_y + m(ev_y) = (1+e)mv_y$$

The horizontal impulse is given by:

$$\mu J = m(v_x - v_x')$$
$$\Rightarrow \mu(1+e)v_y = v_x - v_x'$$
$$\Rightarrow v_x' = v_x - \mu(1+e)v_y$$

The final velocity of Paul's ball can be expressed as:

$$\begin{aligned} v'^2 &= v_x'^2 + v_y'^2 \\ &= (v_x - \mu(1+e)v_y)^2 + (ev_y)^2 \end{aligned}$$

Differentiating with respect to $e$ and equating $\mathrm{d}v'/\mathrm{d}e = 0$, we have:

$$\begin{gathered} 2ev_y^2 + 2(v_x - \mu(1+e)v_y)(-\mu v_y) = 0 \\ \Rightarrow (1+\mu^2)e - \mu\left(\frac{v_x}{v_y} - \mu\right) = 0 \end{gathered}$$

Substituting $v_x/v_y = \tan\theta$ and rearranging gives us back the same expression as the previous solution:

$$e = \frac{\mu(\tan\theta - \mu)}{1+\mu^2}$$

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**

**Half Hour Rush E1: Karma** (3 points)

Gerrard and Ziwen are both point masses standing in the same plane $P$. Gerrard tries to hit Ziwen with a small ball of mass $m = 0.50$ kg by shooting it towards him directly. Unbeknownst to him, the ball has charge $q = 3.0$ C and there is a uniform magnetic field $B = 0.20$ T directed perpendicular to $P$. Determine the time taken $t$ for Gerrard to be hit by his own ball.

*Leave your answer to 2 significant figures in units of* s.

---

*Solution:* We denote the velocity of the ball as $v$. The Lorentz force acting on the moving charged ball is always perpendicular to its velocity, acting as the centripetal force which keeps the ball moving in a circle. We can equate these forces to determine the radius of the circle $r$:

$$Bvq = \frac{mv^2}{r}$$
$$r = \frac{mv}{Bq}$$

The time it takes for Gerrard to be hit by his own ball is the time it takes for the ball to complete one circular revolution. We have:

$$t = \frac{2\pi r}{v} = \frac{2\pi m}{Bq} \approx \boxed{5.2 \text{ s}}$$

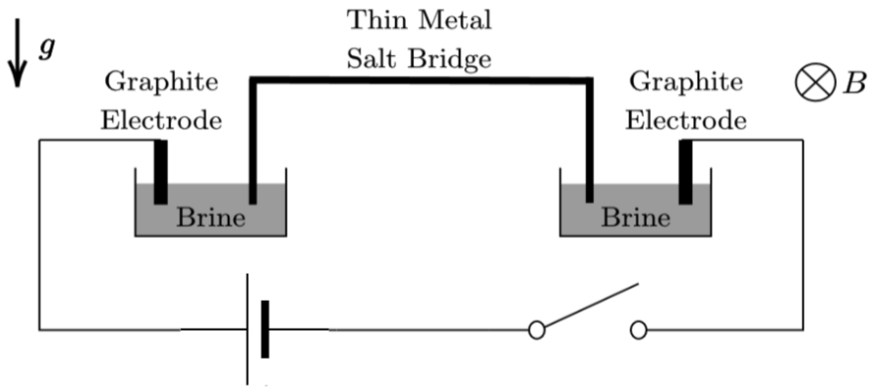
**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush E2: You Need To Calm Down** (4 points)

Claurine is about to manufacture copious amounts of chlorine gas at home via the chemical setup shown below. To make Claurine calm down, her sister Claudine exposes the horizontal section of the thin conducting salt bridge of length $l = 0.20$ m to a uniform magnetic field $B = 0.10$ T directed into the plane containing the salt bridge. The salt bridge of total mass $m = 0.030$ kg is initially at rest but free to move vertically.

Claurine then closes switch $S$ and is shocked to observe the entire salt bridge jump up instantly and rise to a maximum height $h = 2.0$ m above its original position. Find the magnitude of the total charge $q$ that flows in the salt bridge throughout the process.

Assume that the salt bridge loses contact with the brine completely before reaching its maximum height.




*Leave your answer to 2 significant figures in units of* C.

---

*Solution:* We first denote $T$ as the (extremely small) time interval between the instant when switch $S$ is closed and when the circuit is reopened due to the jumping up of salt bridge $KLMN$. During this time interval, a current made up of cations and anions flows in the horizontal section of the salt bridge from point $L$ to point $M$. The interaction between this current and the magnetic field gives rise to a magnetic force on the salt bridge directed vertically upwards as per Fleming's Left-Hand Rule. The magnitude of this magnetic force $F$ at one instant is given by $BIl$, where $I$ is the magnitude of the current in the salt bridge at this instant. The action of $F$ throughout $T$ imparts an impulse $J$ on the salt bridge, with a magnitude given by:

$$J = \int_0^T F \, \mathrm{d}t = \int_0^T BIl \, \mathrm{d}t = Bl \int_0^T I \, \mathrm{d}t = Blq$$

This instantaneous impulse causes the salt bridge to acquire an upward velocity $v$ and then ascend by a height $h = 1.50$ m. Since the salt bridge is initially at rest, the magnitude of this impulse $J$ must be equal to $mv$:

$$Blq = mv$$

It is worthy to note that the salt bridge is no longer subjected to any force by the magnetic field during its upward motion, as (1) there is no more current flowing in it to interact with the magnetic field, and (2) the salt bridge does not form a closed loop for any current to be induced in it. Therefore, we can simply write the kinematic equation $v^2 = 2gh$, and it follows that:

$$q = \frac{mv}{Bl} = \frac{m}{Bl}\sqrt{2gh} \approx \boxed{9.4\text{ C}}$$

In the aftermath, Claudine finds her sister in a state of profound shock, evidently startled by the abrupt jumping up of the salt bridge. It seems unlikely that Claurine is going to calm down anytime soon.

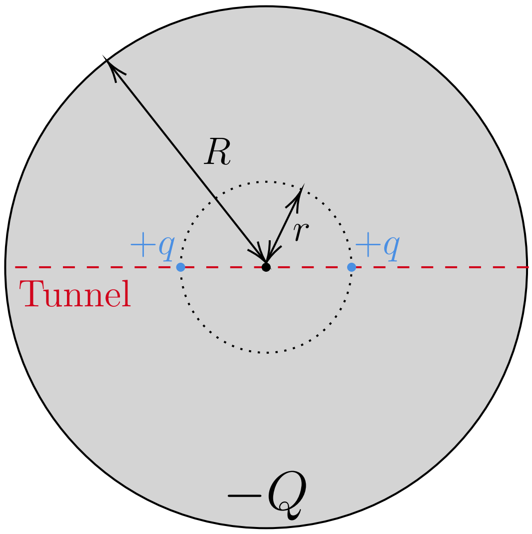
**Setter: Liu Yueyang,** yueyang.liu@sgphysicsleague.org

**Half Hour Rush E3: We Are Never Ever Getting Back Together** (4 points)

An infinitesimally narrow tunnel is drilled through an insulating solid sphere with radius $R$ and uniform charge $-Q$. Romeo and Juliet are two point charges each with charge $+q$, both located inside this tunnel an initial distance $r = 2.00$ m away from the centre of the sphere. Upon release from rest, Romeo and Juliet are separated by a maximum distance $d$ during their subsequent motion. Calculate $d$.

Take $\frac{Q}{q} = \frac{1}{2}$ and $\frac{R}{r} = 2$.

*Leave your answer to 3 significant figures in units of* m.




---

*Solution:* Initially, the point charges will repel each other. However, as the distance between the point charges increases, the repulsive force between them decreases while the attractive force exerted by the charged solid sphere on the point charges increases. Thus, there exists a maximum separation between the point charges. At this moment, the velocity and hence kinetic energy of the point charges is instantaneously zero, so it is convenient to solve this problem via conservation of energy.

Let the electric potential energy at infinity be zero. The initial potential energy of the system $U_i$ consists of two terms: the potential energy between the two point charges, $U_1$, and the potential energy between the solid sphere and each of the point charges, $U_2$. Since the point charges are identical, $U_2$ is just double the potential energy between the solid sphere and one of the point charges.

$U_1$ can be directly found:

$$U_1 = \frac{kq^2}{2r}$$

To find $U_2$, consider the (attractive) electric force $F$ experienced by a point charge inside the solid sphere at radius $r$. By Newton's Shell Theorem, $F$ is exerted by the portion of the solid sphere with radius $r$ and not by the portion outside $r$. $F$ is given by:

$$\begin{aligned} F &= \frac{kQq}{r^2}\frac{r^3}{R^3} \\ &= \frac{kQq}{R^3}r \end{aligned}$$

Thus, $F \propto r$. This is analogous to the spring force, where the restoring force is proportional to the distance extended or compressed. Hence, we can use the potential energy formula $E = \frac{1}{2}Kx^2$, where $K = \frac{kQq}{R^3}$ to find $U_2$.

We add up the work done to bring the point charge from infinity to the surface of the shell, and the work done to then bring it from $R$ to $r$:

$$\begin{aligned} U_2 &= 2(U_{\infty\to R} + U_{R\to r}) \\ &= 2\left(-\frac{kQq}{R} + \left(-\frac{1}{2}\frac{kQq}{R^3}R^2\right) - \left(-\frac{1}{2}\frac{kQq}{R^3}r^2\right)\right) \end{aligned}$$

Thus, the total initial electric potential energy $U_i$ of the system is given by:

$$\begin{aligned} U_i &= U_1 + U_2 \\ &= \frac{kq^2}{2r} + 2\left(-\frac{kQq}{R} - \frac{1}{2}\frac{kQq}{R^3}R^2 + \frac{1}{2}\frac{kQq}{R^3}r^2\right) \\ &= \frac{kq^2}{2r} + \frac{kQq}{R^3}r^2 - \frac{3kQq}{R} \end{aligned}$$

The final electric potential energy of the system at maximum distance $U_f$ is given by[8]:

$$U_f = \frac{kq^2}{d} - \frac{4kQq}{d}$$

Applying conservation of energy, we may solve for $d$ in terms of $r$:

$$\begin{aligned} \frac{kq^2}{2r} + \frac{kQq}{R^3}r^2 - \frac{3kQq}{R} &= \frac{kq^2}{d} - \frac{4kQq}{d} \\ d &= \frac{2q - 8Q}{\frac{q}{r} + \frac{2Q}{R}\left(\frac{r^2}{R^2} - 3\right)} \\ &= \frac{2 - 8\left(\frac{Q}{q}\right)}{1 + 2\left(\frac{Q}{q}\right)\left(\frac{R}{r}\right)^{-1}\left[\left(\frac{R}{r}\right)^{-2} - 3\right]}r \\ &= \frac{32}{3}\text{ m} \\ &\approx \boxed{10.7\text{ m}} \end{aligned}$$

[8]This expression is only valid for a maximum distance that occurs outside the sphere i.e. $d > R$. For a maximum distance that occurs inside the sphere, the expression is more complicated. However, you can show that for the values given in the problem, the maximum distance will occur outside the sphere by showing that $U_i$ is larger than the electric potential energy of the point charge when it reaches the surface of the solid sphere (meaning that the kinetic energy is non-zero).

*Note:* In fact, the subsequent motion of the point charges is an oscillation! Each point charge will oscillate between the point of minimum separation (the initial position) and the point of maximum separation (the distance calculated in this problem).

**Setter: Roger Zhang, roger.zhang@sgphysicsleague.org**

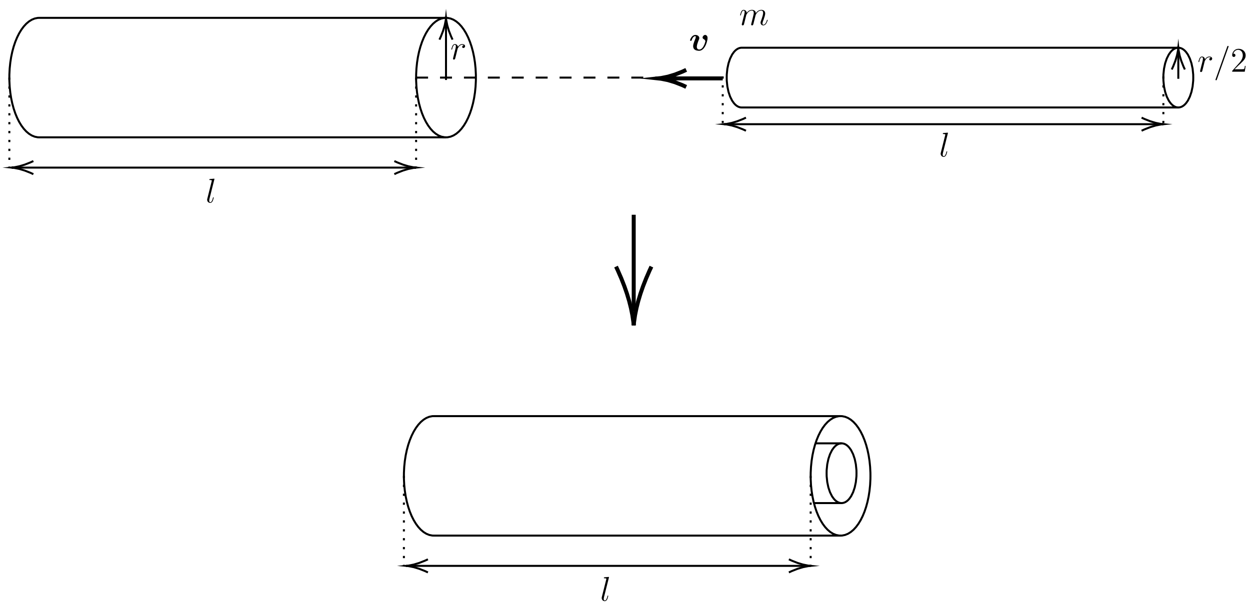
## Half Hour Rush E4: Blank Space (5 points)

A fixed superconducting cylindrical shell A of radius $r = 0.05$ m and length $l = 100$ m initially has uniform current flowing in its azimuthal direction with total magnitude $I = 500$ A. A second superconducting cylindrical shell B of mass $m = 2.5 \times 10^{-6}$ kg, radius $\frac{r}{2}$ and length $l$ initially has no current flowing through it. It is positioned infinitely far away from shell A, and is allowed to move.

Shell B is launched towards shell A with initial velocity $v$. The axes of symmetry of the two cylinders are aligned throughout the subsequent motion of shell B. Determine the velocity $v$ such that shell B will fill the length of the blank space within shell A exactly, after a long time.

*Hint:* The magnetic flux through a superconductor is conserved.

*Leave your answer to 2 significant figures in units of* m s$^{-1}$.




---

*Solution:* By Faraday's Law, a change in magnetic flux through a body induces an electromotive force (EMF). However, since superconductors have zero resistance, any EMF would imply that an infinitely large current is produced. Hence, magnetic flux through a superconductor is conserved.

Firstly, let us determine the magnetic flux through the cylindrical shells. Since $l \gg r$, we can treat the cylindrical shells as infinitely long solenoids with one turn of wire, so the magnetic field $B$ inside the shell along the axis is given by:

$$B = \mu_0 \frac{I}{l}$$

Initially, there is no magnetic flux through shell B, whereas there is a magnetic flux through shell A. In the final state, when shell B fills the length of shell A exactly, there will still be no magnetic flux through shell B due to the conservation of magnetic flux. This implies that the final currents flowing through the two shells are equal and opposite. We shall denote this current as $I'$.

Since the magnetic flux through shell A is also conserved, we have:

$$\pi r^2 I = \pi \left(r^2 - \left(\frac{r}{2}\right)^2\right) I'$$
$$I' = \frac{4}{3} I$$

The smaller cylinder will slow down as it passes through the larger cylinder as its kinetic energy is converted to magnetic field energy within the cylinders. The magnetic field energy density $u$ is given by:

$$u = \frac{B^2}{2\mu_0}$$

Superconductivity implies that energy is conserved, hence:

$$\frac{1}{2} m v^2 + \frac{\left(\mu_0 \frac{I}{l}\right)^2}{2\mu_0} \pi r^2 l = \frac{\left(\mu_0 \frac{I'}{l}\right)^2}{2\mu_0} \pi \frac{3}{4} r^2 l$$
$$\frac{1}{2} m v^2 = \frac{\mu_0 I^2 \pi r^2}{6l}$$

We can finally solve for $v$ to obtain:

$$v = \sqrt{\frac{\mu_0 I^2 \pi r^2}{3ml}} \approx \boxed{1.8\,\mathrm{m\ s^{-1}}}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush X1: Stuck Ball** (3 points)

Tom seals an empty bottle lying horizontally with a solid spherical ball. The opening of the bottle and the ball have the same cross-sectional area $A = 4 \times 10^{-4}$ m$^2$. Jerry then punctures a small hole in the bottle, pumps all the air out of the bottle and reseals it, creating a vacuum. Suppose that static friction between the ball and the bottle is at its maximum value, and is precisely sufficient to keep the ball in place. Determine the minimum force $F$ required for Tom to remove the ball.

You may refer to the Data Sheet for the value of the atmospheric pressure $p_0$.

*Leave your answer to 2 significant figures in units of* N.

---

*Solution:* Atmospheric pressure pushes the ball inwards. Since the inside of the bottle is a vacuum, there is no air pressure pushing the ball outwards, and only static friction at its maximum $f_{\text{max}}$ keeps the ball in place. Hence, its magnitude is equal to the atmospheric pressure multiplied by the cross-sectional area of the ball:

$$f_{\text{max}} = p_0 A$$

As Tom pulls the ball outwards, the static friction changes in magnitude to keep the ball in equilibrium. As $F$ increases beyond the force due to atmospheric pressure, the static friction will act inwards instead to oppose the motion of the ball. Thus, $F$ has to overcome both the static friction and the force due to atmospheric pressure:

$$F = f_{\text{max}} + p_0 A = 2 p_0 A \approx \boxed{81 \text{ N}}$$

**Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org**

**Half Hour Rush X2: My Ball Is Shockingly Bright** (4 points)

Theo has a solid ball of radius $R = 0.690$ m made of the metal rhodium, with charge $Q_0 = +1.11$ nC on it. Theo then shines monochromatic ultraviolet radiation of wavelength $\lambda = 37.0$ nm on the ball. Rhodium has a work function of $\Phi = 4.98$ eV. After a very long time, the charge on the ball stays relatively constant. If the number of photoelectrons that have escaped over the entire duration is $N$, calculate $\log_{10} N$.

*Leave your answer to 2 significant figures.*

---

*Solution:* The electric potential of Theo's ball is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

Electrons will only escape Theo's ball if they have enough energy to do so. Applying the photoelectric equation:

$$\frac{hc}{\lambda} \geq \Phi + eV$$

As electrons leave the ball, the total charge $Q$ increases, and hence the potential $V$ increases as well. Electrons are no longer able to escape Theo's ball when the energy from each photon is insufficient:

$$\frac{hc}{\lambda} = \Phi + eV_f$$

where $V_f$ is the final electric potential of Theo's ball. We can find the final charge on the ball using the final potential:

$$\frac{1}{4\pi\epsilon_0}\frac{Q_f}{R} = \frac{1}{e}\left(\frac{hc}{\lambda} - \Phi\right)$$

$$Q_f = \frac{4\pi R\epsilon_0}{e}\left(\frac{hc}{\lambda} - \Phi\right)$$

We can then find the change in charge and hence find the number of electrons:

$$\Delta Q = \frac{4\pi R\epsilon_0}{e}\left(\frac{hc}{\lambda} - \Phi\right) - Q_0$$

$$N = \frac{1}{e}\left(\frac{4\pi R\epsilon_0}{e}\left(\frac{hc}{\lambda} - \Phi\right) - Q_0\right)$$

$$\log_{10} N = \log_{10}\left(\frac{4\pi R\epsilon_0}{e^2}\left(\frac{hc}{\lambda} - \Phi\right) - \frac{Q_0}{e}\right)$$

$$\approx \boxed{9.8}$$

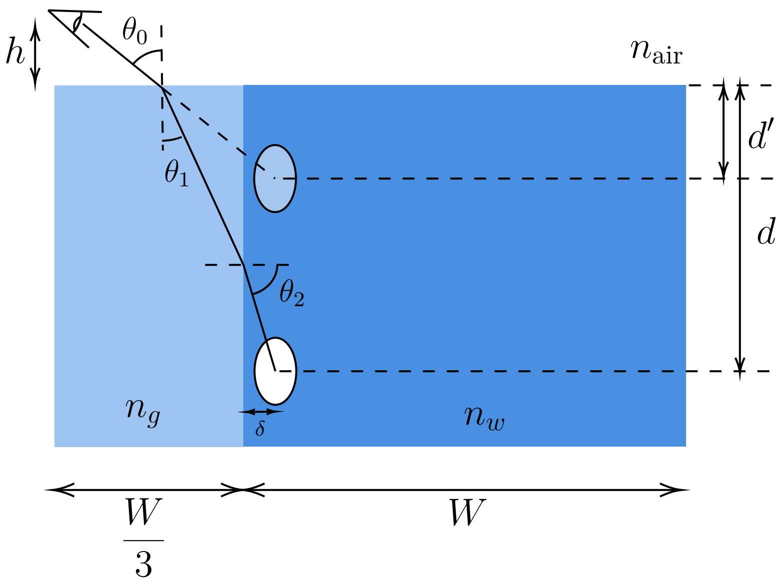
**Setter: Sun Yu Chieh,** yuchieh.sun@sgphysicsleague.org

**Half Hour Rush X3: Ball's Deep** (4 points)

Jed, who has height $h = 1.33$ m, stands at the edge of a deep pool of width $W = 5$ m and glass walls of thickness $W/3$.

Jed accidentally drops his favourite ball into the pool at a small distance $\delta \ll W$ from the edge. Moving to the far edge of the glass wall to get a better look through the glass, he observes that the ball stops sinking at an apparent depth $d'$. Find $d'$.

<u>Refractive indices:</u>
Air: $n_{\text{air}} = 1$
Water: $n_w = 1.33$
Glass: $n_g = 1.50$




*Leave your answer to 2 significant figures in units of* m.

---

*Solution:* When the ball appears to stop sinking, the angles $\theta_0$, $\theta_1$ and $\theta_2$ should be approximately constant. For $\theta_2$ to remain approximately constant while the ball continues to sink, the ball must have sunk to a great depth, with $\theta_2 \approx \pi/2$.

Applying Snell's Law at the water-glass interface, we have:

$$n_g \sin\left(\frac{\pi}{2} - \theta_1\right) = n_w \sin\theta_2$$
$$n_g \cos\theta_1 \approx n_w \sin\left(\frac{\pi}{2}\right)$$
$$= n_w$$

As $\delta \ll W$, we can express the apparent depth $d'$ as:

$$d' = \frac{W}{3\tan\theta_0} - h$$

Finally, considering the glass-air interface, we have:

$$\sin\theta_0 = n_g \sin\theta_1$$

Solving, we get:

$$\tan\theta_0 = \sqrt{\frac{n_g^2 - n_w^2}{1 - n_g^2 + n_w^2}}$$

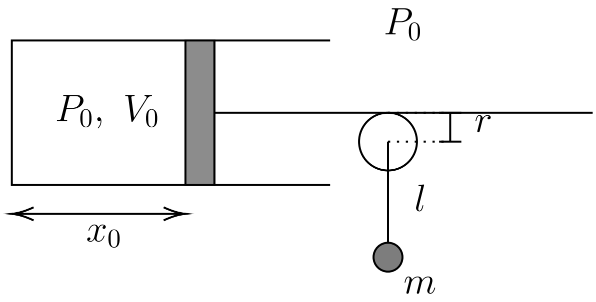
Hence, we can find $d'$:

$$\begin{aligned} d' &= \frac{W}{3}\sqrt{\frac{1 - n_g^2 + n_w^2}{n_g^2 - n_w^2}} - h \\ &\approx \boxed{0.40 \text{ m}} \end{aligned}$$

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**

**Half Hour Rush X4: Pendulous Balls** (5 points)

A rigid massless rod of length $l = 2$ m is fixed to the centre of a massless wheel of radius $r = 0.1$ m on one end, and a mass $m = 10$ kg on the other end, forming a pendulum. The wheel's rotation drives the shaft of a movable piston without slipping. Sealed in the container is an ideal gas initially at atmospheric pressure $P_0 = 1.01 \times 10^5$ Pa and initial volume $V_0 = 1$ m$^3$. Initially, this system is in equilibrium at length $x_0 = 1$ m.

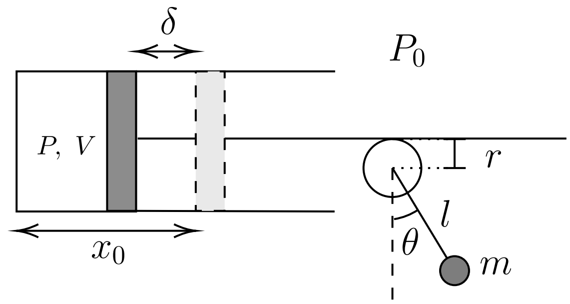



The walls of the container are thermally conductive, causing the gas to stay at constant temperature. Yueyang taps the ball of the pendulum *lightly*. Find the angular frequency $\omega$ of the subsequent small oscillations.

*Leave your answer to 2 significant figures in units of* rad s$^{-1}$.

---

*Solution:* Suppose the wheel has turned an angle $\theta$, driving the piston a distance $\delta$ into the gas. This causes its pressure to rise to $P$.




Considering torques about the centre of the wheel, we have:

$$ml^2\ddot{\theta} = -(P - P_0)Ar - mgl\sin\theta \tag{1}$$

where $A$ is the area of the piston/container.

Since the wheel drives the shaft without slipping, we can express:

$$\delta = r\theta$$

Thus, the new gas volume $V$ is given by:

$$V = V_0 - Ar\theta$$

Since the gas compresses isothermally, we have:

$$PV = P_0 V_0$$
$$\Rightarrow P = \frac{P_0 V_0}{V} = \frac{P_0 V_0}{V_0 - Ar\theta}$$

Substituting this into (1):

$$ml^2\ddot{\theta} = P_0 Ar\left(1 - \frac{V_0}{V_0 - Ar\theta}\right) - mgl\sin\theta$$

Notice that $V_0 = Ax_0$. We can now write:

$$ml^2\ddot{\theta} = P_0 Ar\left(1 - \frac{1}{1 - r\theta/x_0}\right) - mgl\sin\theta \tag{2}$$

As Yueyang only lightly tapped the pendulum, we may make a small angle approximation, with $\sin\theta \approx \theta$ and $r\theta \ll x_0$. By a binomial expansion to first order in $\theta$:

$$\frac{1}{1 - r\theta/x_0} \approx 1 + \frac{r\theta}{x_0}$$

Hence, we can express (2) as:

$$ml^2\ddot{\theta} \approx -\frac{P_0 Ar^2}{x_0}\theta - mgl\theta$$
$$\approx -\left(\frac{P_0 V_0 r^2}{x_0^2} + mgl\right)\theta$$
$$\Rightarrow \ddot{\theta} \approx -\left(\frac{P_0 V_0 r^2}{ml^2 x_0^2} + \frac{g}{l}\right)\theta$$

Therefore, the frequency $\omega$ of small oscillations is:

$$\omega = \sqrt{\frac{P_0 V_0 r^2}{ml^2 x_0^2} + \frac{g}{l}}$$
$$= \boxed{5.5 \text{ rad s}^{-1}}$$

**Setter: Tran Duc Khang, khang.tran@sgphysicsleague.org**