

Problem 1: Kicking Pebbles

(3 points)

After a long day of work, Le Xi walks along a cobbled path at speed $u = 1.4 \text{ m s}^{-1}$. At any time, one of his feet is in contact with the ground, while the other moves at a constant forward velocity slightly off the ground. The moving foot collides with loose pebbles on the ground, kicking them forward at speed v relative to the ground. If the collisions between his shoe and the pebbles are perfectly elastic, determine the maximum value of v .

Leave your answer to 3 significant figures in units of m s^{-1} .

Solution:

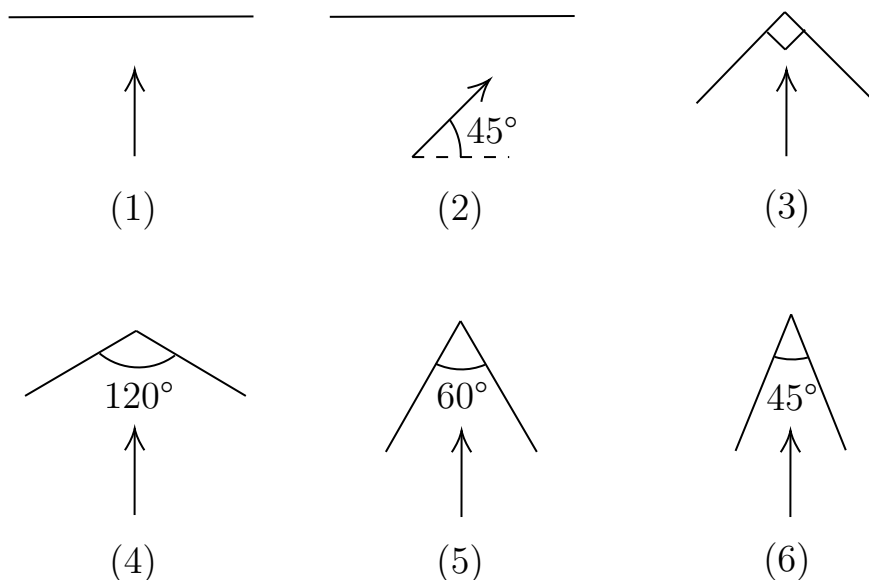
Since each foot is only moving for half the time, and its time-averaged velocity must be equal to Le Xi's walking speed, it must move at $2u$ while off the ground. The pebble starts off stationary and has a much smaller mass than the foot, so its maximum final velocity (relative to the ground) is twice that of the foot. As such, $v = 4u = \boxed{5.6 \text{ m s}^{-1}}$.

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem A: Reflection

(3 points)

In the movie “Mulan”, she looks into a series of reflective surfaces and sings the line “why is my reflection someone I don’t know?”. This is likely because she got an eyepatch on her right eye during the war (not canonical), but her reflection has an eyepatch on its left eye. In the diagrams below, mirrors are represented by the black lines. In which of the diagrams will Mulan be looking directly at an image of herself with a right eyepatch?

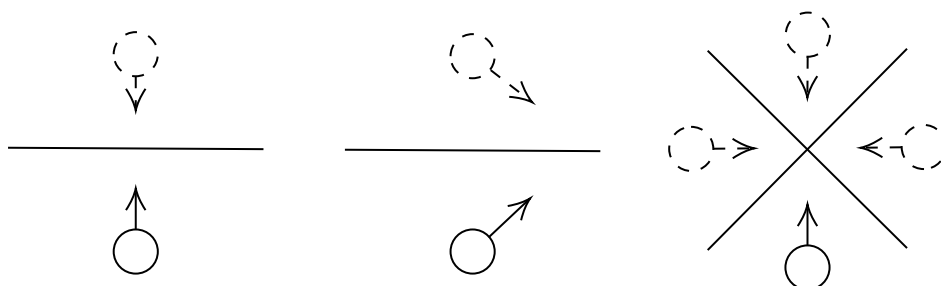


Give your answer by concatenating the diagram numbers above, in ascending order. For example, if you think diagrams (1), (3), and (4) result in her looking directly at an image of herself with a right eyepatch, submit 134.

Leave your answer as an integer.

Solution:

For each of the configurations given, the angle between the mirrors, θ evenly divides the circle into n slices, where $n = 360^\circ/\theta$. The resulting images are shown in the following diagrams:



Mulan is only looking directly at an image of herself with a right eyepatch for diagrams (3) and (6), so the answer is 36.

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Problem B: Dress to Impress

(3 points)

Wonky is sauntering down a street, wearing his favourite hoodie. He takes steps at a frequency of $f = 130$ bpm (beats per minute), in time with Taylor Swift's hit "[Cardigan](#)". He notices that if the drawstring of his hoodie has a certain length l_0 , it will swing from side to side with a large amplitude while he walks.

Assume that the aglet of the drawstring is much more massive than the drawstring itself, causing it to behave like a simple pendulum. Determine the value of l_0 .

Leave your answer to 2 significant figures in units of cm.

Solution:

As Wonky walks, his body will sway from side to side with a frequency half that of his steps, thus driving the oscillations of the drawstring. For the drawstring to have a large amplitude of oscillation, this has to match its natural frequency. The frequency of the pendulum's oscillations is then

$$f_0 = \frac{130 \text{ bpm}}{2 \times (60 \text{ s min}^{-1})} = 1.08 \text{ Hz}.$$

The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l_0}{g}} = \frac{1}{f_0}.$$

Solving for l_0 ,

$$\begin{aligned} l_0 &= \frac{g}{4\pi^2 f_0^2}, \\ &= \boxed{21 \text{ cm}}. \end{aligned}$$

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Problem 2: ATLP120X

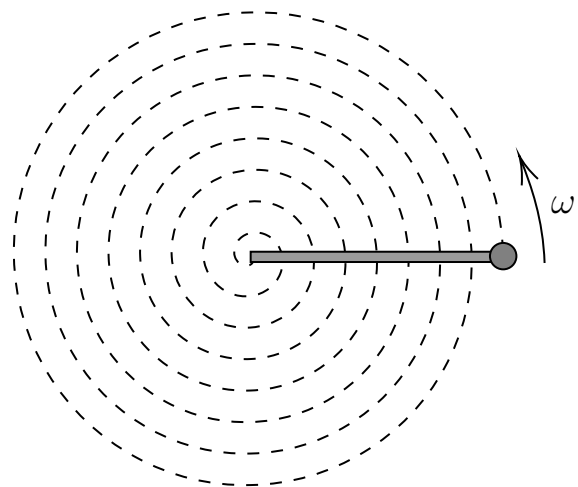
(3 points)

Vinyl records are analog recordings: music is converted into tiny grooves on the disc, which form a spiral inward. To play the record, a record player spins it at a constant angular speed of $\omega = 10 \text{ rev s}^{-1}$. A stylus is placed at the start of the spiral, and slowly moves inwards along the groove. On this particular record, the average number of grooves per radial centimeter is $n = 50.0 \text{ cm}^{-1}$.

The outermost groove is $r_s = 14.0 \text{ cm}$ from the centre of the record, and the innermost groove is $r_l = 6.00 \text{ cm}$ from the centre of the record. Determine the total playtime t of the record.



(The author likes his hobbies.)



Leave your answer to 2 significant figures in units of s.

Solution: As the grooves are tiny, the spiral can be approximated as a large number of concentric rings N . The total playtime is equal to the time taken for the stylus to pass over all the rings.

Therefore,

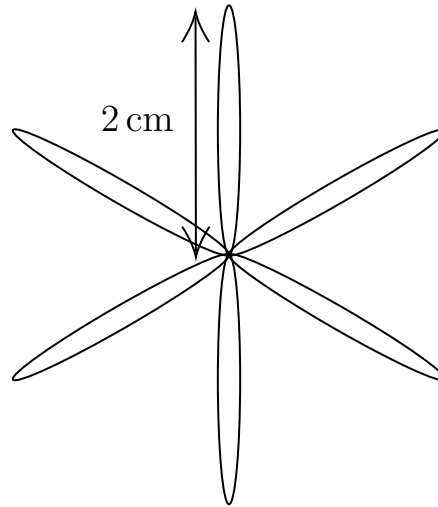
$$\begin{aligned}
 t &= \frac{N}{\omega} \\
 &= \frac{n(r_s - r_l)}{\omega} \\
 &= \boxed{40 \text{ s}}
 \end{aligned}$$

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Problem 3: Hexagon Force

(3 points)

Six identical lenses with diameter $D = 2.0$ cm are arranged in a hexagon, with their edges touching each other:



Determine the maximum focal length f of each lens such that there exists a light beam that travels in a closed loop around the hexagon.

Leave your answer to 3 significant figures in units of cm.

Solution:

At the maximum focal length, this beam forms a hexagon at the edge of the lenses. Thus, for each of the lenses, light enters and leaves the lens at an angle of 30° to the optical axis. We denote the radius of each lens as $r = D/2 = 1.0$ cm.

If we consider a ray travelling along the optical axis, we find that each refraction by a lens is equivalent to an object at $r \cot 30^\circ$ from the lens, and the image at $r \cot 30^\circ$ from the lens, on the opposite side of the lens as the object. Using the thin lens equation, we can write down

$$\begin{aligned} \frac{1}{f} &= \frac{1}{u} + \frac{1}{v} \\ &= \frac{2}{r \cot 30^\circ} \end{aligned}$$

Hence

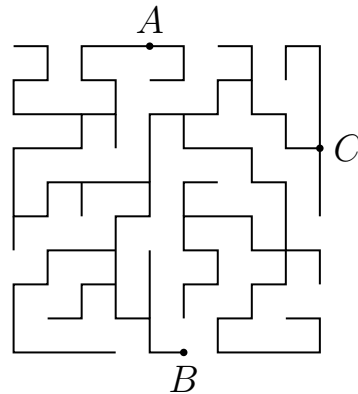
$$\begin{aligned} f &= \frac{r \cot 30^\circ}{2} \\ &= \boxed{0.866 \text{ cm}} \end{aligned}$$

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem C: Lost in the Woods

(3 points)

Tired of having to make solder in GregTech, Wenkai decides to use a soldering iron and a resistive wire with uniform resistance per unit length to make the following maze:

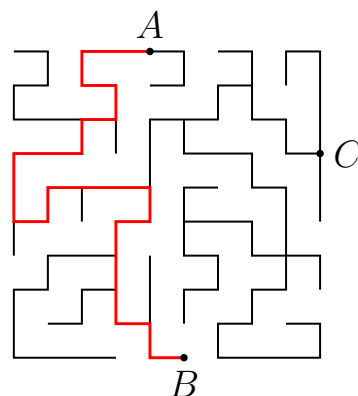


He then connects a 1.0 V voltage source across points A and B of the maze, with point B grounded. What is the voltage of point C with respect to the ground?

Leave your answer to 2 significant figures in units of V.

Solution:

The maze has the following solution:



Since the maze only has 1 solution, any branches away from the solution path are at a uniform voltage equal to that on the solution path from which they branch. Current only flows along the solution path, so the voltage along the solution path decreases linearly with increasing distance from A . The total length of the path is 24 maze segments and the length of the path from B to the point at which the branch containing C begins is 8 maze segments. Thus, the voltage at point C is:

$$V_C = \frac{8}{24} \times 1.0 \text{ V} = \boxed{0.33 \text{ V}}$$

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Problem 4: Simp 1

Amy has a crush on Jake. She has been chasing him for the past 5 years, unsuccessfully. Frustrated by her persistence, Jake gets the help of a highly intelligent dog to finally escape her. The characters play a game of catch on a 2-dimensional plane, with the following rules:

- A “catch” occurs when two characters occupy the same position on the plane.
- If Amy catches Jake before his dog catches her, she wins.
- If Jake’s dog catches Amy before she catches Jake, Jake and his dog win.

The top speeds of Amy, Jake and his dog are v_A , v_J and v_D respectively, and they can change their velocities instantaneously. Jake and his dog start the game close to each other and Amy starts far away from both of them. Jake and his dog must remain a finite, non-zero distance from each other (i.e. Jake cannot camp on his dog). Assume all characters play optimally.

We define v_{crit} such that if $v_A > v_{\text{crit}}$, then Amy can always win.

- (a) If $v_J = 0 \text{ m s}^{-1}$ and $v_D = 1.0 \text{ m s}^{-1}$, what is the value of v_{crit} ?

Leave your answer to 2 significant figures in units of m s^{-1} . (2 points)

- (b) If $v_J = 0.8 \text{ m s}^{-1}$ and $v_D = 0.5 \text{ m s}^{-1}$, what is the value of v_{crit} ?

Leave your answer to 2 significant figures in units of m s^{-1} . (2 points)

Solution:

For both cases, Amy must have a top speed greater than the individual top speeds of Jake and his dog. This can be rationalized as follows:

When Jake’s dog is about to catch Amy (ie. they almost occupy the same position in space), she can outrun the dog and avoid capture only if her top speed is greater than the dog’s. Similarly, Amy can only catch Jake if she outruns him. Therefore, the answers to parts (a) and (b) are

$$(a) \ v_{\text{crit}} = \boxed{1.0 \text{ m s}^{-1}}$$

$$(b) \ v_{\text{crit}} = \boxed{0.8 \text{ m s}^{-1}}$$

This solution makes intuitive sense, but there are a few technicalities that we will lay out below.

Firstly, we note that in the absence of the dog, Amy can catch Jake if and only if she is faster than him, as the rate of separation between Amy and Jake’s positions is only negative if $v_A > v_J$ (assuming both characters play perfectly).

Now, we will consider a slightly different minigame where Amy is trying to travel along a straight line, but the dog begins at some position r_A away from Amy on the line. Her goal is to get past the dog and back on the line without being caught. We note that this is the problem posed in Simp 4, where we obtained the equation

$$\frac{\pi}{2} = \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \ln \left(\frac{r_J}{r_A} \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \right) \equiv u \ln \left(\frac{r_J}{r_A} u \right).$$

In this case, r_J is the distance along the line from the dog's initial position where she will be able to rejoin it, no matter what. Inverting the equation, we get

$$\frac{r_J}{r_A} = \frac{1}{u} \exp \left(\frac{\pi}{2u} \right),$$

so that the ratio between r_J and r_A is always finite for nonzero u , requiring that $v_A > v_D$. As such, for infinitesimal r_A (since Amy can get infinitesimally close to the dog without being caught), r_J is also infinitesimally small, so that this minigame occurs over an infinitesimally small region of the plane.

Thus, if the straight line that she runs along always points towards Jake, then Jake is powerless to help his dog while this minigame is being played (since he is a finite distance away from them, while they finish the game in an infinitesimal amount of time). So, in order for Amy to catch Jake,

1. She must be able to catch Jake normally (requiring $v_A > v_J$).
2. She must be able to win this minigame with the dog (requiring $v_A > v_D$).

As such, v_{crit} is the maximum value of v_J and v_D .

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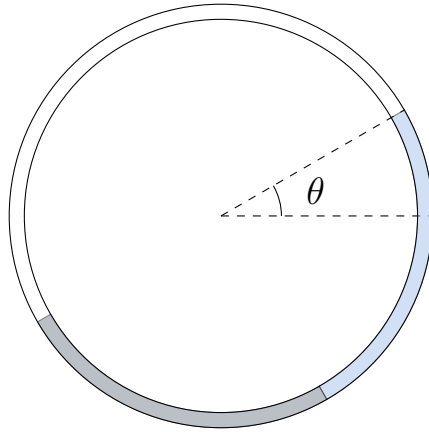
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Problem D: Tilted Liquids

(3 points)

A vertical ring of volume V and of negligible thickness contains 2 immiscible liquids of density ρ_1 and ρ_2 , with $\rho_2 = 3\rho_1$. The volume of each liquid in the ring is $0.25V$. The less dense liquid is floating on top of the denser liquid. At equilibrium, the liquids have the following arrangement:

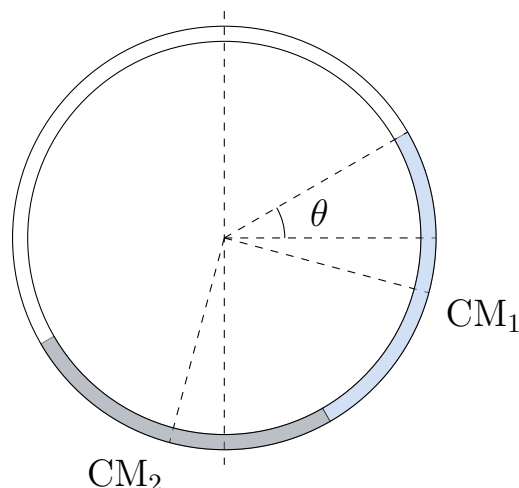


Determine the value of θ .

Leave your answer to 3 significant figures in units of degrees.

Solution:

At equilibrium, the centre of mass must be below the centre of the ring. We can then consider the centre of masses of each of the liquid layers:



The horizontal distances of the centre of masses from the centreline are $r \cos(45^\circ - \theta)$ and $r \sin(45^\circ - \theta)$ for the lighter and heavier fluids respectively.

Thus,

$$\rho_1 r \cos(45^\circ - \theta) = \rho_2 r \sin(45^\circ - \theta)$$

$$\tan(45^\circ - \theta) = \frac{\rho_1}{\rho_2}$$

$$45^\circ - \theta = \arctan \frac{1}{3}$$

$$\theta = 45^\circ - \arctan \frac{1}{3}$$

$$= \boxed{26.6^\circ}$$

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Problem E: Choo Choo

(3 points)

Thomas is a uniform train of length $l = 50$ m. He wishes to complete the frictionless track below, starting at $s = 0$ m with initial velocity v along the track. What is the minimum initial velocity $v_{\min}$ required for Thomas to reach the end of the track at $s = 600$ m? Neglect air resistance.



Leave your answer to 3 significant figures in units of m s^{-1} .

Solution:

Since Thomas is not a point mass, the change in its gravitational potential energy depends on the position of his centre of mass. The obvious points of consideration are at the highest peak at $s = 140$ m and at the horizontal plateau from $s = 380$ m to $s = 480$ m.

Since the gradients of the track before and after the peak at $s = 140$ m are equal in magnitude, the maximum height h_{cm} of the centre of mass as the train passes through this point is:

$$h_{\text{cm}} = 100 - \frac{25}{2} \cdot \frac{20}{\sqrt{20^2 + 40^2}} \approx 94.410 \text{ m}$$

This is slightly higher than when the whole train is on the plateau of height $h = 94 \text{ m}$.

As the starting height is $h_0 = 46 \text{ m}$, we can write down the energy balance:

$$\begin{aligned} mgh_0 + \frac{1}{2}mv_{\text{min}}^2 &= mgh_{\text{cm}} \\ v_{\text{min}} &= \sqrt{2g(h_{\text{cm}} - h_0)} \\ &= \boxed{30.8 \text{ m s}^{-1}} \end{aligned}$$

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Problem 5: Oceans Rise, Amperes Fall

(3 points)

A wire with resistance $R = 1.50\ \Omega$ and length $l = 2.00\ \text{m}$ is connected vertically between two plates in Earth's uniform gravitational field g . The top plate is held at $V_1 = 0\ \text{V}$ and the bottom plate at $V_1 = 10\ \text{V}$. An ammeter is attached to the wire and measures current I_1 . The entire setup is then turned upside down and, strangely, a slightly different current I_2 is measured. Determine $|I_1 - I_2|$.

Leave your answer to 2 significant figures, in units of nA.

Solution: The difference in current is due to a voltage caused by gravity. An electron in the wire experiences a gravitational force $m_e g$. This produces an additional electromotive force

$$\varepsilon_g = \pm \frac{m_e g l}{e}$$

Where the sign changes when the setup is inverted. Thus, the difference in current is simply

$$\frac{2\varepsilon_g}{R} = \frac{2m_e g l}{eR} = \boxed{0.15\ \text{nA}}$$

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Problem 6: Simulated Universe

(4 points)

Oon Han finds himself as the first inhabitant of a simulated universe, standing on an Earth with an identical gravitational field as ours. In this universe, time progresses in steps of duration $\Delta t = 6.40 \times 10^{-15}$ s and kinematics is described by the recurrence relations

$$\begin{aligned}\mathbf{r}(t + \Delta t) &= \mathbf{r}(t) + \mathbf{v}(t)\Delta t \\ \mathbf{v}(t + \Delta t) &= \mathbf{v}(t) + \mathbf{a}(t)\Delta t\end{aligned}$$

where $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ denote position, velocity and acceleration respectively. Unfortunately, this results in total energy not being conserved, so his universe will soon meet its impending doom.

Oon Han lets an orange of mass $m = 0.140$ kg fall from rest and catches it $t_0 = 0.600$ s later. The total mechanical energy of an object is defined as the sum of its kinetic and potential energies. The change in total mechanical energy of the orange from the time it was released to right before it was caught is ΔE J. Determine $\ln |\Delta E|$.

Leave your answer to 3 significant figures.

Solution: Let subscript n denote the value of a quantity at time $t = n\Delta t$ and let y , v denote the falling distance and speed of the orange respectively. The orange accelerates under gravity, so

$$\begin{aligned}y_{n+1} &= y_n + v_n\Delta t \\ v_{n+1} &= v_n + g\Delta t\end{aligned}$$

with initial conditions $y_0 = 0$ and $v_0 = 0$. The speed obeys an arithmetic progression,

$$v_n = v_0 + ng\Delta t = ng\Delta t$$

while the distance can be determined to be

$$y_n = y_0 + (v_0 + v_1 + \cdots + v_{n-1})\Delta t = (0 + 1 + \cdots + (n-1))g\Delta t^2 = \frac{1}{2}gn(n-1)\Delta t^2$$

Then, taking the initial potential energy to be zero, the mechanical energy is

$$E_n = \frac{1}{2}mv_n^2 - mgy_n = \frac{1}{2}m(ng\Delta t)^2 - mg \cdot \frac{1}{2}gn(n-1)\Delta t^2 = \frac{1}{2}mg^2n\Delta t^2$$

Substituting $t_0 = n\Delta t$, we get the change in mechanical energy is

$$E = \frac{1}{2}mg^2t_0\Delta t = 2.59 \times 10^{-14} \text{ J}$$

so

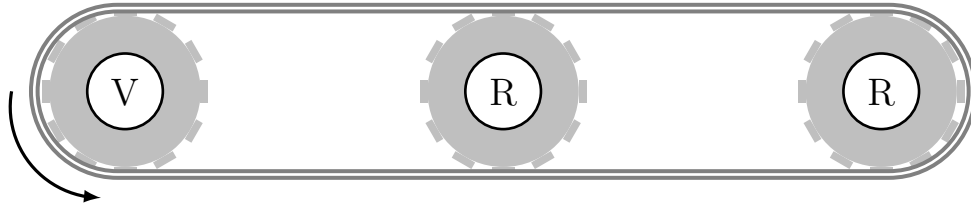
$$\ln |\Delta E| = \ln(2.59 \times 10^{-14}) = \boxed{-31.3}$$

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Problem 7: Spintronics

In *Spintronics*, electrical circuit components are replaced by mechanical gears and chains. You may assume that all gears have radius $r = 5.0$ cm and negligible mass, and that chains do not slip.

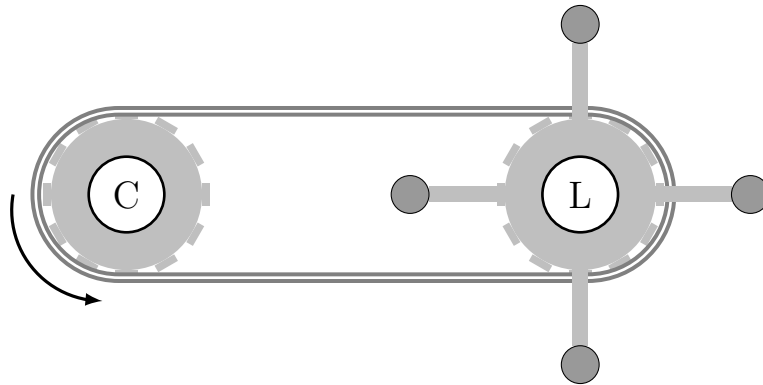
- (a) Let us consider a circuit, as shown below, with 3 gears and a chain. Gear V is a motorised gear, which exerts a constant force of $F_V = 10$ N on the chain. Each free gear R exerts a resistive force $F_R = -bv$ on the chain, where $b = 2.0 \text{ kg s}^{-1}$ is a constant and v is the linear velocity of the chain.



In the above setup, what is the final linear velocity of the chain?

Leave your answer to 2 significant figures in units of m s^{-1} . (2 points)

- (b) In the next setup, shown below, gear C is attached to a torsional spring, which obeys Hooke's law. Thus, it exerts a resistive torque $\tau_C = -k\theta r$ when rotated, where $k = 0.60 \text{ kg m s}^{-2}$ is a constant and θ is the angular displacement from its equilibrium position, in radians. Gear L has four massless rods of length $l = 5.0$ cm connected normal to the gear, with a point mass $m = 0.20$ kg at the end of each rod.



Gear C is rotated to some $\theta = \theta_0$ and released. What is the time t for gear C to first return to its equilibrium position?

Leave your answer to 2 significant figures in units of s. (3 points)

Solution:

- (a) The motorised gear represents a voltage source, while the free gears represent resistors.

Although this setup looks like a parallel electronics circuit, the two resistors are actually in series when we look at the forces acting on the chain. Balancing forces,

$$F_V + 2F_R = 0$$

$$F_V = 2bv$$

$$v = \boxed{2.5 \text{ m s}^{-1}}$$

- (b) Gear C represents a capacitor and the other gear represents an inductor. We can consider torque to obtain

$$I\ddot{\theta} = -k\theta r$$

where $I = 4m(l + r)^2$ is the moment of inertia of the gear representing the inductor. This circuit represents a LC circuit, and the above equation represents simple harmonic motion. Therefore,

$$\theta = \theta_0 \cos(\omega t)$$

where $\omega = \sqrt{kr/I}$. So, when θ is first 0,

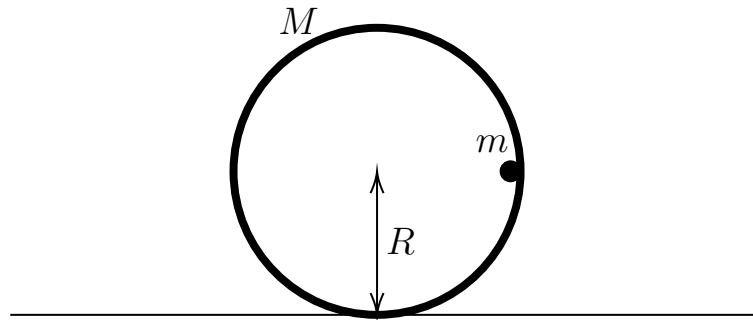
$$t = \frac{\pi}{2} \sqrt{\frac{I}{kr}} = \boxed{0.81 \text{ s}}$$

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Problem 8: Do A Barrel Roll

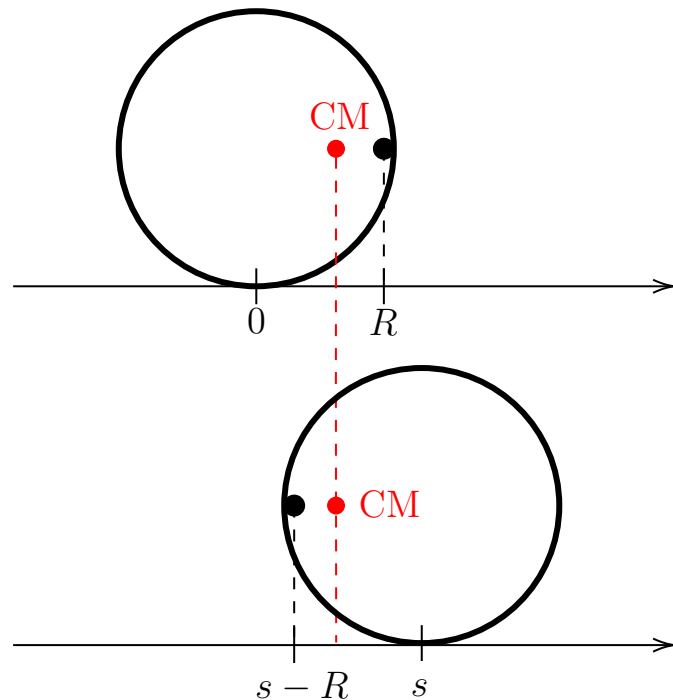
(3 points)

A hollow frictionless cylinder of mass $M = 5.0$ kg and radius $R = 1.0$ m is placed on horizontal ground. A tiny sphere of mass $m = 1.0$ kg is placed halfway up the inner side of the cylinder, and the whole setup is released from rest. Determine the amplitude A of the sideways oscillations of the cylinder's centre.



Leave your answer to 2 significant figures in units of m.

Solution: The horizontal position of the centre of mass of the cylinder + sphere system does not change. This is because the ground is frictionless, so there are no horizontal external forces on the system.



Let the initial position of the centre of the cylinder be $x = 0$. The x -coordinate of the centre of mass of the system, x_{CM} , is given by:

$$x_{CM} = \frac{mR}{M + m}$$

At the turning point of the oscillation, the bead is now on the left side of the cylinder.

Let the position of the centre of the cylinder here be $x = s$. Then x_{CM} is:

$$x_{CM} = \frac{Ms + m(s - R)}{M + m}$$

Equating the two expressions for x_{CM} (as x_{CM} does not change), we obtain:

$$s = \frac{2m}{M + m}R$$

Finally note that the amplitude A is half of the distance between turning points:

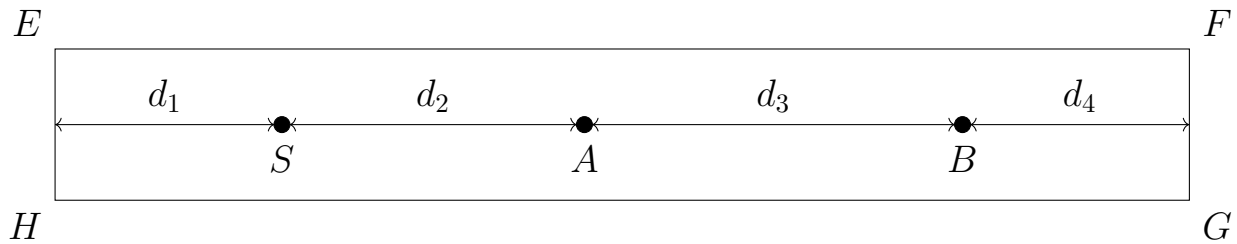
$$A = \frac{s}{2} = \frac{m}{M + m}R \approx \boxed{0.17 \text{ m}}$$

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Problem 9: Audio Cue

(5 points)

Physicist S is playing a game with friends A and B using omnidirectional speakers (i.e., speakers that emit sound uniformly in all directions). Physicist S also has a microphone. All three stand along a straight line in a room of length L and negligible width. Physicist S, friend A, and friend B stand at points S , A , and B respectively. The distances d_1 to d_4 are defined as shown in the diagram below:



Their positions satisfy $d_1 < \min(d_2, d_3, d_4)$. Walls EF and GH are lined with acoustic foam, which absorbs all incident sound. Walls FG and EH reflect all incident soundwaves perfectly.

The game proceeds as follows:

1. Physicist S starts the microphone. After a delay of $t_{\text{start}} = 1.00$ s, he plays a single note of frequency $f_S = 500$ Hz from his speaker.
2. Upon first hearing S's note, Friend A immediately plays a note of frequency $f_A = 1200$ Hz.
3. Upon first hearing S's note, Friend B immediately plays a note of frequency $f_B = 1800$ Hz.
4. Physicist S stops the recording shortly after his microphone has captured each of the three notes at least three times (including the note at $t = t_{\text{start}}$).

Volume warning: The audio may be loud.

The audio file Physicist S recorded can be found [here](#). You may find it helpful to use an online tone generator to familiarise yourself with the three frequencies.

Determine the value of $d_1 + d_2 + 2d_3 + 2d_4$.

Leave your answer to 2 significant figures in units of km.

Solution: We analyse the times at which each sound is recorded. The sound physicist S plays at $f = f_S$ is immediately recorded at $t = 1$ s. The next 2 times he hears the sound correspond to reflections from the walls: one from wall EH and the other from

wall FG. Hence, these echoes occur at $t = 1 + 2\frac{d_1}{v_s}$ and $t = 1 + 2\frac{d_2 + d_3 + d_4}{v_s}$.

Applying similar reasoning to the notes emitted by A and B, we obtain the following table of recording times:

Frequency	1st sound (s)	2nd sound (s)	3rd sound (s)
$f = f_S$	1	$t = 1 + 2\frac{d_1}{v_s}$	$t = 1 + 2\frac{d_2 + d_3 + d_4}{v_s}$
$f = f_A$	$t = 1 + 2\frac{d_2}{v_s}$	$t = 1 + 2\frac{d_1 + d_2}{v_s}$	$t = 1 + 2\frac{d_2 + d_3 + d_4}{v_s}$
$f = f_B$	$t = 1 + 2\frac{d_2 + d_3}{v_s}$	$t = 1 + 2\frac{d_1 + d_2 + d_3}{v_s}$	$t = 1 + 2\frac{d_2 + d_3 + d_4}{v_s}$

This ordering is guaranteed because each distance must be positive and $d_1 < \min(d_2, d_3, d_4)$. Hence, we can use a stopwatch to record the time taken after the 1st sound at f_S until

1. The second time we hear a sound at f_S .
2. The first time we hear a sound at f_A .
3. The first time we hear a sound at f_B .
4. The third time we hear a sound at f_B , which happens to be when all 3 sounds coincide.

We can obtain the following values after timing:

$$\begin{aligned}
 2\frac{d_1}{v_s} &= 2.94 \text{ s} \\
 2\frac{d_2}{v_s} &= 4.71 \text{ s} \\
 2\frac{d_2 + d_3}{v_s} &= 15.88 \text{ s} \\
 2\frac{d_2 + d_3 + d_4}{v_s} &= 19.41 \text{ s}
 \end{aligned}$$

We can then solve the values of d_1 to d_4 to be 0.50 km, 0.80 km, 1.90 km, and 0.60 km respectively. Hence,

$$d_1 + d_2 + 2d_3 + 2d_4 = \boxed{6.3 \text{ km}}$$

An error bar of 0.2 km was given.

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Problem 10: Linear or Quadratic?

(4 points)

E-Jay, a basketball enthusiast, is interested in determining the nature of drag experienced by basketballs so he can improve the accuracy of his shots. He hypothesises that the drag force acting on a basketball is either always linear or always quadratic, i.e:

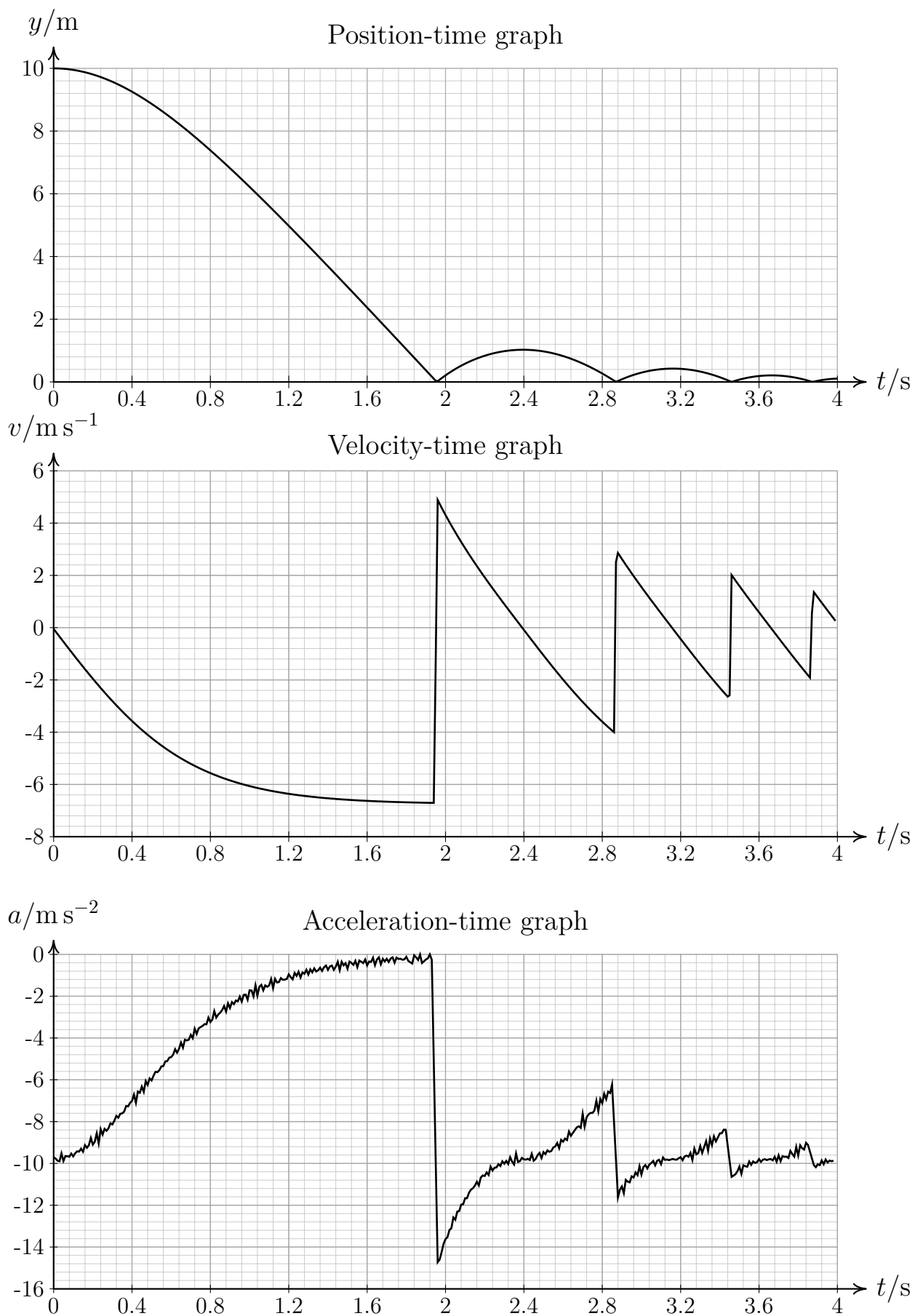
$$\underset{\text{(linear)}}{F_{\text{D}} = bv} \quad \text{or} \quad \underset{\text{(quadratic)}}{F_{\text{D}} = bv^2}$$

where F_{D} is the magnitude of the drag force, v is the speed and b is a constant.

To determine which form it is, E-Jay conducts an experiment. He drops a basketball of mass $m = 0.600$ kg from rest at some height above the ground and tracks its motion. He obtains the position, velocity and acceleration data for the basketball and plots graphs of their values against time, which are shown on the following page. By considering whether the drag force is linear or quadratic, determine b .

Leave your answer to 2 significant figures in units of either kg s^{-1} or kg m^{-1} .

If you think there is not enough information to determine the form of drag, input -1 .



Solution: The forces acting on the basketball are the gravitational force mg and the drag force F_D . By Newton's second law,

$$ma = -mg + F_D \quad \implies \quad a = -g + \frac{F_D}{m}$$

To differentiate between the two forms, consider the time derivative of acceleration (also known as the jerk) at $t = 0$. The initial velocity is zero and the initial acceleration is $-g$ since the basketball starts at rest. If the drag force is linear, then

$$a = -g - \frac{b}{m}v \quad \implies \quad \dot{a} = -\frac{b}{m}\dot{v} = -\frac{b}{m}a \quad \implies \quad \dot{a}(0) = \frac{bg}{m}$$

If the drag force is quadratic, then

$$a = -g + \frac{b}{m}v^2 \quad \implies \quad \dot{a} = \frac{b}{m}(2v\dot{v}) = \frac{2b}{m}av \quad \implies \quad \dot{a}(0) = 0$$

Looking at the acceleration-time graph, the tangent at $t = 0$ is flat, so the time derivative of acceleration is clearly zero. Therefore, the nature of the drag force must be *quadratic*.

Now, we can look at both the velocity-time graph and the acceleration-time graph. Right before the first bounce, the velocity-time graph is almost flat and the acceleration is almost zero. This implies at the ball has essentially reached terminal velocity v_t .

Reading the velocity-time graph, we have $v_t \approx -6.8 \text{ m s}^{-1}$. Therefore, since the acceleration is zero,

$$-g + \frac{b}{m}v_t^2 = 0 \quad \implies \quad b = \frac{mg}{v_t^2} \approx \boxed{0.13 \text{ kg m}^{-1}}$$

The value used in the simulation to generate the graphs is $b = 0.129 \text{ kg m}^{-1}$.

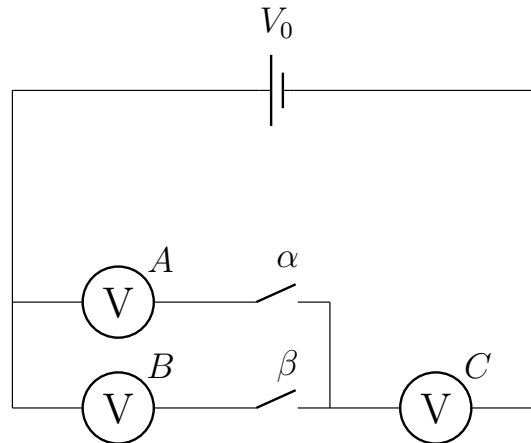
Setter: Oon Han, oon.han@sgphysicsleague.org

Problem 11: Nuts and Volts

(3 points)

Scrat has three voltmeters, two switches and a battery of voltage $V_0 = 2.00$ V. He labels the three voltmeters A, B, C and the two switches α, β . Unfortunately, in a scramble to retrieve his acorn, he accidentally breaks the displays of voltmeters A and B , so he can only see the voltage displayed by C .

Scrat connects the components in a circuit as shown below, where both switches are initially open. He closes β and reads that the voltage measured by C is $V_1 = 1.28$ V. Keeping β closed, he closes α and reads that the voltage measured by C is $V_2 = 1.40$ V.



Just as Scrat opens β and tries to catch a glimpse of the new voltage V_3 displayed by C , the ice cracks below his feet and he falls into the abyss beneath the surface. Determine V_3 .

Leave your answer to 3 significant figures in units of V.

Hint: The voltmeters are non-ideal.

Solution: Each voltmeter has their own resistance, which we shall let be R_A, R_B, R_C respectively. When α is open and β is closed,

$$V_1 = \frac{R_C}{R_B + R_C} V_0 \quad \Rightarrow \quad \frac{R_B}{R_C} = \frac{V_0}{V_1} - 1$$

by the potential divider rule. When α and β are both closed, the two voltmeters A and B are in parallel. The effective resistance is

$$R_{\text{eff}} = \left(\frac{1}{R_A} + \frac{1}{R_B} \right)^{-1}$$

and the voltage of C is

$$V_2 = \frac{R_C}{R_{\text{eff}} + R_C} V_0 \quad \Rightarrow \quad \frac{R_{\text{eff}}}{R_C} = \frac{V_0}{V_2} - 1$$

by the potential divider rule. We can now solve for R_A/R_C as

$$\frac{R_A}{R_C} = \left(\frac{R_C}{R_{\text{eff}}} - \frac{R_C}{R_B} \right)^{-1}$$

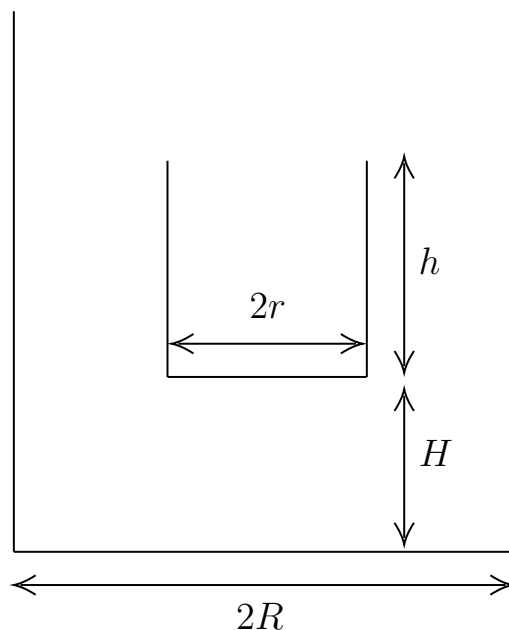
so when α is closed and β is open, by the potential divider rule,

$$V_3 = \frac{R_C}{R_A + R_C} V_0 = \boxed{0.714 \text{ V}}$$

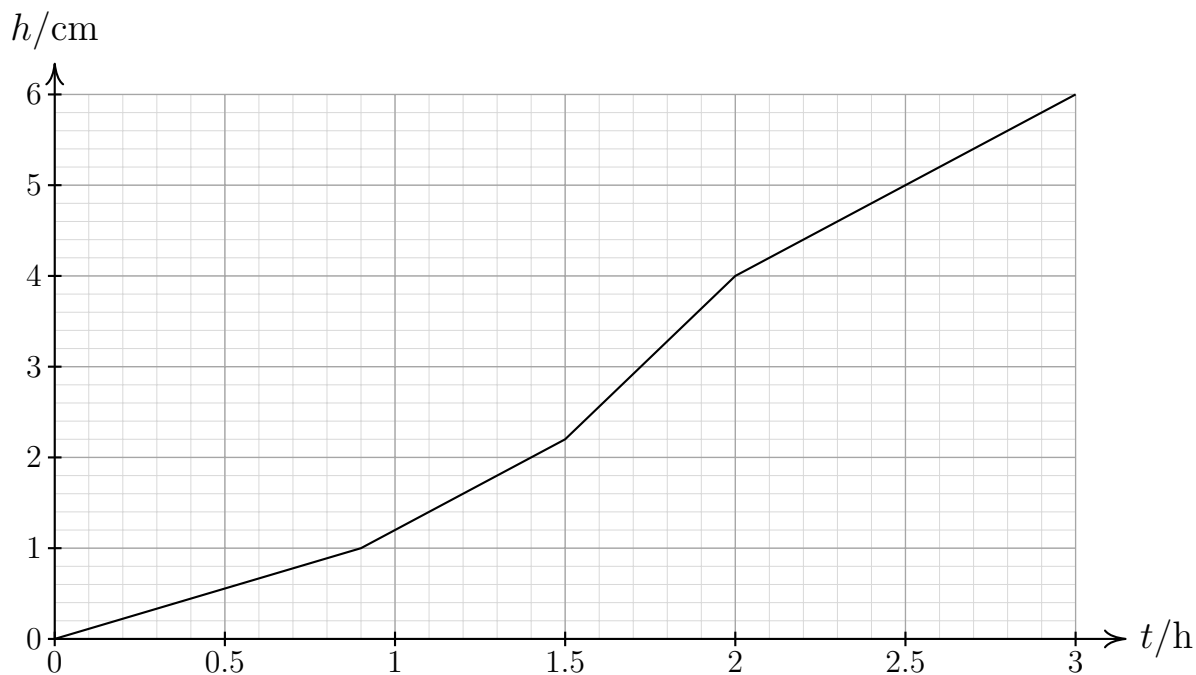
Setter: Oon Han, oon.han@sgphysicsleague.org

Problem 12: Let the Rain Pour

In a spark of engineering mania, Sid decides to build a (somewhat useless) device to collect rainwater. The device consists of two cylinders with closed bottoms, where the smaller cylinder is suspended inside the larger cylinder. The following diagram shows a cross-section of the device.



During a particularly long rainstorm, the water level of the outer cylinder is measured as a function of time to obtain the following graph. Assume that the rate of rainfall is constant.



- (a) Determine the rate of rainfall v .

Leave your answer to 2 significant figures in units of cm h^{-1} .

(2 points)

It turns out that from the data alone, we cannot uniquely determine the values of H and h – there are 2 possible sets of values which can produce this graph.

(b) Determine the average of the 2 possible values of H .

Leave your answer to 2 significant figures in units of cm. (2 points)

Solution:

At any point in time, the volumetric flow rate of rainwater into the outer cylinder is vA , where A is the area capturing rainwater that leads into the outer cylinder. Initially, when the inner cylinder is not overflowing, $A = \pi(R^2 - r^2)$. After the inner cylinder begins overflowing, $A = \pi R^2$.

If the water level of the outer cylinder is less than H or greater than $H + h$, the area of its surface is πR^2 . If the water level of the outer cylinder is between these heights, the area of its surface $\pi(R^2 - r^2)$. We will denote this surface area S .

At any point in time, the rate of change of the water level of the outer cylinder is

$$\frac{dh}{dt} = \frac{A}{S}v.$$

Initially, the water level increases at a rate

$$\frac{dh}{dt} = \frac{R^2 - r^2}{R^2}v.$$

This case corresponds to the first segment of the graph.

After a very long time, the water level in the outer cylinder becomes higher than the top of the inner cylinder. Here, the water level increases at a rate

$$\frac{dh}{dt} = v.$$

This case corresponds to the fourth segment of the graph.

Between these 2 extremes, there are 2 possible scenarios: either the inner cylinder overflows first or the water level in the outer cylinder reaches the bottom of the inner cylinder first.

If the inner cylinder is not overflowing and the water level in the outer cylinder has reached the bottom of the inner cylinder, the water level increases at a rate

$$\frac{dh}{dt} = v.$$

If the inner cylinder is overflowing and the water level in the outer cylinder has not yet reached the bottom of the inner cylinder, the water level also increases at a rate

$$\frac{dh}{dt} = v.$$

These cases correspond to the second segment of the graph.

If the inner cylinder is overflowing and the water level in the outer cylinder has reached the bottom of the inner cylinder, the water level increases at a rate

$$\frac{dh}{dt} = \frac{R^2}{R^2 - r^2}v.$$

This case corresponds to the third segment of the graph.

The water level in the outer cylinder reaches the bottom of the inner cylinder, H , on one side of the second segment of the graph. As such, H is either 1.0 cm or 2.2 cm.

Thus, from our analysis,

- (a) The rainfall rate is the slope of the graph in the last segment,

$$v = \frac{6 - 4}{3 - 2} = \boxed{2.0 \text{ cm h}^{-1}}.$$

- (b) The average value of the 2 possible heights is

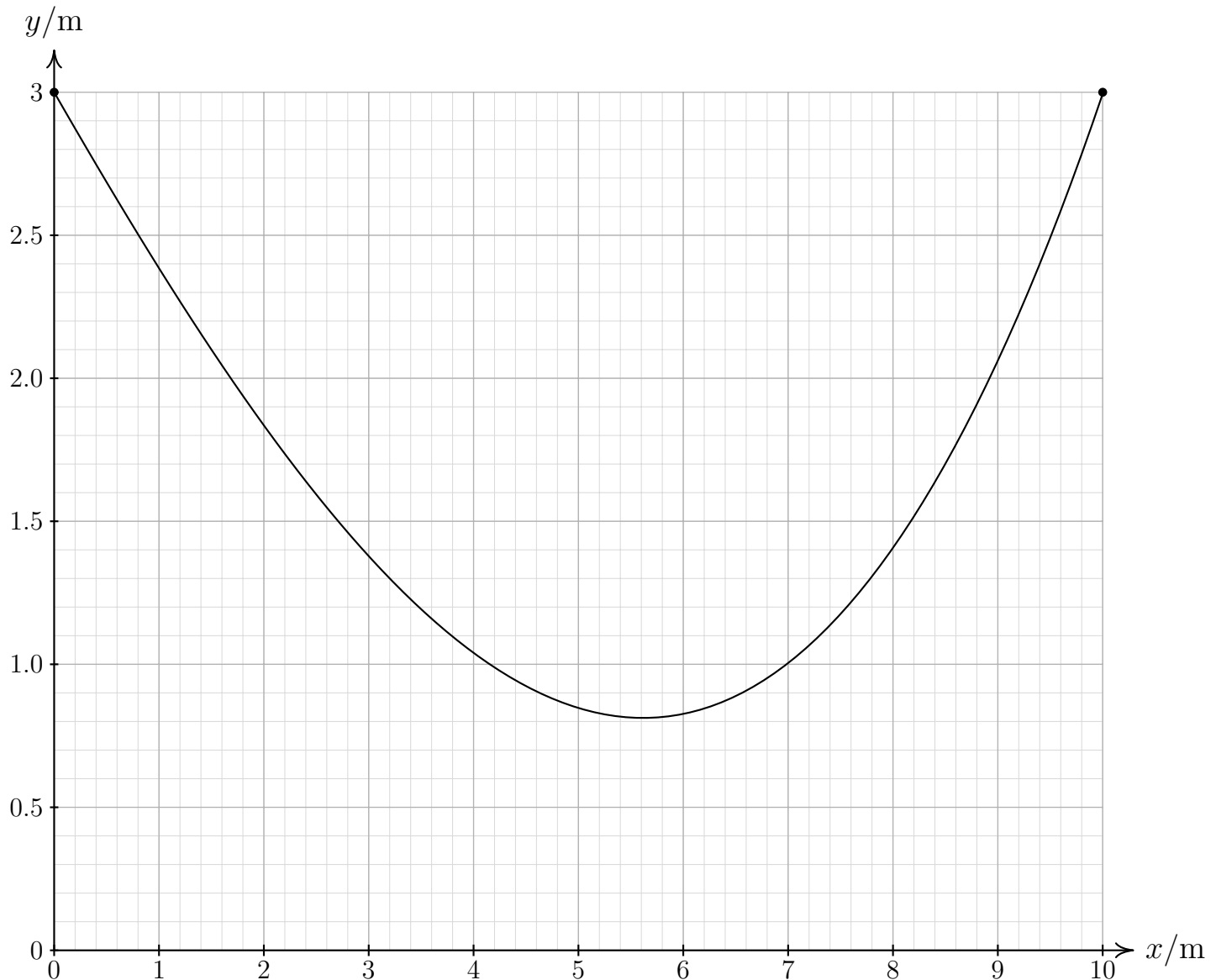
$$H_{\text{ave}} = \frac{1.0 + 2.2}{2} = \boxed{1.6 \text{ cm}}.$$

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem 13: Hanging Rope

(3 points)

An inextensible, non-uniform rope is hung between two supports at $x = 0$ and $x = 10$ m. Its linear density, measured from one end of the rope, is given by $\lambda(l) = al^b$, where a and b are unknown constants. A plot of the rope profile at equilibrium is shown below.



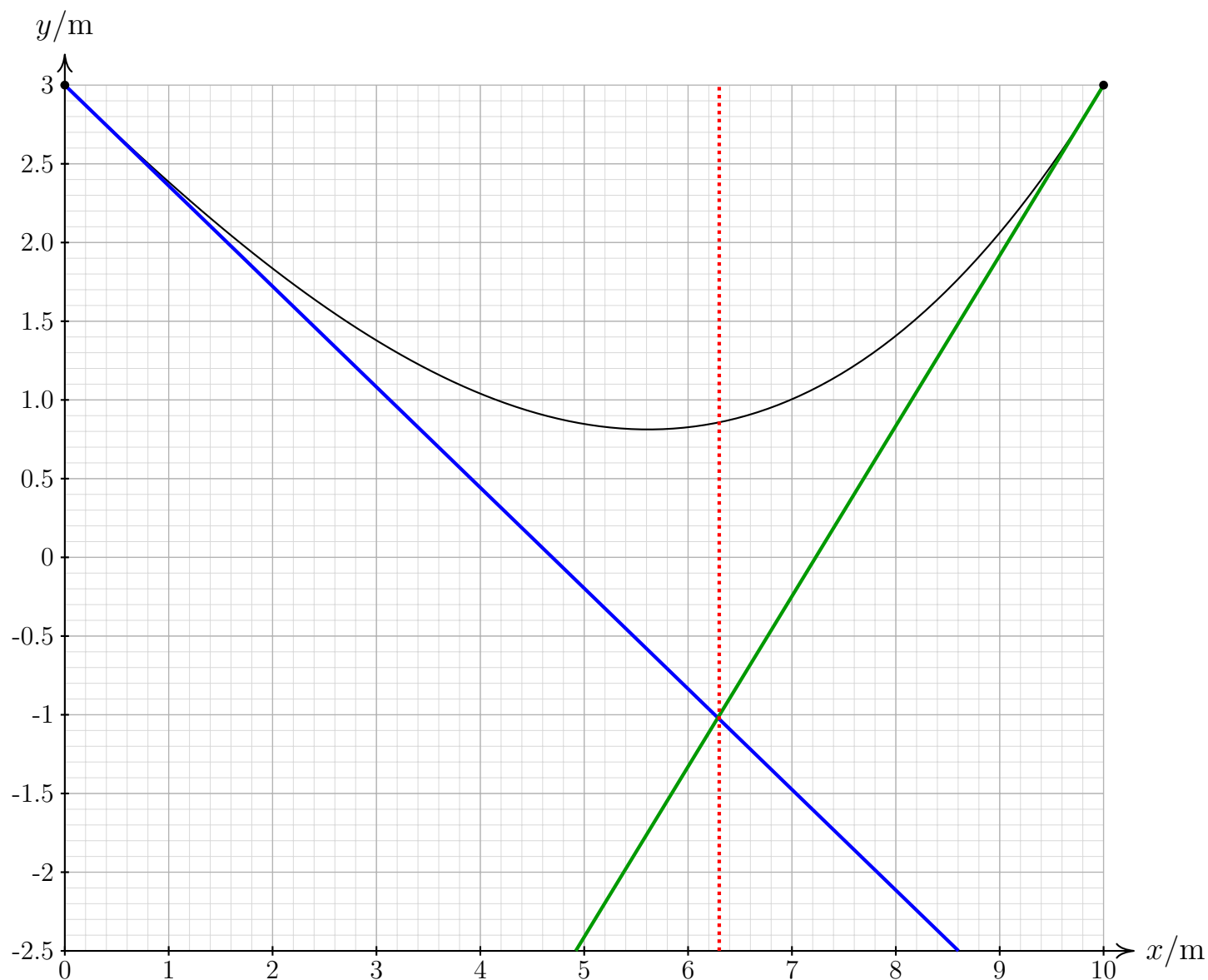
Find the x -coordinate of the center of mass of the rope.

Leave your answer to 2 significant figures in units of m.

Solution:

For equilibrium to be maintained, the lines of action of all forces acting on the rope must intersect at a single point. It follows that the center of mass lies on the vertical

line through the point of intersection of the tangents to the rope at its endpoints.



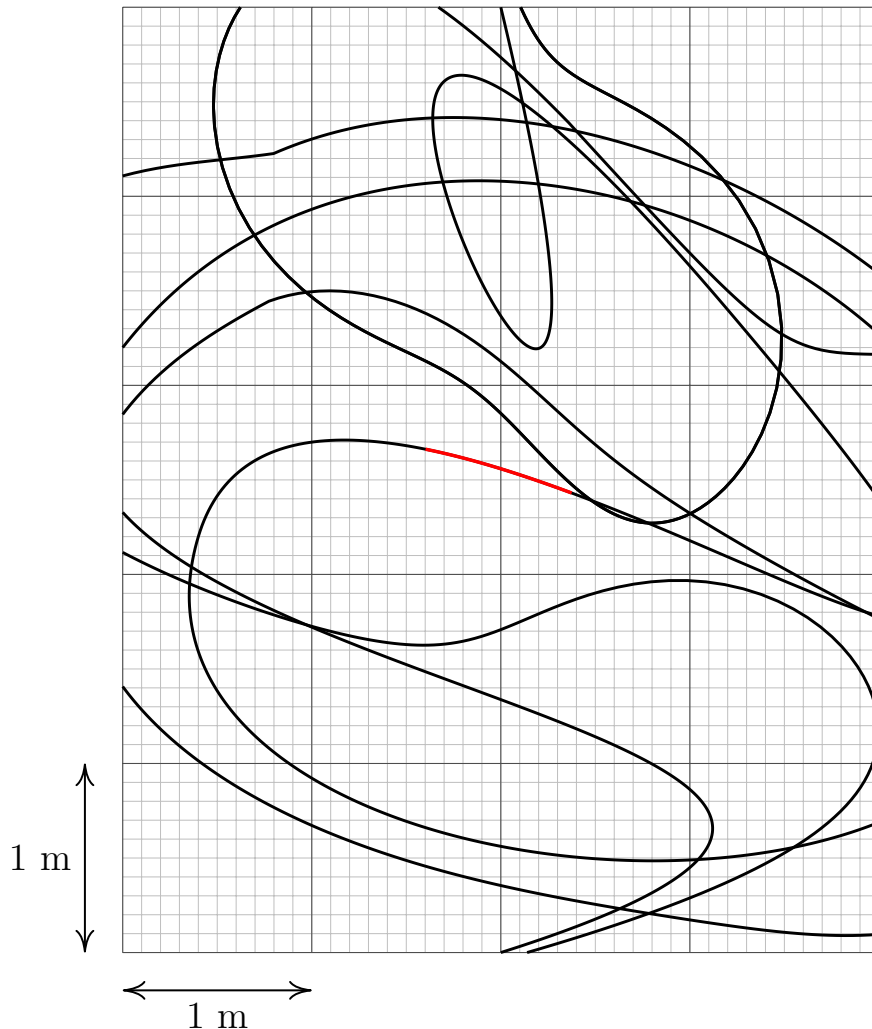
From the graph, the center of mass is at $x = \boxed{6.3 \text{ m}}$. The mass density is a red-herring.

Setter: Aaron Ng, aaron.ng@sgphysicsleague.org

Problem 14: Cycle Madness B-Side

(5 points)

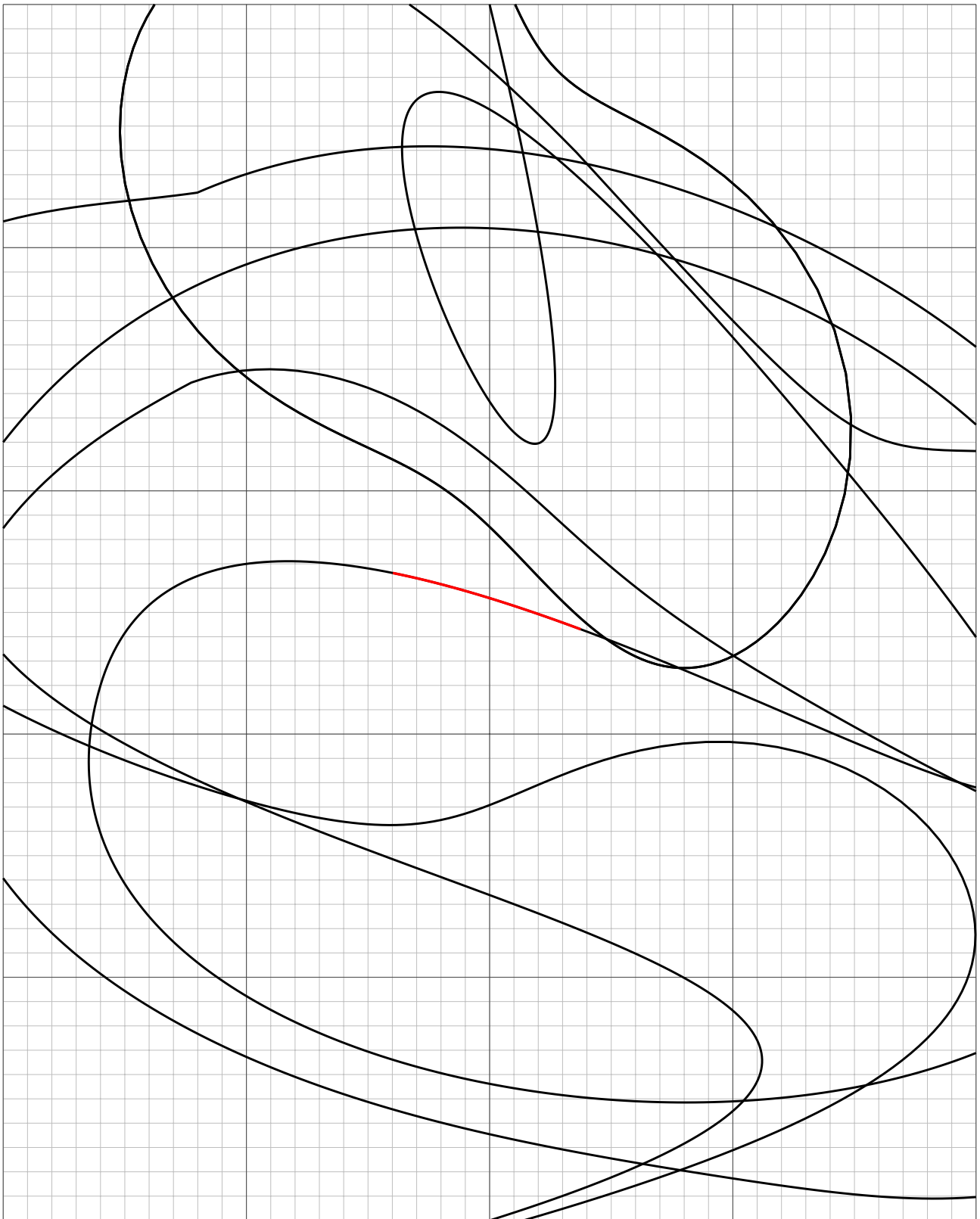
Madeline is cycling when she accidentally cycles into a puddle of water. As a result, both her wheels are wet and hence leave some tracks on the ground. The figure below shows part of the tracks that were formed (you may refer to the next page for a larger version of the same figure). Each major grid square represents a $1\text{ m} \times 1\text{ m}$ square.



Refer to next page for larger version.

The distance between the centres of the two wheels of Madeline's bicycle is l . Assume that Madeline's wheels do not slip. Let $\alpha = +1$ if the wheel that traced out the red segment is moving left to right, and $\alpha = -1$ if it is moving right to left. Given that $l < 2\text{ m}$ and that Madeline is always cycling forwards, determine $\alpha \times l$.

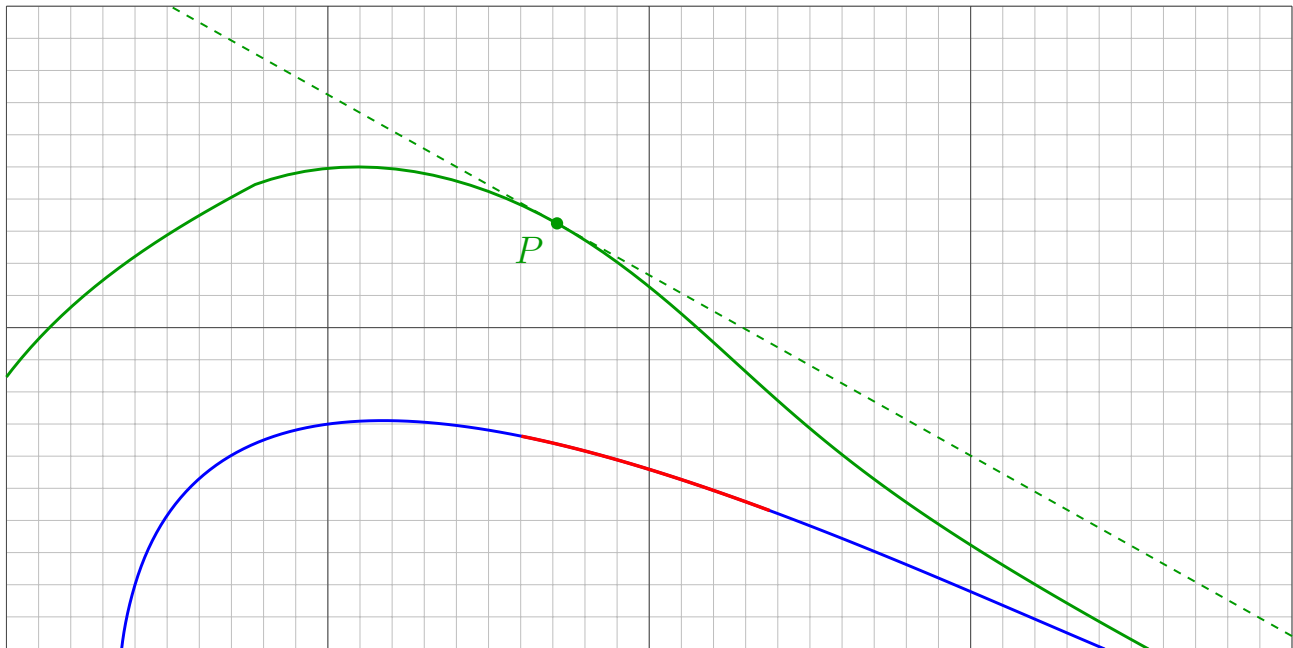
Leave your answer to 2 significant figures in units of m.



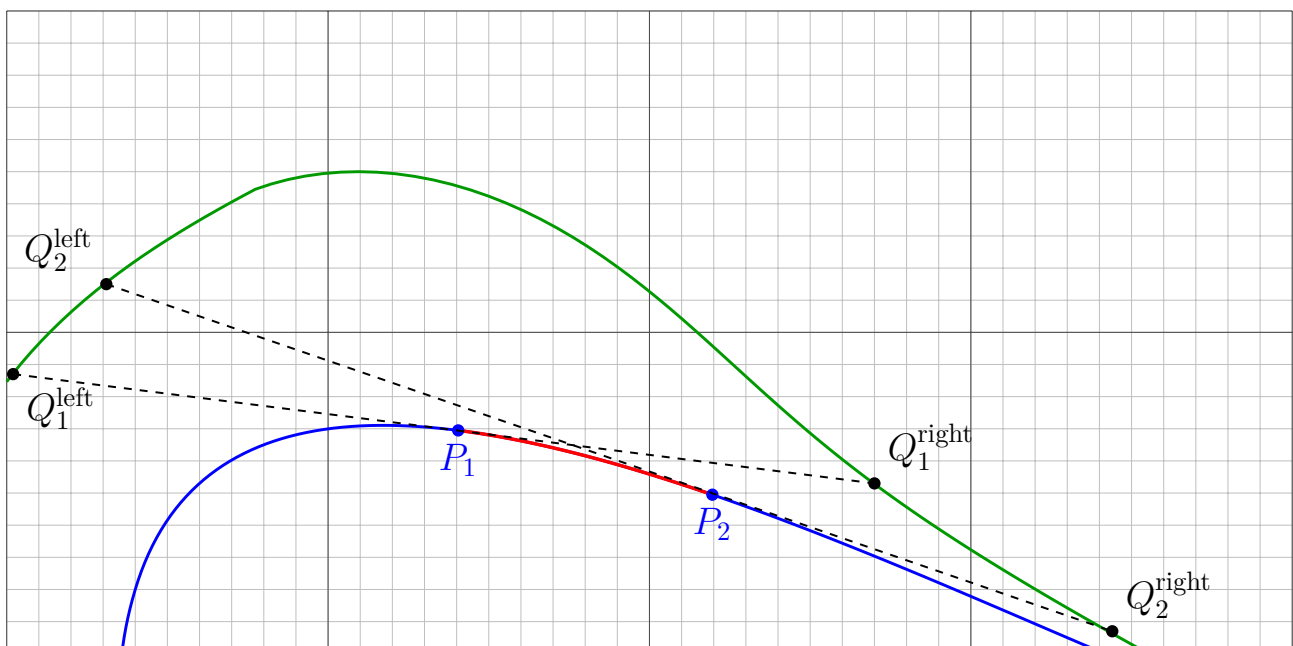
Solution: The key is to realise that for a bicycle, the back wheel has to always roll towards the front wheel, while the front wheel can roll in any direction. Hence, for any point B on the track of the back wheel, the corresponding front wheel position is always a distance l along the tangent to the back wheel's track at B .

To figure out l , we can start by identifying a pair of possible front wheel and back wheel tracks (these can be identified by pairs of curves that have similar-looking shapes). Here, we examine the curve containing the red segment (in blue) and another track that is likely to be the corresponding track (in green). To reduce clutter, all the other tracks are hidden.

Note that the green track cannot be the back wheel track for this pair. To see why, consider the tangent to the green track at P . Notice that it does not intersect the blue curve at any point within 2 m of P , which is the largest possible value of l .



Therefore, the blue track has to be the back wheel track and the green track is the front wheel track. We can then draw a few tangent lines to the blue track (here, only 2 will be drawn):



We now need to measure some lengths, which can either be done by using the provided grid and the Pythagorean theorem, or by taking a physical ruler, measuring lengths on the screen and then scaling by the appropriate amount.

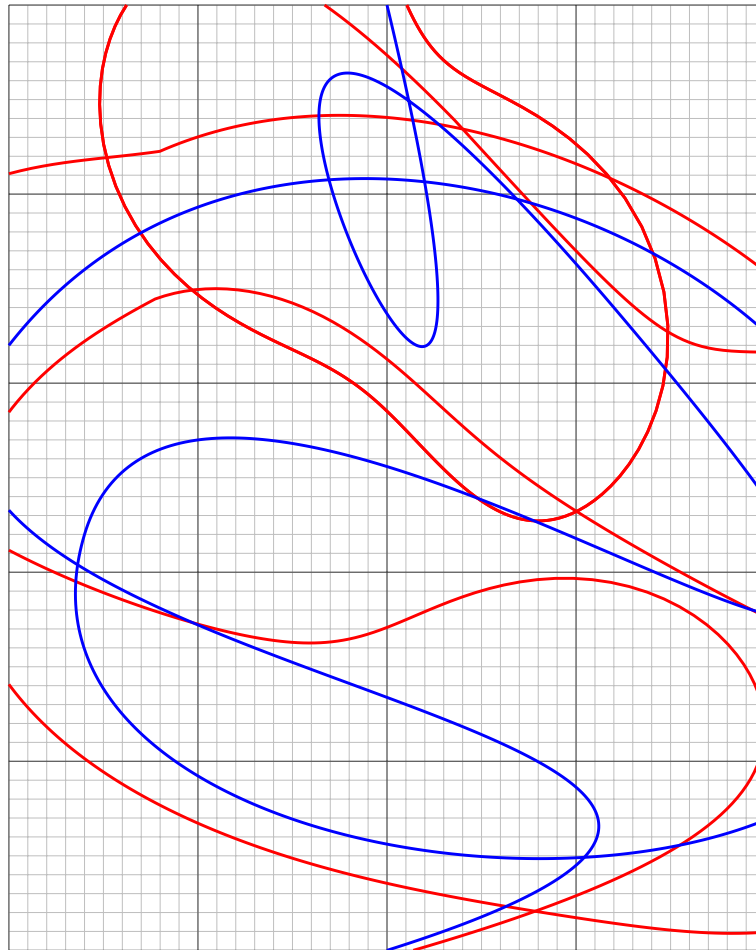
If the back wheel is travelling towards the left along the red segment, then the corresponding front wheel positions to P_1 and P_2 are at Q_1^{left} and Q_2^{left} respectively. However, it is clear that the lengths $P_1Q_1^{\text{left}}$ and $P_2Q_2^{\text{left}}$ are different in length, when they are supposed to both be equal to l , which is a contradiction.

If instead, the back wheel is travelling towards the right along the red segment, then the corresponding front wheel positions to P_1 and P_2 are at Q_1^{right} and Q_2^{right} respectively. It can be verified that $P_1Q_1^{\text{right}}$ and $P_2Q_2^{\text{right}}$ have approximately the same lengths, which is what we expected. Hence, $\alpha = +1$ and $l \approx 1.3$ m.

Therefore,

$$\alpha \times l \approx \boxed{1.3 \text{ m}}$$

Note: Using a similar procedure, it is possible to identify all the tracks as either a front wheel track (**red**) or a back wheel track (**blue**).

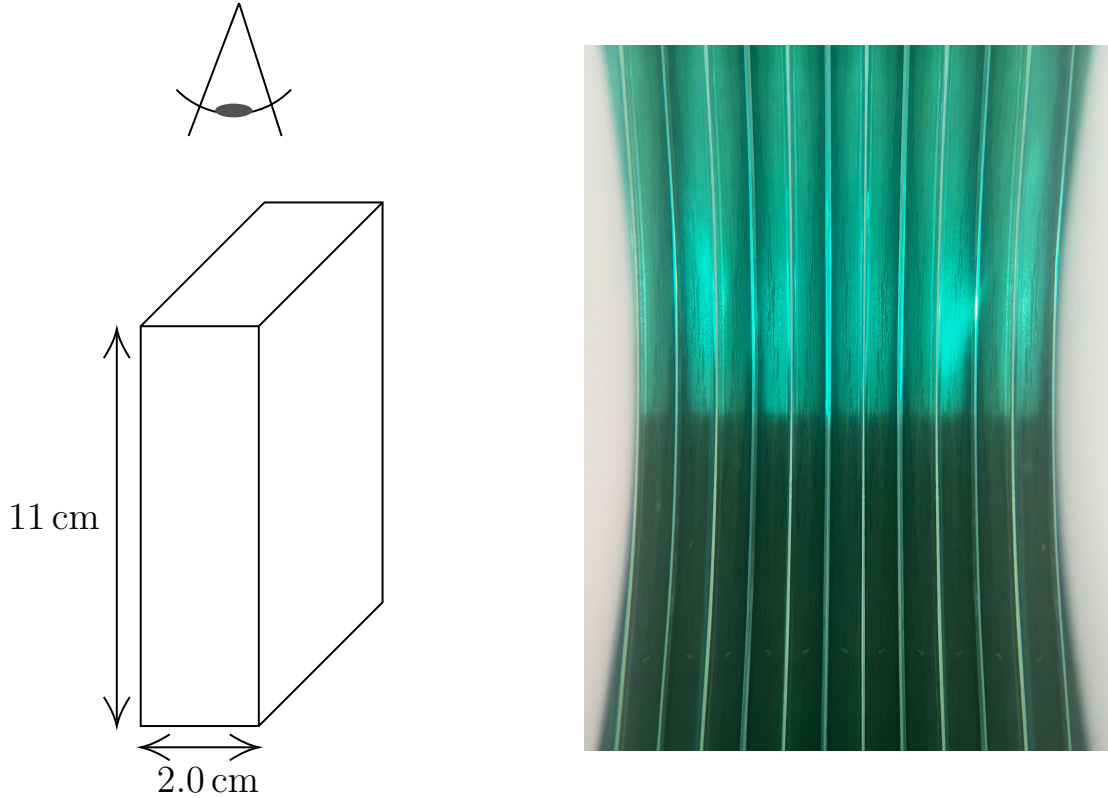


Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org

Problem 15: Through the Looking Glass

(3 points)

Ryab was polishing glass panels (as is his pastime), when he noticed something interesting. Looking into a panel from the top, its base appears to be laterally duplicated several times. A photo of what he saw is shown below.



The glass panels have height $h = 11$ cm and width $w = 2.0$ cm. Estimate the refractive index of the glass, n .

Leave your answer to 2 significant figures.

Solution: The multiple images come from light bouncing between the two surfaces of the glass panel. Below the critical angle, light escapes rather than undergoing Total Internal Reflection, and no further images are formed.

From figure (b), about $i = 11$ images can be seen. Hence, we can find the critical angle as

$$\theta_c = \tan^{-1} \left(\frac{2h}{iw} \right)$$

Writing down the relation between critical angle and reflective index, we have:

$$\sin^{-1} \left(\frac{1}{n} \right) = \tan^{-1} \left(\frac{2h}{iw} \right)$$

$$n = \sqrt{1 + \left(\frac{2h}{iw}\right)^2}$$
$$\approx \boxed{1.4}$$

This question was inspired by Suki-Ya's glass partitions.

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Problem 16: Disappearing Water

(3 points)

To fulfil his combat medic obligations, Le Xi needs to learn how to administer IV therapy. He heats water to exactly 100 °C and fills a syringe with 30 mL of it, taking care to avoid trapping any air. He then seals the nozzle and pulls the plunger from the 30 mL to the 31 mL mark. After a long time, Le Xi notices that a very small amount of the water has gone missing!

Assume the syringe is perfectly thermally insulating. The density of water at 100 °C is 0.958 g/mL. Determine $|\Delta V_w|$, the change in the volume of liquid water.

Leave your answer to 2 significant figures in units of μL .

Solution: The key is to remember that water boils at 100 °C under a pressure of 1 atm. That means in the syringe, the water at 100 °C will reach an equilibrium if the pressure of the evaporated water vapour is $p_0 = 1$ atm. For a volume of $V = 1$ mL the number of moles of water that need to evaporate is

$$n = \frac{p_0 V}{RT} = 3.27 \times 10^{-5} \text{ mol}$$

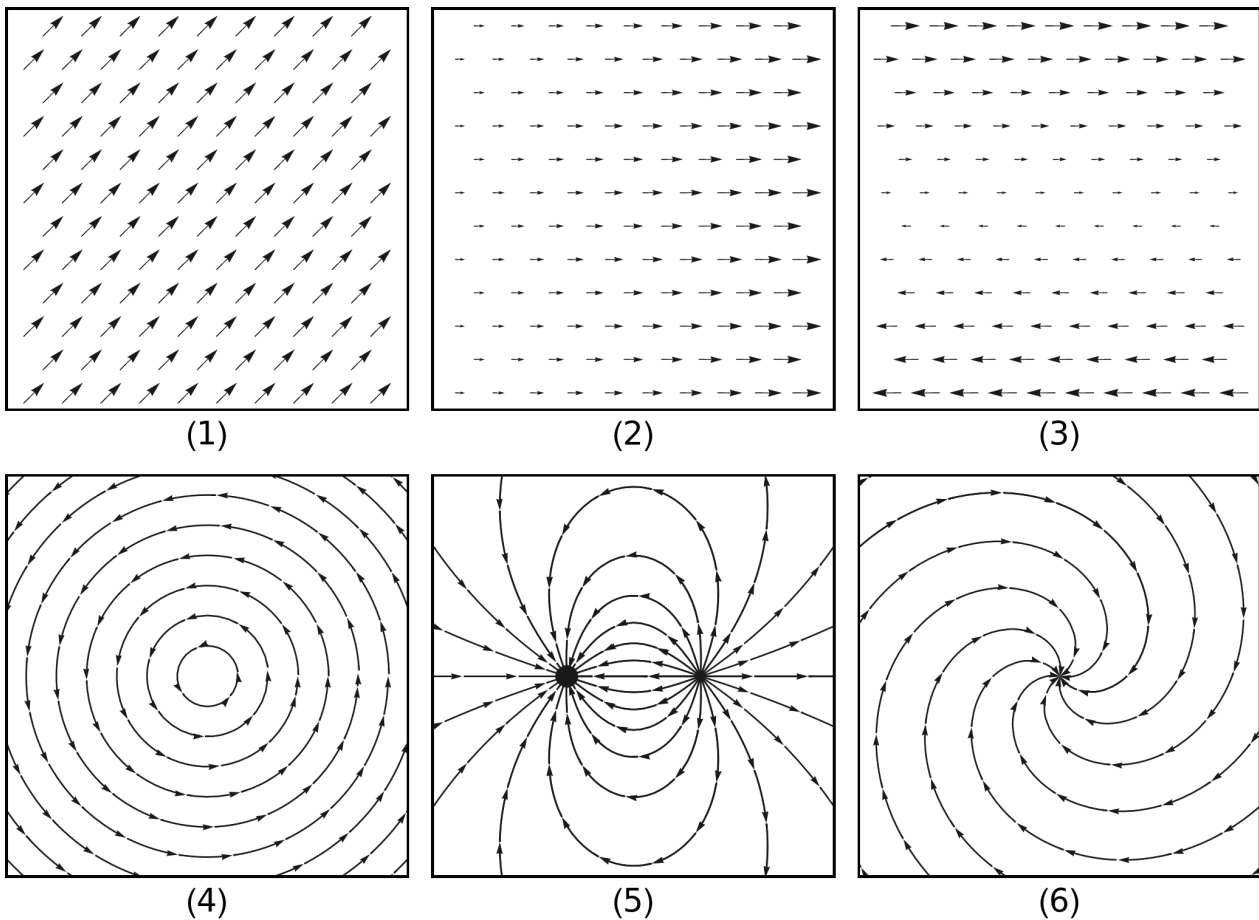
Hence, the change in the volume of liquid water is

$$|\Delta V_w| = \frac{n * 18.02 \text{ g/mol}}{\rho} = \boxed{0.61 \mu\text{L}}$$

Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Problem 17: Shape of You

A series of vector fields are shown below. Diagrams (1-3) are vector plots while diagrams (4-6) are streamline plots.



For each of the following parts, give your answer as a concatenation of the numbers of the diagrams above, in ascending order. For example, if you think the answer is diagrams (1), (2), and (3), then submit 123.

- (a) Which of these represents a possible part of the electric field that can be produced by a static configuration of electric charges?

Leave your answer as an integer. (2 points)

- (b) Which of these represents a possible part of the velocity field in a sealed 2D tank of water?

Leave your answer as an integer. (2 points)

Solution:

- (a) An electric field produced by a static configuration of electric charges cannot have any “circulation”, as that would imply that one could gain energy simply by following the electric field in a loop and returning to the same point.

Clearly, diagrams (3) (consider a rectangular loop around the centre), (4) and (6) do not satisfy this, while diagrams (1), (2) and (5) do. Thus the answer is 125.

- (b) Since water is incompressible, the velocity field in a tank of water must not have any sources or sinks. For instance, (2) cannot be a velocity field. Consider a rectangular box - more flux is exiting the right than entering the left, which is impossible in a sealed water tank.

The diagrams which satisfy this condition are (1), (3), (4) so the answer is 134.

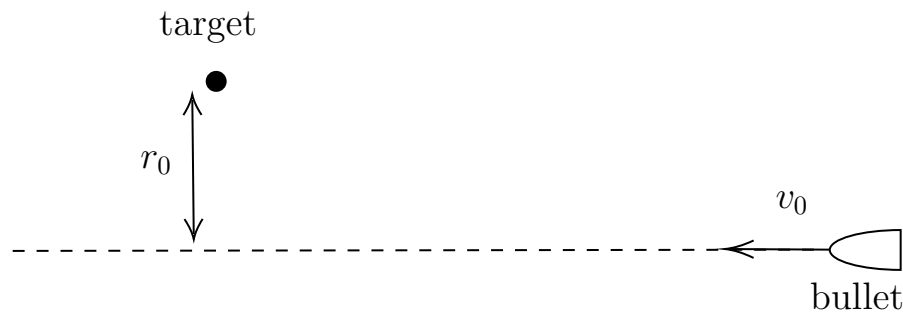
These ideas of “circulation” and “sources” can be formalised as the mathematical concepts of curl and divergence.

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Problem 18: Aimbot

In the game *Valorant*, a first-person shooter, player A's character, the "target", keeps getting hit by bullets fired by the opponent. Fed up, he decides to investigate the situation and observes two scenarios where the bullet strayed from a straight line to hit the target. He hypothesizes that this is because of electrostatic forces. The target, teammate, and bullet can be treated as points, and both cases occur in a vacuum with no gravity.

- (a) He figures that the reason why the bullet keeps hitting the target is that they attract each other. Assume that the fixed target is charged with $q = 0.50 \text{ mC}$ and the bullet is charged with $-q$. The bullet of mass $m = 40 \text{ g}$ is fired from a gun far away from the target, at an initial horizontal velocity v_0 , with $r_0 = 1.6 \text{ m}$.

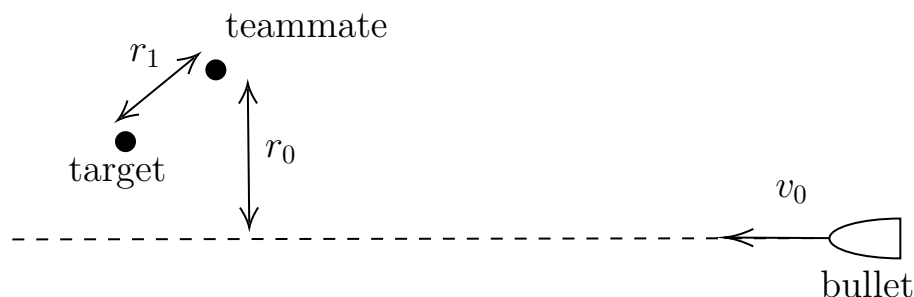


Determine the value of v_0 that allows the bullet to hit the target.

Leave your answer to 3 significant figures in units of m s^{-1} . (2 points)

Input -1 if no value of v_0 exists.

- (b) Next, player A decides to blame his stationary teammate for being charged with $+q$ instead, while the fixed target is a neutral insulator. The bullet of mass $m = 40 \text{ g}$ is fired from a gun far away from the target, at an initial horizontal velocity $v_0 = 500 \text{ m s}^{-1}$, with $r_0 = 1.6 \text{ m}$.



Let r_1 be the distance between the target and the stationary teammate. Given that the bullet hits the target at its maximum velocity, determine the value of r_1 .

Leave your answer to 3 significant figures in units of m . (3 points)

Input -1 if no value of v_0 exists.

Solution:

- (a) By conservation of angular momentum, v_0 must be $\boxed{0.00 \text{ m s}^{-1}}$ for the bullet to collide with the target. This is intuitive, but we can show it as follows:

Taking the target as the reference point, the bullet initially has angular momentum $L = mv_0 r_0$ and kinetic energy $KE = \frac{1}{2}mv_0^2$. Let v and r be the velocity of the bullet and the perpendicular distance from the bullet's velocity to the target respectively. As L is conserved, vr is conserved, so $v^2 \propto 1/r^2$, but electric potential energy $E \propto 1/r$, so energy is not conserved as $r \rightarrow 0$.

- (b) The electric force is proportional to $1/r^2$ where r is the distance between two charges, similar to universal gravitation. Hence, the bullet follows a hyperbolic orbit as it starts off with excess kinetic energy at infinity. Since the bullet hits the target at its maximum velocity, r_1 is the distance of closest approach.

Let us define $\mu = \frac{q^2}{4\pi\epsilon_0}$. Conservation of energy and conservation of angular momentum gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 - \frac{\mu}{r_1}$$

and

$$r_0 v_0 = r_1 v_1 \implies v_1 = \frac{r_0}{r_1} v_0$$

respectively. Then,

$$\begin{aligned} mv_0^2 r_1^2 + 2\mu r_1 - mv_0^2 r_0^2 &= 0 \\ \implies r_1 &= \frac{-\mu + \sqrt{\mu^2 + m^2 v_0^4 r_0^2}}{mv_0^2} = \boxed{1.39 \text{ m}} \end{aligned}$$

where the negative solution has been omitted.

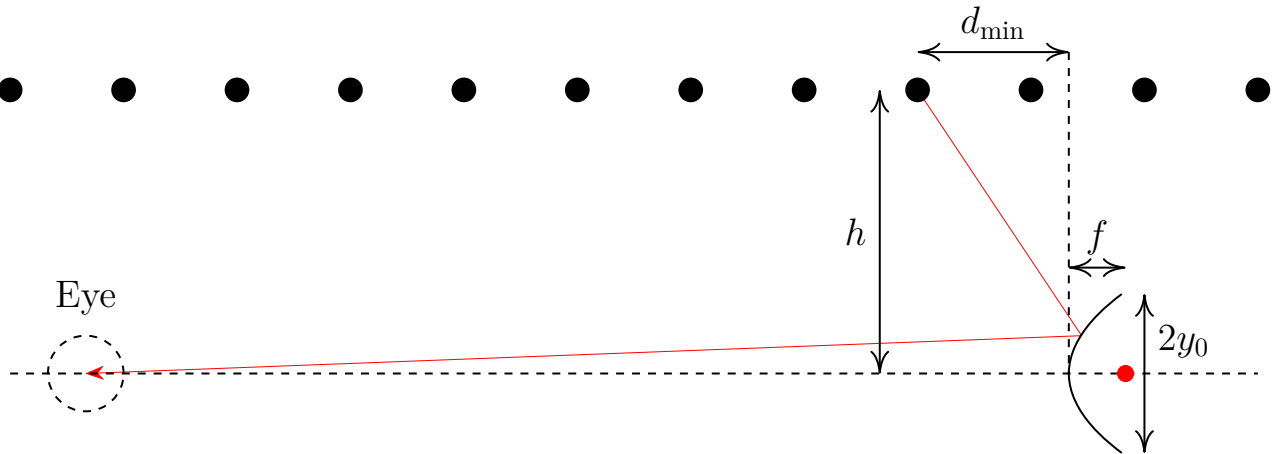
Setter: Liu Zhi, liu.zhi@sgphysicsleague.org

Problem 19: Tunnel Vision

(4 points)

Jamie is driving far behind Gindon on the KPE and notices reflections of the many overhead lights on his rear window.

We can model the rear window as a parabolic mirror whose axis of symmetry is horizontal, as shown in the figure below. Its focal length is $f = 2.5$ m and its height is $y_0 = 0.30$ m. Jamie's eye lies on the axis of symmetry. The tunnel lights are directly above the cars at a height $h = 5.0$ m above the midpoint of the window. Assume each tunnel light is a point source and that they are spaced very closely together.



Find the value of $d_{\min}$, the horizontal distance between the midpoint of the rear window and the nearest tunnel light behind Gindon whose reflection can be seen by Jamie. Assume the window is small enough that only lights located behind it are reflected.

Hint: A vertical parabola of focal length f has equation

$$y = \frac{x^2}{4f}$$

Leave your answer to 2 significant figures in units of m.

Solution: The key insight here is that since Jamie is very far from Gindon, any reflected rays that reach her are effectively horizontal. Hence, she only sees rays reflected parallel to the axis of the parabola.

For parabolic mirrors, all incident light rays travelling towards the focal point are reflected parallel to the optical axis. Therefore, to determine which tunnel lights Jamie sees, we consider light rays from the tunnel lights towards the focus.

Let us set the midpoint of Gindon's rear window as the origin, with the positive x -axis pointing forward and the positive y -axis pointing upwards. A light ray from a tunnel light at $(-d, h)$ that travels towards the focus has the equation

$$y = -\frac{h}{d+f}(x-f)$$

Since this parabola is horizontal, from the provided hint it has equation

$$x = \frac{y^2}{4f}$$

Substituting this into the previous equation gives the point where the ray strikes the mirror, as well as the distance of the associated light source:

$$y = -\frac{h}{d+f} \left(\frac{y^2}{4f} - f \right)$$
$$d = \frac{h}{y} \left(f - \frac{y^2}{4f} \right) - f$$

The ray from the nearest light source passes through the top of the mirror, i.e,

$$y = y_0$$

Hence, we can compute

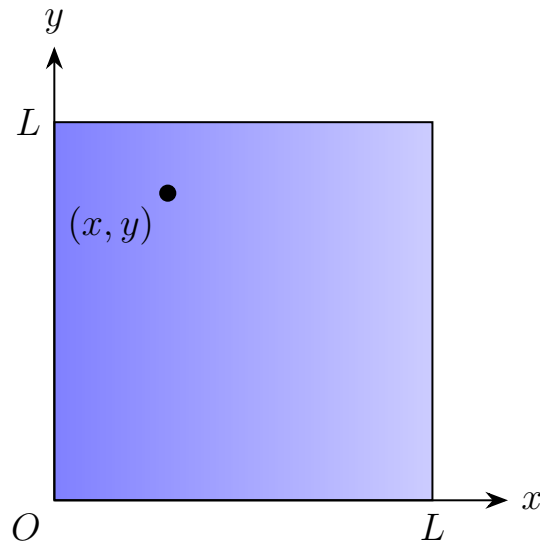
$$d_{\min} = \boxed{39 \text{ m}}$$

Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Problem 20: Plate Moment

(5 points)

Keith has a thin **non-uniform** square plate with side length $L = 1.000$ m. By using a special machine, Keith is able to measure the moment of inertia of the plate along any axis passing through the plate at the coordinates (x, y) and perpendicular to the plate.

**Format**

You are to input the coordinates of your query x and y to **3 decimal places** in units of m with $0 \leq x, y \leq L$. The simulation will output I , in units of kg m^2 to **5 significant figures**.

Hypothetical Example		
x (m)	y (m)	Output: I (kg m^2)
0.3000	0.8123	1.6969

Find the mass of the plate, m .

Leave your answer to 2 significant figures in units of kg.

You are given 3 simulation queries per problem attempt.

Solution: We have 4 unknown variables: the coordinates of the centre of mass (X, Y) , the moment of inertia about the centre of mass I_0 and the mass of the plate m . The parallel axis theorem allows us to write $I(x, y)$ as a function of (x, y) and these unknown variables:

$$I(x, y) = I_0 + m((x - X)^2 + (y - Y)^2)$$

With 4 queries, we have should have enough infomation to determine all the unknowns. However, it is possible to optimise down to only 3 queries if we only need m . Consider

the points $(0, 0)$, $(L/2, 0)$ and $(L, 0)$. Then:

$$\begin{aligned} I(0, 0) - I(L/2, 0) &= m \left(LX - \frac{L^2}{4} \right) \\ I(0, 0) - I(L, 0) &= m(2LX - L^2) \end{aligned}$$

We can then solve for m :

$$m = \frac{2}{L^2} \left(2(I(0, 0) - I(L/2, 0)) - (I(0, 0) - I(L, 0)) \right)$$

Using the queries $(x, y) = (0.000, 0.000)$, $(0.500, 0.000)$ and $(1.000, 0.000)$, we will obtain the outputs $I = 2.3868$, 1.9429 and 3.3489 respectively, and hence:

$$m \approx \boxed{3.7 \text{ kg}}$$

The actual value used in the simulation is $m = 3.737 \text{ kg}$.

Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org

Problem 21: Simp 2

(4 points)

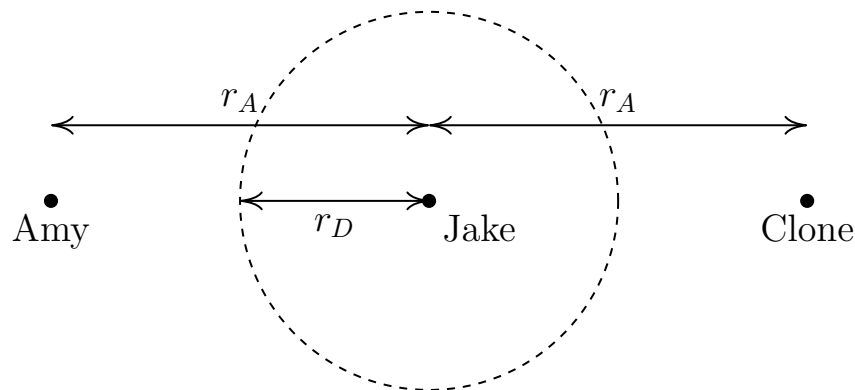
Amy, Jake, and his dog decide to play the same game of catch on a 2-dimensional plane. However, Amy has brought a clone of herself to make things a little harder for Jake and his dog (and for you!). They play with the same rules as before:

- A “catch” occurs when two characters occupy the same position on the plane.
- If Amy or her clone catches Jake before his dog catches her, she wins.
- If Jake’s dog catches both Amy and her clone before either catches Jake, Jake and his dog win.

Having barely escaped Amy in the previous game, Jake is now exhausted and cannot move. He also does not have the energy to entertain his hyperactive hyper-intelligent dog. As such, they add the following rules:

- Jake must remain stationary.
- Jake’s dog can only catch either Amy at a distance greater than r_D from Jake.

Jake and his dog begin at the origin, while Amy and her clone begin on opposite sides of Jake and his dog, at a distance r_A . The Amys have a top speed v_A , and the dog has a top speed v_D . All of them can change their velocities instantaneously.



They start the game with $r_A/r_D = 2$. Assuming all characters play optimally, Amy wins if $v_A/v_D > \chi$. Determine χ .

Leave your answer to 3 significant figures.

Solution:

If all characters play optimally, they will never stray from the line connecting the starting points of Amy and her clone. If they did, then their distance from Jake would increase/decrease at a slower rate than their top speed, allowing the opposing team to gain an advantage.

The optimal strategy for the Amys is as follows:

- If the dog is moving away from an Amy, she should run towards Jake at v_A .
- If the dog is moving towards an Amy, she should run away from the dog at v_A .

The optimal strategy for the dog is to run towards one Amy, catch her, then run towards the other Amy. If the dog manages to catch the second Amy before she reaches r_D , it wins.

Since the relative speed of the dog and the first Amy is $v_D - v_A$, and the dog has to catch up by a distance r_A , the time taken for the dog to catch the first Amy is

$$t_1 = \frac{r_A}{v_D - v_A}.$$

After this, the dog turns and runs in the opposite direction. The time that it takes to reach the origin is t_1 , and the time that it takes to reach the point r_D from Jake on the opposite side is r_D/v_D . Thus, the total time that the dog takes to complete its journey is

$$t_D = 2\frac{r_A}{v_D - v_A} + \frac{r_D}{v_D}.$$

The second Amy takes

$$t_A = \frac{r_A - r_D}{v_A}$$

to reach the same point. At her minimum speed, $t_D = t_A$. Therefore,

$$2\frac{r_A}{v_D - v_A} + \frac{r_D}{v_D} = \frac{r_A - r_D}{v_A}.$$

Letting $v_A/v_D = \chi$ and $r_A/r_D = \eta$, this reduces to

$$\frac{2\eta}{1 - \chi} + 1 = \frac{\eta - 1}{\chi}.$$

Multiplying both sides by $\chi(1 - \chi)$, expanding, and simplifying,

$$\chi^2 - 3\eta\chi + \eta - 1 = 0.$$

This is a quadratic equation, which we can solve to give

$$\chi = \frac{3\eta - \sqrt{9\eta^2 - 4\eta + 4}}{2} = \boxed{0.172}.$$

Once again, there are some technicalities to this solution. The optimal strategy for Amy is actually as follows:

- If the dog is moving away from an Amy, she should run towards Jake at v_A .

- If the dog is moving towards an Amy, she should run away from the dog at $v_A(1 - \epsilon)$, where $\epsilon \ll 1$.

This strategy forces the dog's optimal strategy to be as above (committing to one Amy, and then the other). To prove this, we can consider an arbitrary strategy where the dog switches targets after some time t .

When the dog comes back to the origin, both Amys will be at a distance $r_A - \epsilon v_A t$ from the origin, effectively restarting the game with both Amys slightly closer to the origin and thus a lower value of η . Clearly, this is a better situation for Amy (left as an exercise to the reader).

To maximise the time that the dog takes to catch the first Amy, she should choose ϵ to be as small as possible, giving the limiting case where $\chi \rightarrow 0.172$ as $\epsilon \rightarrow 0$.

Setters: Luc Mazereeuw, Ng Le Xi, Liu Wenkai

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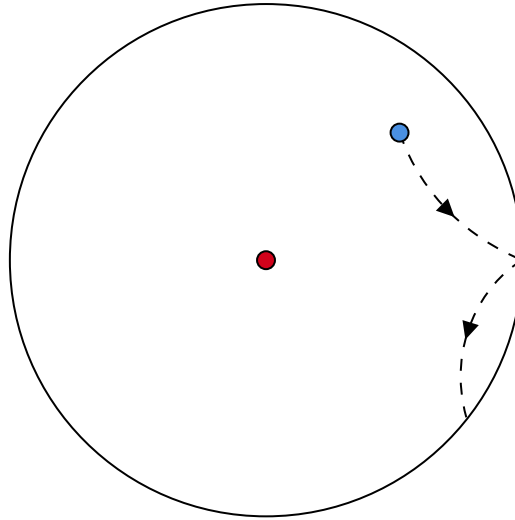
lexi.ng@sgphysicsleague.org

wenkai.liu@sgphysicsleague.org

Problem 22: Boing Boing

(3 points)

A proton is placed at an unknown position within a circular container. An fixed positive point charge of unknown magnitude is located at the centre of the circle.



At $t = 0$, the proton is projected with an unknown velocity $\mathbf{v}_0$. It subsequently moves within the container, following the dashed arrow shown in the figure and colliding inelastically with the walls, where each collision has coefficient of restitution e . Aaron, standing outside the container, is able to hear each impact and records the times of the first 15 collisions as shown in the table below:

No.	1	2	3	4	5	6	7	8
$t/\mu\text{s}$	2.0450	7.3476	12.679	17.724	22.254	26.162	29.433	32.117
No.	9	10	11	12	13	14	15	16
$t/\mu\text{s}$	34.289	36.031	37.421	38.525	39.401	40.094	40.642	41.075

From the data, estimate e . You may neglect gravity.

Leave your answer to 2 significant figures.

Solution: Between each bounce, the proton travels in a hyperbola with its axis of symmetry along the radial axis $\hat{r}$. The key is to realise that when the proton is very close to the edge of the cylinder, its acceleration is approximately constant, which means its trajectory approximates a parabola. This happens for the last few bounces where the proton bounces very rapidly.

For these small bounces considered, the cylinder's wall is approximately straight. This means the time between bounces is proportional to the radial component of the velocity, v_r :

$$\Delta t = \frac{2v_r}{a_r} \propto v_r$$

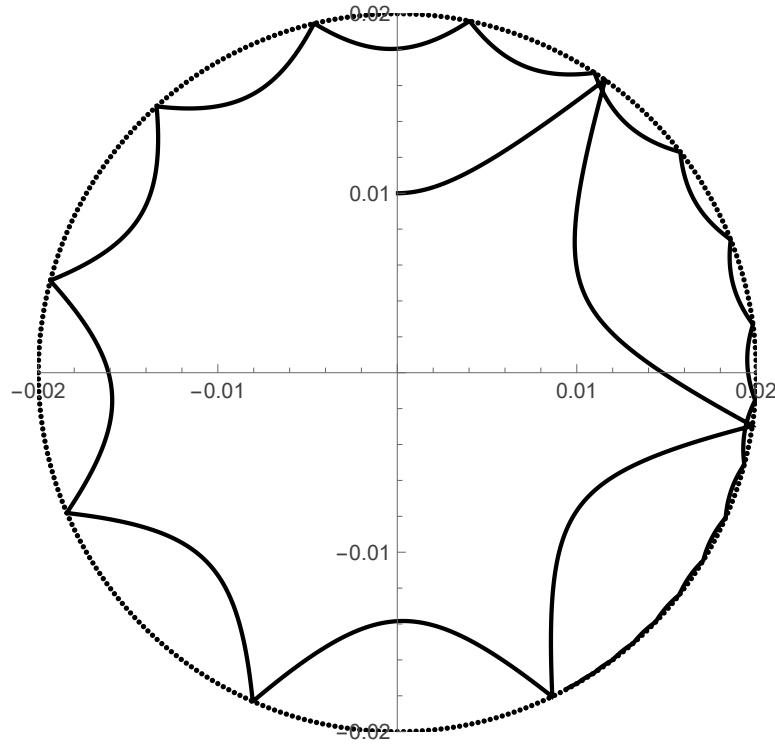
Thus, as $e = v_{r,f}/v_{r,i}$, we can take the ratio of time intervals between the last few bounces. Taking bounces 14-15, and 15-16, we have

$$e \approx \frac{41.075 - 40.642}{40.642 - 40.094} = 0.790$$

$$e \approx \boxed{0.79}$$

The value of e used to generate this data was indeed 0.790.

Here is the proton's trajectory, courtesy of Mathematica:



Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

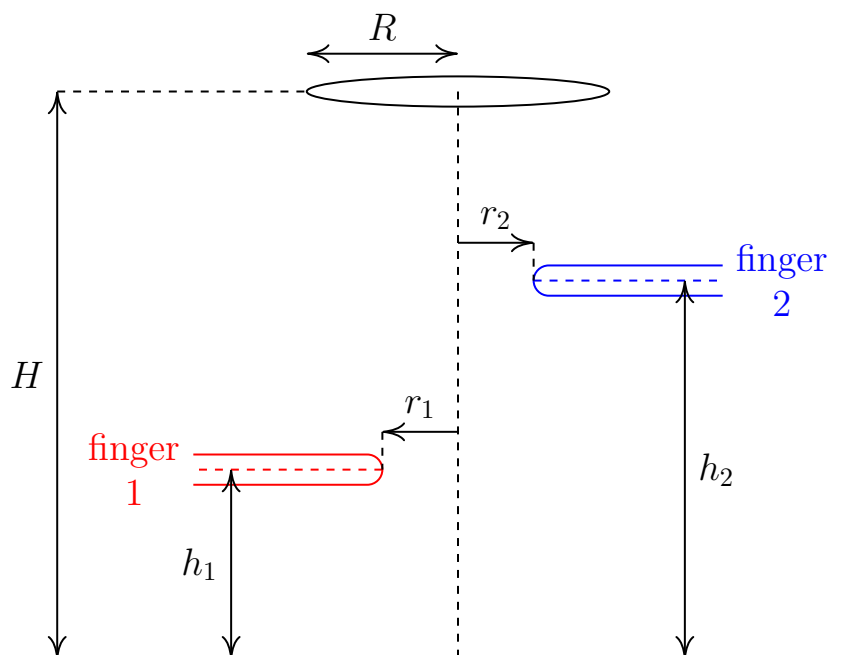
Problem 23: Magnetic Shadows

(4 points)

When two objects form shadows due to a non-point light source, the shadows may seem to “attract” each other as they get closer. Note that when we use the term “attract”, we refer specifically to the completely unilluminated parts of the shadows expanding. The image below shows an example.



Imagine a circular light source of radius $R = 4.0$ cm, positioned $H = 15$ cm above the ground, shining directly downwards.



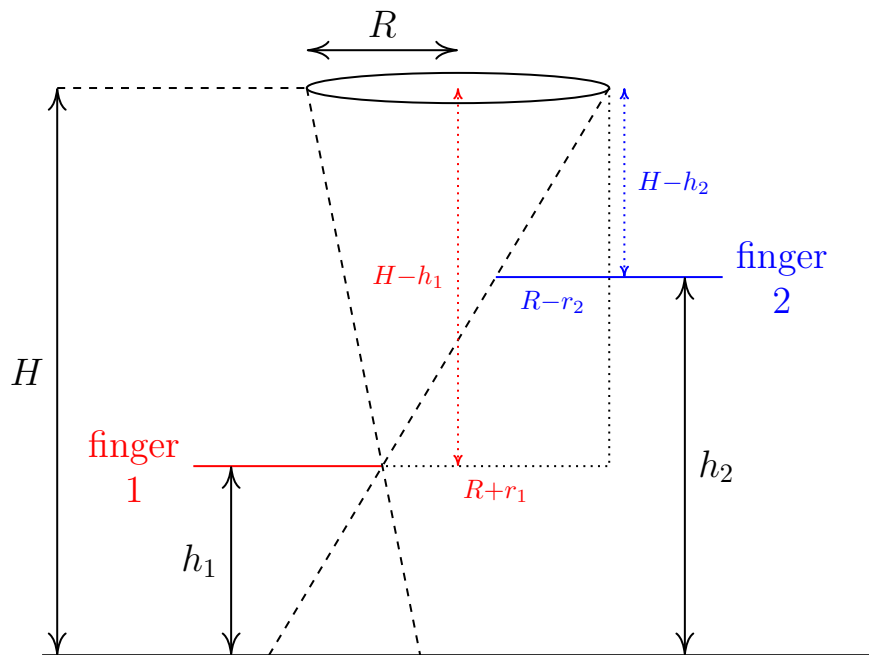
A person puts two fingers below the light source, pointing towards each other. From the top view, both fingers are aligned to the same diameter of the light source. Finger 1 is $h_1 = 5.0$ cm above the ground, and finger 2 is $h_2 = 10$ cm above the ground. The horizontal distance between the tip of the lower finger and the centre of the light source is $r_1 = 2.0$ cm.

Keeping h_1 and h_2 constant, he slowly moves finger 2 towards the centre. What is the minimum horizontal distance r_2 between the centre and the tip of finger 2 before the

two shadows start to “attract” each other?

Leave your answer to 2 significant figures in units of cm.

Solution: The shadows appear to “attract” each other because of geometry. When the shadow is caused by a light source that is not a point source, an umbra and a penumbra are formed. The umbra is the area where no light can reach it, resulting in complete darkness. The penumbra is the area where only some light can reach, resulting in partial darkness.



The figure above shows the cross section of the setup. As the higher finger moves closer to the centre, we observe that when the penumbra of the upper finger reaches the edge of the umbra of the lower finger, the area below the lower finger where zero light can reach becomes bigger. Thus, the shadow of the higher finger appears to "attract" the shadow of the lower finger.

By similar triangles, we can write down

$$\frac{R - r_2}{R + r_1} = \frac{H - h_2}{H - h_1}$$

Therefore, the minimum distance of the higher finger to the centre of the light source is

$$r_2 = R - (R + r_1) \left(\frac{H - h_2}{H - h_1} \right) = \boxed{1.0 \text{ cm}}$$

This phenomenon is known as “shadow blistering” and is pretty cool :)

Setter: Liu Zhi, liu.zhi@sgphysicsleague.org

Problem 24: Shower Troubles

(4 points)

A shower water heater consists of a large well-mixed tank of volume V and inner surface area A , as well as an outlet. The tank walls are maintained at temperature T_h by a heating system, while heat is transferred to the water with a convective heat transfer coefficient h . When the shower is turned on, the heated water exits the showerhead at a mass flow rate q , and is replaced by cold water at room temperature $T_0 = 25^\circ\text{C}$.

The shower has 2 unmarked control knobs: one controls T_h , while the other controls q . Each knob has 2 fixed settings labelled *low* and *high*, but the numerical values are unknown. The shower is allowed to reach steady state under 3 different knob settings, and the outlet temperatures are recorded below:

Temperature setting	Flow rate setting	Outlet temperature/ $^\circ\text{C}$
low	low	$T_{11} = 36.7$
low	high	$T_{12} = 33.3$
high	low	$T_{21} = 43.9$
high	high	$T_{22} = ???$

Unfortunately, the shower broke before the final measurement could be taken. Using the available data, determine the unknown temperature, T_{22} .

Leave your answer to 3 significant figures in units of $^\circ\text{C}$.

Solution: At steady state, the heat entering the tank via convection and the inlet must be equal to the heat leaving the tank via the outlet. Letting the outlet temperature be T , we have

$$qc_w T = qc_w T_0 + hA(T_h - T)$$

$$T = \frac{hAT_h + qc_w T_0}{hA + qc_w}$$

We can write down

$$\alpha = \frac{hA}{hA + qc_w}$$

$$T = \alpha T_h + (1 - \alpha)T_0$$

Where α is a function of q . Notice that this function is linear in both T_h and α , with $(T_h, \alpha) = (0, 0)$ corresponding to T_0 . Hence,

$$\frac{T_{22} - T_0}{T_{21} - T_0} = \frac{T_{12} - T_0}{T_{11} - T_0}$$

$$T_{22} = T_0 + \frac{(T_{12} - T_0)(T_{21} - T_0)}{T_{11} - T_0}$$

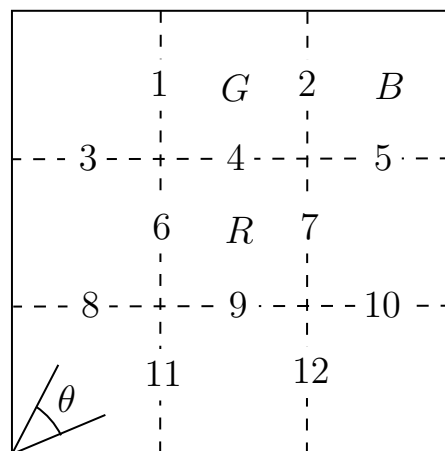
$$= \boxed{38.4^\circ\text{C}}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Problem 25: Mirror Box

A square room is divided into a 3×3 grid. A red ball (R), blue cylinder (B) and green cube (G) are placed as shown in the diagram below, at different heights. The walls, floor, and ceiling of the room are mirrors, and the ceiling of the middle cell is a light source.

A camera is placed at one corner of the room, looking horizontally at the opposite corner. The horizontal field of view of a camera is defined as the angle between the leftmost and rightmost rays of light that are captured by the camera, and is labelled by θ in the diagram below. The camera used has a field of view which is a multiple of 10, when measured in units of degrees.



Two photos are taken by the camera, shown on the following page:

- Photo 1: A photo with the objects in the room as described above.
 - Photo 2: A photo with the objects in the same position, but with mirrors on 2 of the grid segments (numbered in the diagram above).
- (a) What is the horizontal field of view of the camera?

Leave your answer to 1 significant figure in units of degrees. (2 points)

- (b) Which 2 grid segments have mirrors placed on them in Photo 2? Concatenate the numbers of the segments in your final answer, in ascending order. For example, if you think the mirrors are at grid segments 4 and 12, give your answer as 412.

Leave your answer as an integer. (3 points)

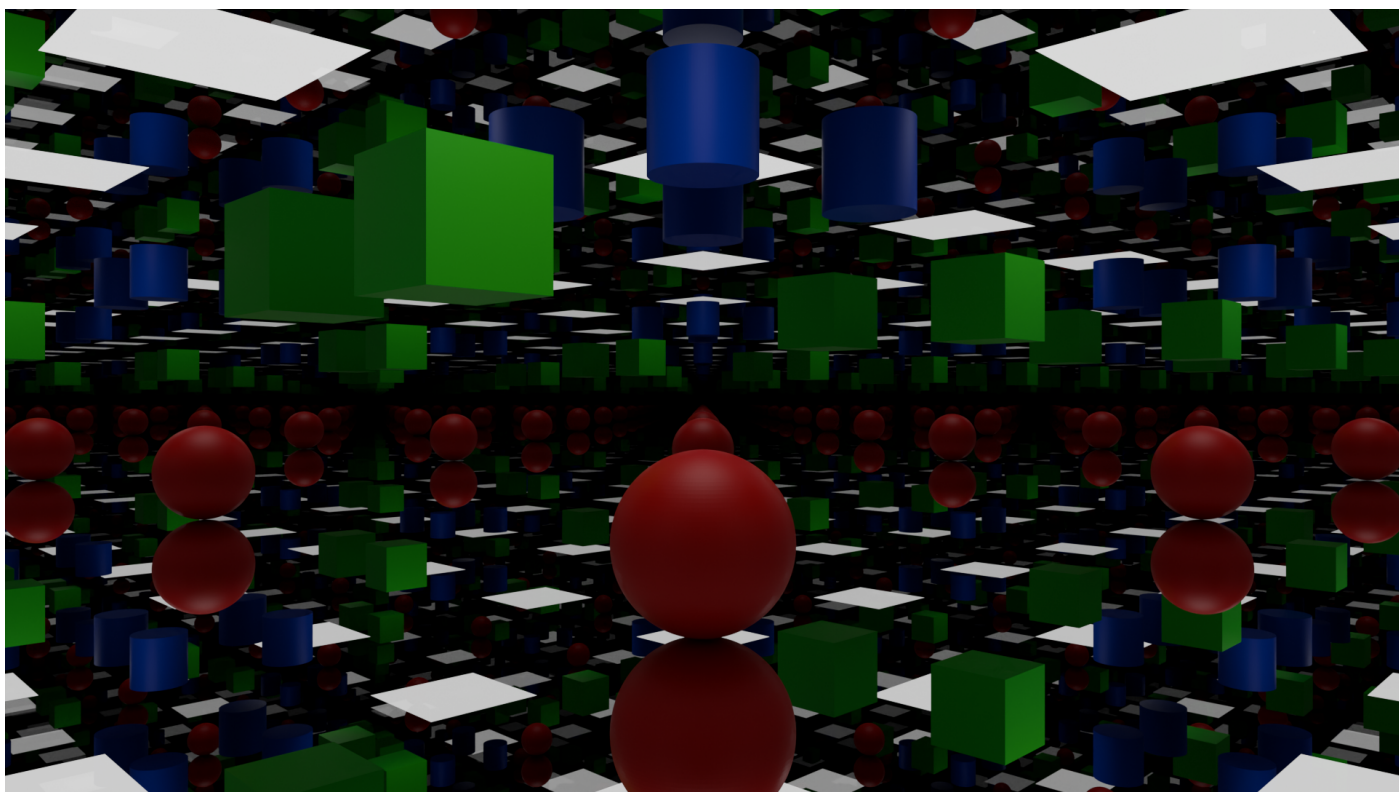


Photo 1: No added mirrors

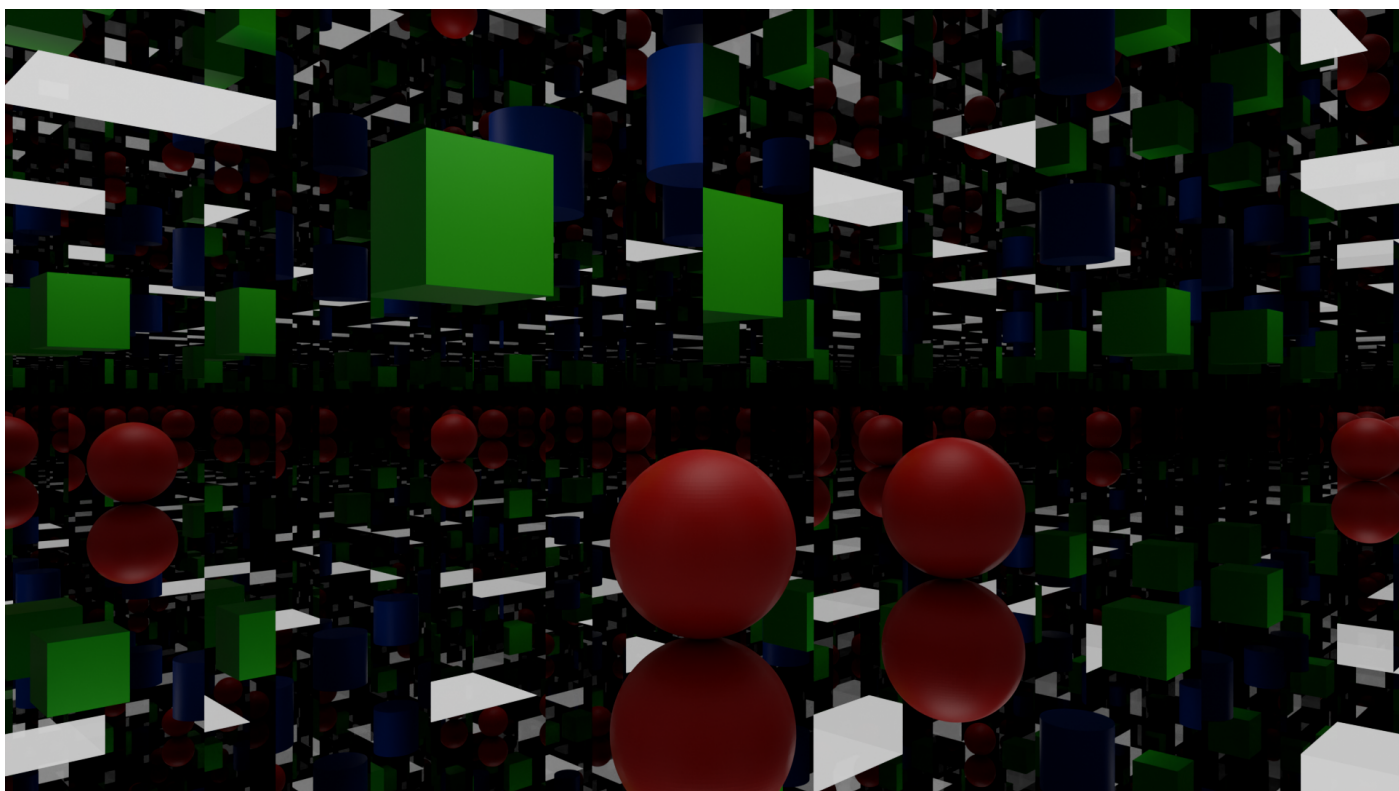


Photo 2: With added mirrors

Solution:

- (a) We can use the images of the objects to determine the field of view of the camera. For convenience, we will use the first photo and the images of the red ball.

We can barely see the second image of the red ball, which is present at coordinates $(2.5L, 0.5L)$. The angle that the light ray coming from this image makes with the side of the room is

$$\alpha = \tan^{-1} \frac{1}{5} = 11.3^\circ.$$

As such, the field of view of the camera is approximately $90^\circ - 2\alpha = 67.4^\circ$. Rounding this up to the nearest 10° , we find that the field of view of the camera is $\theta = \boxed{70^\circ}$.

- (b) In the second image, the blue cylinder is half blocked by a mirror, while the red ball is not blocked by a mirror. Thus, segment 7 has a mirror.

The green cube has also been partially blocked by a mirror. The reflection of the green cube has disappeared, and the reflections of the red ball on the left of the image have been shifted. All of these pieces of evidence point to segment 3 being a mirror.

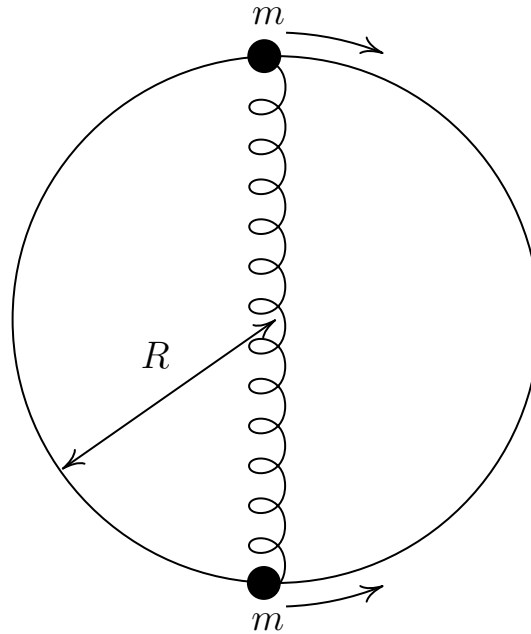
Thus, the answer is $\boxed{37}$.

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem 26: Rubber Slingshot

(3 points)

Raphael has a frictionless circular disk of radius $R = 15.0$ cm and a rubber band that he wants to launch forwards. To do so, he holds the disk horizontally, wraps the rubber band around the diameter of the disk and gives it a small impulse in the x -direction, as shown below.



The rubber band may be modelled as two point masses of mass $m = 13.0$ g connected by a spring of spring constant $k = 60.0$ N m⁻¹ and zero rest length. You may ignore gravity. Determine the maximum horizontal velocity v the rubber band achieves during its subsequent motion.

Leave your answer to 2 significant figures in units of m s⁻¹.

Solution: Let the angle that each mass makes with the vertical be θ and the speed of each mass be u . Conservation of energy gives

$$\frac{1}{2}k(2R)^2 = 2 \left(\frac{1}{2}mu^2 \right) + \frac{1}{2}k(2R \cos \theta)^2 \quad \Rightarrow \quad u^2 = \frac{2kR^2}{m} \sin^2 \theta$$

The rubber band's maximum vertical speed is achieved when it leaves the disk at θ_c , i.e. the normal force exerted by the disk is zero. When that occurs, the only force acting on each mass is the spring force, so the component of the force towards the centre should be equal to the centripetal force, so

$$\frac{mu^2}{R} = k(2R \cos \theta_c) \cos \theta_c \quad \Rightarrow \quad \sin^2 \theta_c = \cos^2 \theta_c \quad \Rightarrow \quad \theta_c = 45^\circ$$

Therefore, the maximum vertical speed of the masses is

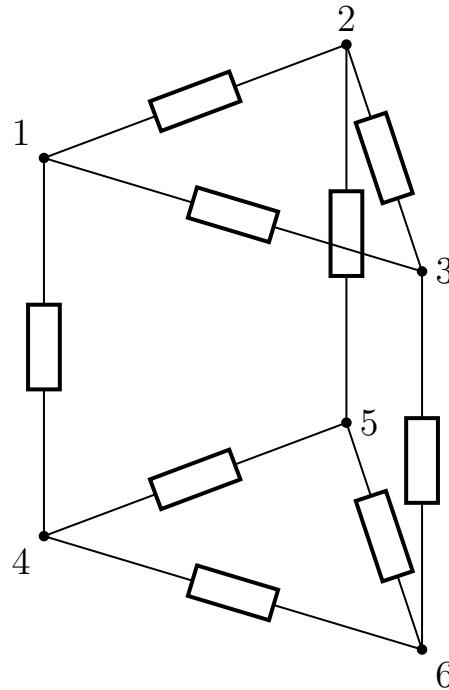
$$v = u \cos \theta_c = \sqrt{\frac{2kR^2}{m}} \sin \theta_c \cos \theta_c = \sqrt{\frac{k}{2m}} R = \boxed{7.2 \text{ m s}^{-1}}$$

Setter: Oon Han, oon.han@sgphysicsleague.org

Problem 27: Sherlock Ohms

(6 points)

There are 9 resistors arranged in the form of a prism, with vertices labelled 1 to 6. Eight resistors have the same resistance R , while one has a different resistance R_{odd} . You are given an ohmmeter that can measure the resistance between 2 vertices.



Determine R_{odd} .

Leave your answer to 2 significant figures in units of Ω .

Format

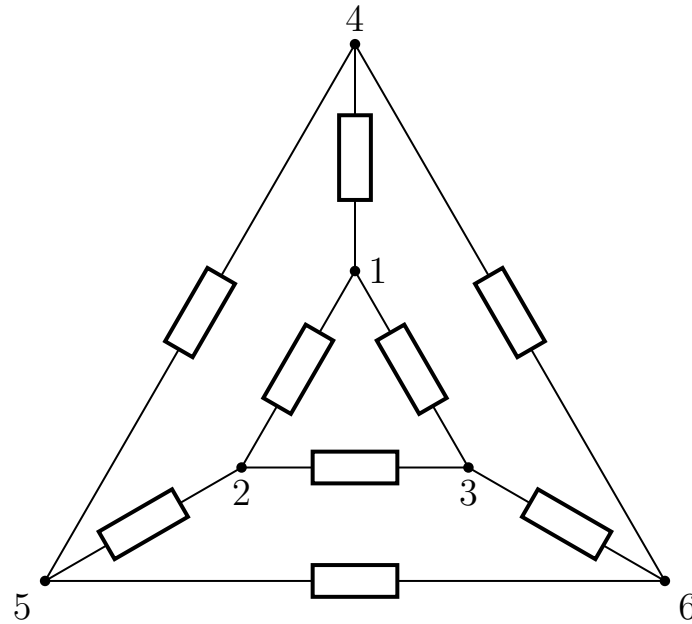
You are to input a **2 digit number** corresponding to the labels of the 2 vertices $\overline{ab}$ that the ohmmeter will be attached to. Input the vertices in ascending order with $a < b$. The simulation will output R , the resistance measured, in units of Ω to **4 significant figures**.

Hypothetical Example	
Vertices	Output: R (Ω)
12	1.234

You are given 5 simulation queries for your first attempt, and 2 more per subsequent attempt.

Solution: We will present a method to obtain the position and the resistance of the resistor within 5 simulation outputs. Let us visualise the circuit in 2D, and the resistance measured across any 2 vertices V_1, V_2 be denoted as $R_{V_1V_2}$. In this solution, each

individual resistor is represented by their edge i.e. 14 refers to the resistor connected to vertices 1 and 4.



Define the edges 14, 36 and 25 to be X edges.

Group each X edge to the 2 non X edges that are not connected to it. E.g. 14, 23, 56 are grouped.

We may make the following observations:

- (a) If we connect the ohmmeter to an X edge which is in the same group as the odd resistor (e.g. edge 14), no current will pass through the other edges in the same grouping (edges 23 and 56) by symmetry.
- (b) If we connect the ohmmeter to a non X edge $\mathbf{E}$, e.g. edge 56 or 23, and the odd resistor is in the same group as $\mathbf{E}$ or is $\mathbf{E}$, no current will pass through the X edge in the same group as $\mathbf{E}$ (edge 14) by symmetry.

We can hence implement the following method:

1. Use 3 simulation inputs to connect the ohmmeter to each of the X edges. Due to the symmetry of the configuration, 2 of the 3 readings obtained will be equal regardless of the position of the odd resistor. The different measurement indicates that the odd resistor is in the same group as that X edge. For example, if 36 produces a different reading, the odd resistor is either 36, 12 or 45.
2. After measuring, edge 14 is found to have a greater resistance than the other 2 X edges. Hence, the odd resistor is either 14, 23 or 56. Then, we will measure both edge 23 and 56.
3. If $R_{56} = R_{23}$, then the odd resistor is 14 since no current passes through this resistor. Since they are not equal, the odd resistor is either 23 or 56. Furthermore,

$R_{14} < R_{25}, R_{36}$ which implies that the odd resistance is greater than the rest as it does not influence the measurement of R_{14} by observation (a).

If you measure the resistance of edge 56 and edge 23, you should notice that edge 56 has a larger resistance, so this means that edge 56 is the odd resistor. A quick explanation for this would be that edge 56 contributes a much larger contribution to the total resistance when the ohmmeter is connected across it than when the ohmmeter is connected across edge 23.

To convince you rigorously we need to do some casework for R_{56} . Since in all cases no current passes through the X edge 14, it is easy to find the resistance. Let's assume first that all the regular resistors have resistance $1\ \Omega$, and the odd resistor has resistance x .

If the odd resistor is edge 14, resistance measured should be

$$R_{56} = \frac{1}{\frac{1}{1} + \frac{1}{2} + \frac{1}{2+\frac{2}{3}}} = \frac{8}{15}\Omega$$

If the odd resistor is edge 23, resistance measured should be

$$R_{56} = \frac{1}{\frac{1}{1} + \frac{1}{2} + \frac{1}{2+\frac{2x}{2+x}}} = \frac{4x+4}{7x+8}\Omega$$

If the odd resistor is edge 56, resistance measure should be

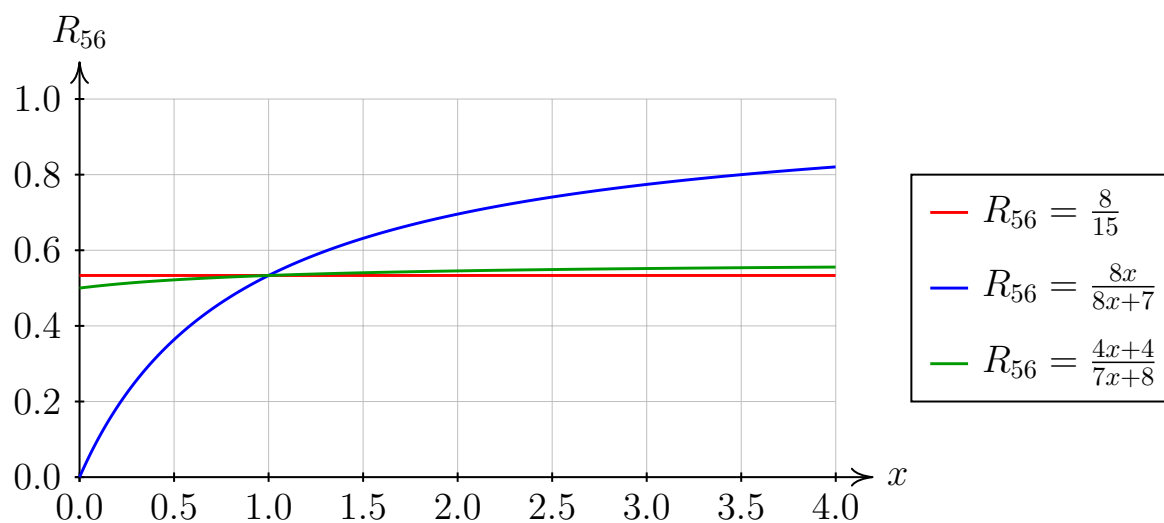
$$R_{56} = \frac{1}{\frac{1}{x} + \frac{1}{2} + \frac{1}{2+\frac{2}{3}}} = \frac{8x}{7x+8}\Omega$$

Since we deduced that the odd resistor is not 14, the resistance across 14 is

$$R_{14} = \frac{1}{1 + \frac{1}{3} + \frac{1}{3}} = \frac{3}{5}\Omega$$

From your measurement of edge 14, you should get a value of $1.200\ \Omega$, which means the value of a normal resistor is $2\ \Omega$ and we need to scale everything down by a factor of 2 to use the equations above. From your measurement of $R_{56} = 1.538\ \Omega$, you should find that the odd resistor has resistance $4.7\ \Omega$.

Plotting the 3 functions of R_{56} out, we have:



Note: It should be possible to reduce this number of steps to 4 by comparing the ratios of R_{23} or R_{56} to the values of R_{14} and R_{25} , but the solution is too computationally complex for our purposes.

Setter: Lv Jindong, jindong.lv@sgphysicsleague.org

Problem 28: Dripping Faucet

(3 points)

Being the careless person he is, Joan has left the tap in his kitchen dripping. The tap can be modelled as a cylindrical tube with inner radius $a = 0.50$ cm. Water drips periodically from the tap, with $T = 3.0$ s between drops. The surface tension of water is $\sigma = 0.071$ N m⁻¹. Assuming the contact angle between the water and inner surface of the tap is $\theta = 0^\circ$, find the average velocity u (over a cross-section) of the water within his tap.

Leave your answer to 2 significant figures in units of mm s⁻¹.

Solution: Droplet breakoff occurs when the weight of the droplet exceeds the maximum surface tension force that adheres the droplet to the tap. The surface tension force is given by

$$F_{\text{s.t.}} = \sigma l$$

Where $l = 2\pi a$ is the length of the contact line. Balancing forces on a droplet of volume V , we may write

$$mg = F_{\text{s.t.}} \implies \rho V g = 2\pi a \sigma$$

i.e.

$$V = \frac{2\pi a \sigma}{\rho g}$$

Hence, the tap's volume flow rate is

$$Q = \frac{V}{T} = \pi a^2 u$$

This gives us an average velocity of

$$u = \frac{V}{T\pi a^2} = \frac{2\sigma}{\rho g a T} = \boxed{0.97 \text{ mm s}^{-1}}$$

Setter: Tan E-Jie, tan.ejie@sgphysicsleague.org

Problem 29: Chill Out

(4 points)

A box of helium gas is initially at pressure $P_0 = 1.01 \times 10^5$ Pa and volume $V_0 = 0.590$ m³. The gas is isothermally compressed to a pressure of $2P_0$ and then adiabatically expands back to pressure P_0 . This sequence is repeated infinitely many times. Assume the helium always behaves as an ideal monatomic gas and all processes are reversible. Determine the net heat Q lost by the gas throughout the entire process.

Leave your answer to 3 significant figures in units of kJ.

Solution: Let V_n be the volume of the gas at the start of the $(n+1)^{\text{th}}$ sequence. During the next isothermal compression, PV is constant, so the volume V after compression can be determined as

$$P_0 V_n = 2P_0 V \quad \implies \quad V = \frac{V_n}{2}$$

Then, during the next adiabatic expansion, PV^γ is constant, so

$$2P_0 V^\gamma = P_0 V_{n+1}^\gamma \quad \implies \quad V_{n+1} = 2^{\frac{1}{\gamma}} V = 2^{\frac{1}{\gamma}-1} V_n$$

Therefore, V_n obeys a geometric progression,

$$V_n = 2^{(\frac{1}{\gamma}-1)n} V_0$$

Since there is no heat transfer during adiabatic expansion, the heat lost during the $(n+1)^{\text{th}}$ sequence is that of the isothermal compression,

$$Q_n = -P_0 V_n \ln \frac{V}{V_n} = P_0 V_0 2^{(\frac{1}{\gamma}-1)n} \ln 2$$

Therefore, the total heat lost is

$$Q = \sum_{n=0}^{\infty} Q_n = (P_0 V_0 \ln 2) \sum_{n=0}^{\infty} 2^{(\frac{1}{\gamma}-1)n} = P_0 V_0 \frac{\ln 2}{1 - 2^{\frac{1}{\gamma}-1}}$$

Since helium is monatomic, $\gamma = \frac{5}{3}$, so

$$Q = \frac{\ln 2}{1 - 2^{-\frac{2}{5}}} P_0 V_0 = \boxed{171 \text{ kJ}}$$

Setter: Oon Han, oon.han@sgphysicsleague.org

Problem 30: As It Was

(4 points)

In the music video for “[As It Was](#)”, there is a scene where Harry Styles runs on a horizontal rotating disk with a central axle. The axle is fixed in place and the disk can rotate freely about it. The disk has radius $R = 1.8$ m and mass $M = 100$ kg; Harry Styles has a mass $m = 70$ kg. Harry Styles and the disk start at rest.

If Harry Styles has a maximum running speed of $v_{\max} = 10$ km h⁻¹ relative to the surface of the disk, what is the maximum angular velocity ω that he can impart to the disk? Ignore air resistance, and assume that Harry Styles does not jump off the disk.

Leave your answer to 3 significant figures in units of rad s⁻¹.

Solution:

Since the only external force acting on the system with the disk and Harry Styles comes from the axle, the angular momentum of the centre about the centre of the disk must be conserved (and equal to 0). Thus,

$$mrv = \frac{1}{2}MR^2\omega,$$

where r is the distance between Harry Styles and the centre of the disk, v is Harry Styles' tangential speed around the centre of the disk (relative to the ground), and ω is the angular velocity of the disk.

To maximise ω , it is obvious that Harry Styles should run tangentially to the disk. The relative speed between Harry Styles and the disk is determined by his maximum running speed, so that

$$v_{\max} = v + r\omega.$$

From these equations, we can eliminate v to give

$$mr(v_{\max} - r\omega) = \frac{1}{2}MR^2\omega.$$

Solving for ω ,

$$\omega = \frac{mrv_{\max}}{\frac{1}{2}MR^2 + mr^2}.$$

Now, we have to find the value of r at which this is maximised. Setting its derivative with respect to r to 0,

$$0 = \frac{1}{\frac{1}{2}MR^2 + mr^2} - \frac{2mr^2}{\left(\frac{1}{2}MR^2 + mr^2\right)^2}.$$

Simplifying and solving, we find that

$$r = \sqrt{\frac{M}{2m}} R.$$

Since $M < 2m$, this is smaller than R and lies inside the disk. Substituting this into the expression for ω ,

$$\begin{aligned}\omega &= \frac{v_{\max}}{R} \sqrt{\frac{m}{2M}}, \\ &= \boxed{0.913 \text{ rad s}^{-1}}\end{aligned}$$

Harry *styles*.

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem 31: Unknown Particle in an Unknown Box

An unknown particle is confined in a one-dimensional box of length L with impenetrable barriers (i.e., the potential is infinite at the boundaries).

You have a magical device that can measure the amplitude of the particle's wavefunction ψ at a certain position. The position is defined by the dimensionless length $\bar{x} = \frac{x}{L}$, where x is the distance from one end of the box. It is known that the particle occupies a state corresponding to an odd quantum number (e.g. $n = 1, n = 3, n = 5, \dots$).

(a) Determine the length L of the box.

Leave your answer to 3 significant figures in units of nm. (2 points)

You are given 1 simulation query per problem attempt.

Another unknown particle of mass m is in a different 1 dimensional box. The particle is in one of the 5 lowest energy levels ($n \in \{1, 2, 3, 4, 5\}$), and the energy level it is at has energy $E = 4.244 \times 10^{-22}$ J.

(b) Determine the mass m of the particle.

Leave your answer to 3 significant figures in units of 10^{-26} kg. For example, if your answer is 2.56×10^{-26} kg, input 2.56. (3 points)

You are given 3 simulation queries per problem attempt.

Format

You are to input the dimensionless length $\bar{x}$ at which the amplitude of the wavefunction will be measured to **5 significant figures** where $0 \leq \bar{x} \leq 1$. The simulation will output the amplitude of the wavefunction ψ to **5 significant figures**.

Hypothetical Example	
Input: $\bar{x}$	Output: $\psi(\bar{x}) / \text{m}^{-1/2}$
0.65838	-18270
0.46537	71333

Hint: The following Wikipedia [link](#) may be helpful.

Solution:

Throughout this solution, we will be making use of the well-known solution of the wavefunction of a particle in a one dimensional box:

$$\psi_n = A \sin\left(\frac{n\pi}{L}x\right) \quad (1)$$

And the energy of energy levels for a particle in a one dimensional box:

$$E_n = \frac{n^2 h^2}{8mL^2} \quad (2)$$

- (a) If the quantum number is odd, the wavefunction has an antinode at $\bar{x} = 0.5$. Also, by the normalisation condition,

$$\int_0^L |\psi(x)|^2 dx = 1$$

By substituting equation (1) into the above integral, we know that $A = \sqrt{\frac{2}{L}}$. At the antinode, the amplitude is equal to A . Hence, we need to obtain the amplitude of the wavefunction at $\bar{x} = 0.5$.

Suggested Query	
Input: $\bar{x}$	Output: $\psi(\bar{x}) / \text{m}^{-1/2}$
0.50000	-68925

Hence,

$$\begin{aligned}
 \sqrt{\frac{2}{L}} &= 68925 \\
 \frac{2}{L} &= 68925^2 \\
 L &= \frac{2}{68925^2} \\
 &= 4.21 \times 10^{-10} \text{ m} \\
 &= \boxed{0.421 \text{ nm}}
 \end{aligned}$$

(b) We can again first get the amplitude at $\bar{x} = 0.5$. There are only a few cases from here:

- If $\psi(0.5)$ is non-zero, we can get the value of L , then by trial and error we can check whether its $n = 1$, $n = 3$ or $n = 5$.
 - If $\psi(0.33333) = 0$, then $n = 3$.
 - If $\psi(0.20000) = 0$, then $n = 5$.
 - Otherwise, $n = 1$
- If $\psi(0.5) = 0$, then we know that $n = 2$ or $n = 4$. We can next test $\bar{x} = 0.25000$.
 - If $\psi(0.25000) = 0$, then $n = 4$. We can next get $\psi(0.12500)$ to obtain the amplitude.
 - If $\psi(0.25000)$ is non-zero, then $n = 2$, and that is the amplitude.

Following the above plan, we can have the following queries:

Suggested Query	
Input: $\bar{x}$	Output: $\psi(\bar{x}) / \text{m}^{-1/2}$
0.50000	0.0000
0.25000	0.0000
0.12500	73324

The length of the box is therefore

$$\begin{aligned}
 L &= \frac{2}{|\psi(0.12500)|^2} \\
 &= 3.720 \times 10^{-10} \text{ m}
 \end{aligned}$$

Hence, the mass is:

$$\begin{aligned}
 m &= \frac{n^2 h^2}{8L^2 E} \\
 &= \boxed{1.50 \times 10^{-26} \text{ kg}}
 \end{aligned}$$

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Problem 32: Squashing

(4 points)

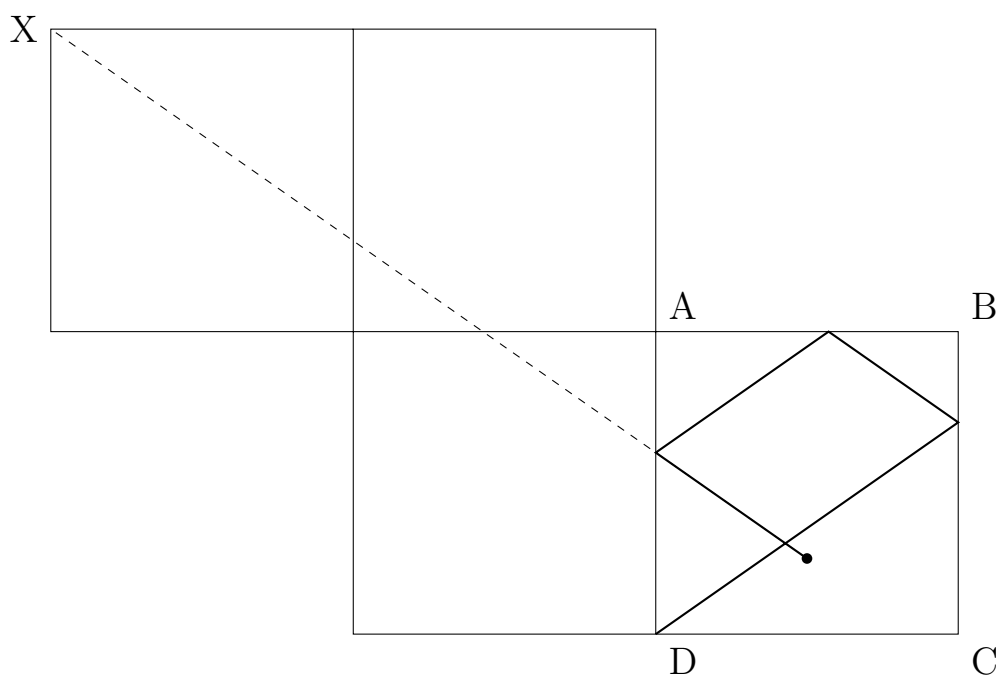
Mario is studiously practising squash shots for his upcoming match against Luigi in a 10 m by 10 m square court ABCD with a very high ceiling. He wants to master his *ultimate shot*: the squash ball will bounce off wall DA, then wall AB, then wall BC before landing at corner D in that specific order.

Mario's ultimate shot can be carried out from many positions on the court. However, Mario is weak, so the maximum initial velocity he can launch the squash ball with is $v = 16 \text{ m s}^{-1}$. What area A of the court can he perform this *ultimate shot* from?

Assume that Mario launches the squash ball from ground level and that all collisions are perfectly elastic.

Leave your answer to 3 significant figures in units of m^2 .

Solution: Since all bounces are elastic, each collision reverses the direction of the x or y component of the velocity. Hence, we can treat the walls like mirrors, and reflect the path of the ball during collision.

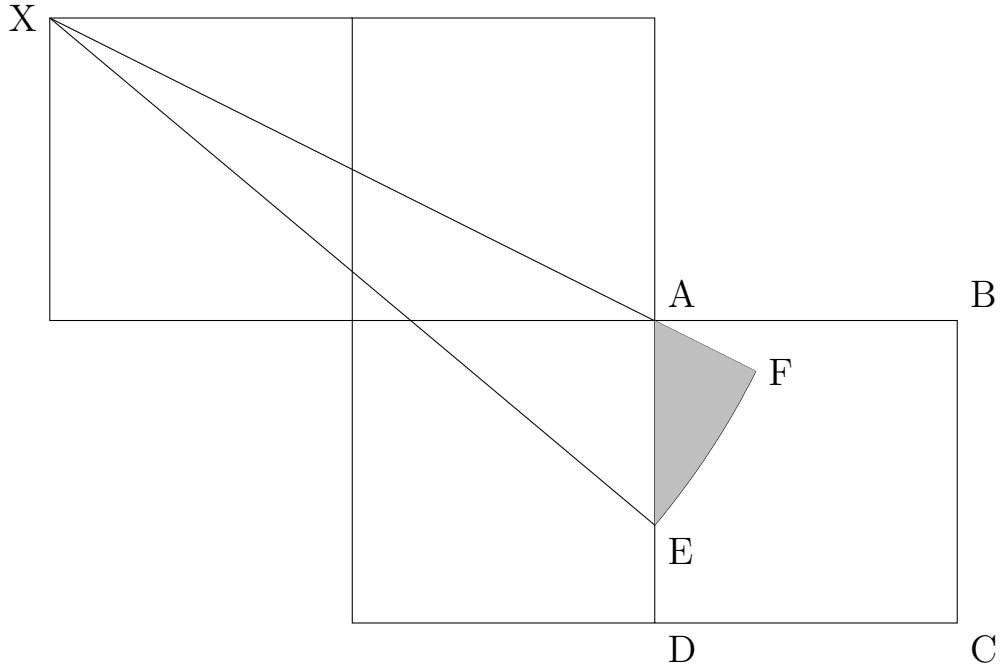


Tiling the square court, we can convert the path to that of a straight line that ends at point X, which corresponds to the endpoint D. In order for the ball to bounce off 3 walls, it must pass through 3 walls from the starting position to X.

Since there is a maximum initial velocity, the maximum range R_{max} is restricted.

$$R_{\text{max}} = \frac{v^2 \sin 2\theta}{g} \leq \frac{v^2}{g}$$

Hence, the *ultimate shot* is only possible if launched from the shaded region.



With $R_{\max} = \frac{v^2}{g}$, the area A can be calculated from the difference in area between sector XFE and $\triangle XAE$. The angle subtended by the sector is given by

$$\theta = \arctan\left(\frac{20 \text{ m}}{10 \text{ m}}\right) - \left(\arcsin \frac{20 \text{ m}}{R_{\max}}\right)$$

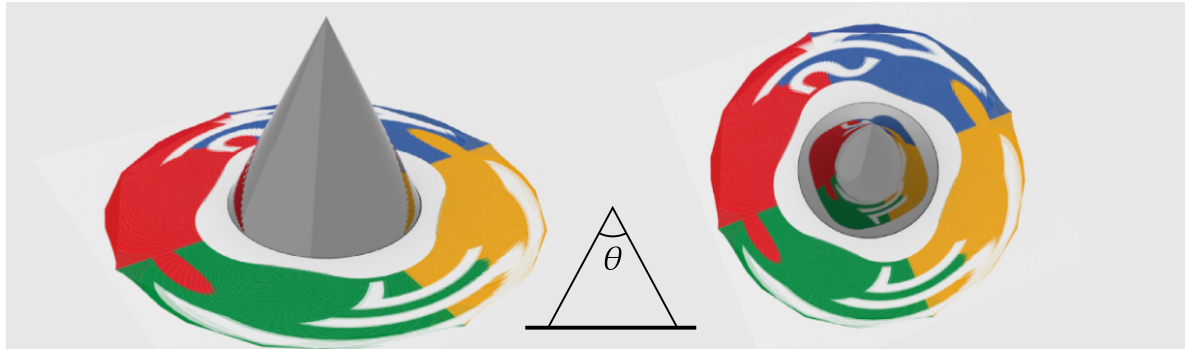
$$A = \pi R_{\max}^2 \left(\frac{\theta}{2\pi}\right) - \frac{1}{2} R_{\max} \sqrt{(10 \text{ m})^2 + (20 \text{ m})^2} \sin \theta \approx \boxed{12.0 \text{ m}^2}$$

Setter: Lv Jindong, jindong.lv@sgphysicsleague.org

Problem 33: Anamorphic SPhL

(4 points)

Anamorphosis is a perspective technique where the distortion in the object disappears when reflected off a curved mirror surface. The image below shows the distorted SPhL logo placed beneath a perfectly reflective cone.



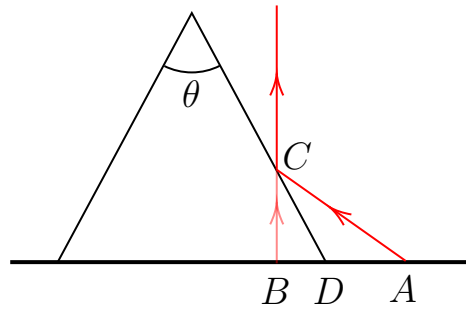
When the cone is viewed directly from above, and from a sufficiently far distance away, the resultant image formed by the reflection off the cone looks perfectly undistorted (the black circle denotes the outer edge of the base of the cone). By making suitable measurements on the diagram below, find the apex angle of the cone θ .



Leave your answer to 2 significant figures in units of degree.

Solution:

Let us consider the path of a light ray that travels from a point on the distorted image A to the eyes of an observer looking at the cone directly from above.



The light reflects off the surface of the cone at point C , while the observer thinks that the light is coming from point B . Notice that $\angle BCD = \angle DCA = \frac{\theta}{2}$. Let us consider $\triangle ABC$ and $\triangle BCD$ which share the same height BC . We have

$$\tan \frac{\theta}{2} = \frac{BD}{BC}, \quad \tan \theta = \frac{BA}{BC} \implies \frac{\tan \frac{\theta}{2}}{\tan \theta} = \frac{BD}{BA}$$

We can easily measure BD and BA from the diagram by choosing matching points in the original (distorted) and reflected images. For example, we can choose the set of points below, giving us $\theta \approx \boxed{40^\circ}$. Answers that are off by a degree will also be accepted.



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Problem 34: Flow State

(4 points)

A cylindrical tube of infinite height with radius $r = 1.00$ m has a tiny leaking hole at height $z = 5.00$ m above bottom of area $A = 0.100$ m². At its bottom end, water is pumped in at a constant gauge pressure $P = 1800$ kPa.

Find h , the equilibrium water level in the pipe. Neglect the effects of turbulence, and assume that the open top of the cylindrical tube and the leaking hole are exposed to atmospheric pressure.

Leave your answer to 3 significant figures in units of m.

Solution: We may apply Bernoulli's Principle:

$$\frac{v^2}{2} + gz + \frac{P}{\rho} = \text{constant}$$

Let v_x be the flow speed and A_x be the cross-sectional area at some point x . Let ρ_w be the density of water and P_{atm} be atmospheric pressure.

Let us consider three points: (1) at the bottom of the pipe, (2) at the tiny leaking hole and (3) at the highest point of the water. First, comparing points (1) and (2)

$$\frac{v_1^2}{2} + \frac{P}{\rho_w} = \frac{v_2^2}{2} + gz$$

Bearing in mind that at the equilibrium, all the water pumped in must be leaving through the hole,

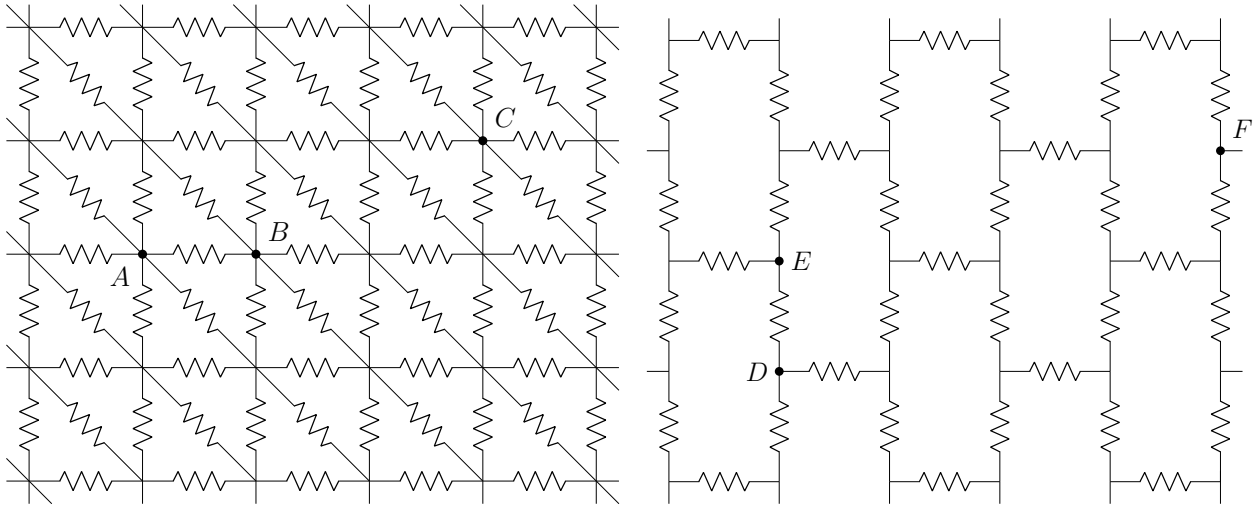
$$\begin{aligned} v_1 A_1 &= v_2 A_2 \Rightarrow v_1 = \frac{v_2 A}{\pi r^2} \\ \frac{A^2 v_2^2}{2\pi^2 r^4} + \frac{P}{\rho_w} &= \frac{v_2^2}{2} + gz \\ v_2^2 \left(\frac{A^2}{2\pi^2 r^4} - \frac{1}{2} \right) &= gz - \frac{P}{\rho_w} \\ \Rightarrow v_2^2 &= \frac{2\pi^2 r^4}{A^2 - \pi^2 r^4} \left(gz - \frac{P}{\rho_w} \right) \end{aligned}$$

Now, comparing points (2) and (3)

$$\begin{aligned} \frac{v_2^2}{2} + gz + \frac{P_{\text{atm}}}{\rho_w} &= gh + \frac{P_{\text{atm}}}{\rho_w} \\ \Rightarrow h &= \frac{v_2^2}{2g} + z \approx \boxed{183 \text{ m}} \end{aligned}$$

Problem 35: Resistor Grids

Two infinite resistor grids made of identical resistors are as shown below. Let R_{XY} denote the effective resistance between nodes X and Y that are on the same grid.



- (a) Determine $\frac{R_{AB}}{R_{DE}}$.

Leave your answer to 2 significant figures.

(2 points)

- (b) Determine $\frac{R_{AC}}{R_{DF}}$.

Leave your answer to 2 significant figures.

(4 points)

Solution:

- (a) Consider injecting current I into node A in the left grid. Due to symmetry, the current splits evenly to its six neighbours, so a current of $\frac{I}{6}$ flows from A to B . Similarly, if we consider extracting current I from node B , a current of $\frac{I}{6}$ also flows from A to B . Hence, by superposition, if we inject current I into node A and extract current I from node B , a current of $\frac{2I}{6} = \frac{I}{3}$ flows from A to B . Thus, by equating the potential difference between nodes A and B obtained by directly considering just the resistor connecting the two nodes and by considering the whole grid as an effective resistor,

$$\frac{IR}{3} = IR_{AB} \quad \Rightarrow \quad R_{AB} = \frac{R}{3}$$

By the same logic, injecting current I into node D in the right grid, the current splits evenly to its three neighbours, so a current of $\frac{I}{3}$ flows from D to E . Then injecting current I into D and extracting current I from E causes current $\frac{2I}{3}$ to

flow from D to E , so

$$\frac{2IR}{3} = IR_{DE} \implies R_{DE} = \frac{2R}{3}$$

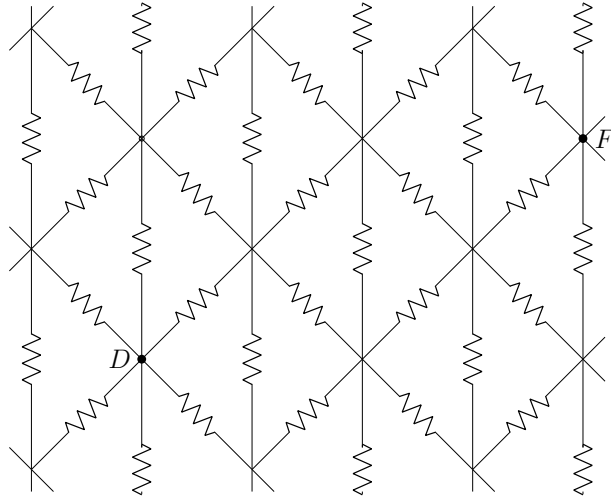
Therefore,

$$\frac{R_{AB}}{R_{DE}} = \frac{\frac{R}{3}}{\frac{2R}{3}} = \frac{1}{2} = \boxed{0.50}$$

- (b) Note that the right grid can be thought of as consisting of tessellated three-resistor stars (in the shape of a “T” rotated clockwise by 90°). Using the Y- Δ transformation, we can transform each star into a triangle made of resistors of resistance

$$R' = \frac{R \cdot R + R \cdot R + R \cdot R}{R} = 3R$$

Hence, we can transform the entire right grid as shown, where each resistor has resistance $3R$:



Now, this is structurally equivalent to the left grid, with the relative positions of nodes D and F in this new grid corresponding to those of nodes A and C in the left grid! The only difference is that the resistors in this new grid are of resistance $3R$ instead of R , so we must have $R_{DF} = 3R_{AC}$, i.e.

$$\frac{R_{AC}}{R_{DF}} = \frac{1}{3} = \boxed{0.33}$$

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Problem 36: Simp 3

(4 points)

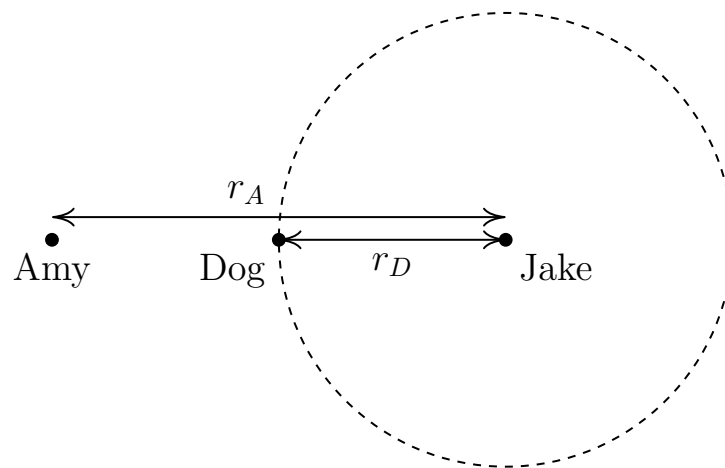
Unfortunately, Amy's clone was caught by the dog. Unfazed, Amy decides to play another round of the game with Jake and his dog, with the same rules as before:

- A “catch” occurs when two characters occupy the same position on the plane.
- If Amy catches Jake before his dog catches her, she wins.
- If Jake's dog catches Amy before she catches Jake, Jake and his dog win.

After what happened to Amy's clone, Jake has decided to keep his dog on a rigid leash. Meanwhile, Amy concludes that planning is the key to success, and commits to a fixed route she will follow regardless of the dog's movements. Unfortunately for her, Jake's dog manages to access her plan before the game begins. The game is therefore governed by the following additional rules:

- Jake remains stationary.
- Jake's dog must remain a fixed distance r_D from him, while Amy can move anywhere in the plane.
- Amy plans her path to catch Jake before the game starts and follows it regardless of the dog's movements.
- Jake's dog knows Amy's planned path before the game starts and can anticipate her movements.

Amy, Jake, and his dog begin the game in a line, with the dog between Amy and Jake. The starting distance between Amy and Jake is r_A . Amy and the dog's top speeds are v_A and v_D respectively, and they can change their velocities instantaneously.

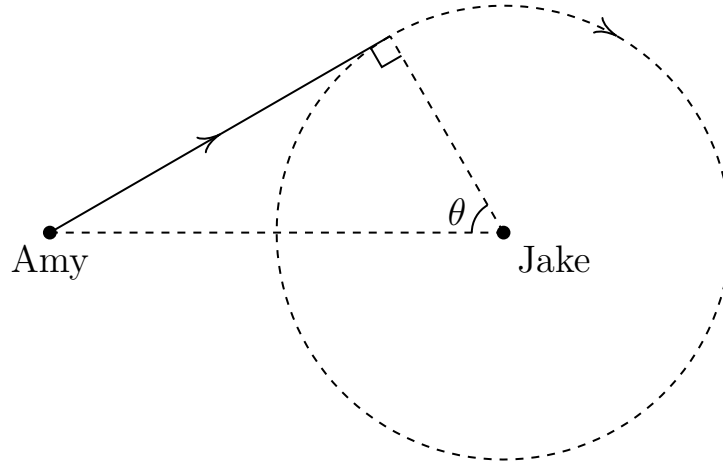


They start the game with $r_A/r_D = 2$. Assuming all characters play optimally, Amy wins if $v_A/v_D > \chi$. Determine χ .

Leave your answer to 3 significant figures.

Solution:

Notice that if Amy manages to cross the circle of radius r_D , she wins - therefore, her goal is to reach a point on the circle that the dog is unable to get to in time. Thus, her optimal plan is as follows



Where she first runs along a tangent to the circle, then circles an infinitesimal distance outside r_D , and finally crosses the circle at the point diametrically opposite the dog's starting position.

From the diagram,

$$\cos \theta = \frac{r_D}{r_A},$$

and the distance that Amy has to run is

$$d_A = \sqrt{r_A^2 - r_D^2} + r_D (\pi - \theta).$$

As such, the time that Amy takes to reach the crossing point is (after substituting in the expression for θ)

$$t_A = \frac{1}{v_A} \left[\sqrt{r_A^2 - r_D^2} + r_D \left(\pi - \cos^{-1} \frac{r_D}{r_A} \right) \right].$$

The time that the dog takes to reach the crossing point is simply

$$t_D = \frac{\pi r_D}{v_D}.$$

The condition for Amy to be able to reach the crossing point before the dog is $t_A < t_D$, which becomes

$$\frac{1}{v_A} \left[\sqrt{r_A^2 - r_D^2} + r_D \left(\pi - \cos^{-1} \frac{r_D}{r_A} \right) \right] < \frac{\pi r_D}{v_D}.$$

Rearranging and simplifying,

$$\frac{v_A}{v_D} > \frac{1}{\pi} \left[\sqrt{\left(\frac{r_A}{r_D}\right)^2 - 1} + \left(\pi - \cos^{-1} \frac{r_D}{r_A}\right) \right].$$

Substituting in the value for r_A/r_D , we arrive at our answer

$$\frac{v_A}{v_D} > \chi = \boxed{1.22}$$

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Problem 37: Hot Creep

(4 points)

While taking his customary midday rooftop siesta, Jed left his laptop in the sun and forgot to retrieve it. As the laptop warms during the day and cools at night, it repeatedly expands and contracts, causing it to slowly creep down the roof.

His roof is sloped at an angle $\theta = 20^\circ$ above the horizontal. You may treat the laptop as a square aluminium plate with side length $a = 30$ cm, linear thermal expansion coefficient $\alpha = 2.4 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$, and coefficient of friction with the roof $\mu = 0.9$. The difference in temperature between day and night is $\Delta T = 10^\circ\text{C}$.

After one year of neglect, how far down the slope has the laptop moved?

Leave your answer to 3 significant figures in units of m.

Solution: During each expansion, there is a line of points on the plate that remain stationary, with the points above moving upwards and the points below moving downward. Let the distance from the bottom of the laptop to this line be l . As the laptop does not accelerate during this process, we may write down the equilibrium force balance:

$$\begin{aligned} f_{\text{below}} - f_{\text{above}} &= mg \sin \theta \\ \mu gl \cos \theta - \mu g(a - l) \cos \theta &= ga \sin \theta \\ l &= \frac{a(\tan \theta + \mu)}{2\mu} \end{aligned}$$

Equivalently, during contraction, the stationary line is l from the top of the plate. Consider the bottom of the laptop. During each cycle, it moves downwards by

$$\begin{aligned} \delta &= l\alpha\Delta T - (a - l)\alpha\Delta T \\ &= \frac{a\alpha\Delta T \tan \theta}{\mu} \end{aligned}$$

Therefore, over a year, the total displacement is

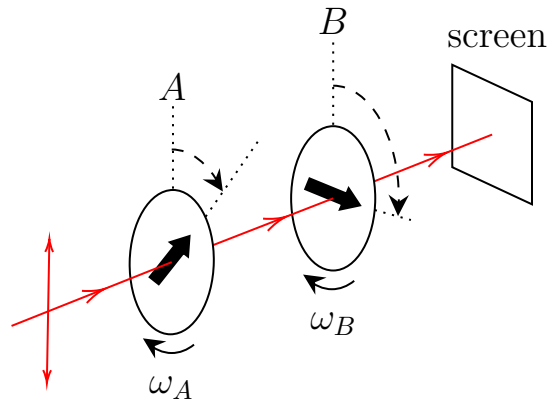
$$365\delta = \boxed{0.0106 \text{ m}}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Problem 38: Spin-2?

(4 points)

A beam of linearly polarised light travels towards 2 ideal linear polarisers A and B , arranged in a row. Initially, both polarisers are oriented upright, in the same direction as the light polarisation. Shanay decides to spin the two polarisers with constant angular velocities ω_A and ω_B respectively, starting at the same time. Both polarisers are spinning in the same direction.



Shanay observes [this](#) on his screen. Unfortunately, Shanay did not start recording immediately after spinning the polarisers, but he has managed to record a perfect loop¹ that is 24.00 s long. Let the periods of rotation of the polarisers be T_A and T_B . If $T_A > T_B$, determine $T_A - T_B$.

Leave your answer to 3 significant figures in units of s.

Solution: Since we know the video is a perfect loop, T_A/T_B has to be rational. The intensity of light on the screen is:

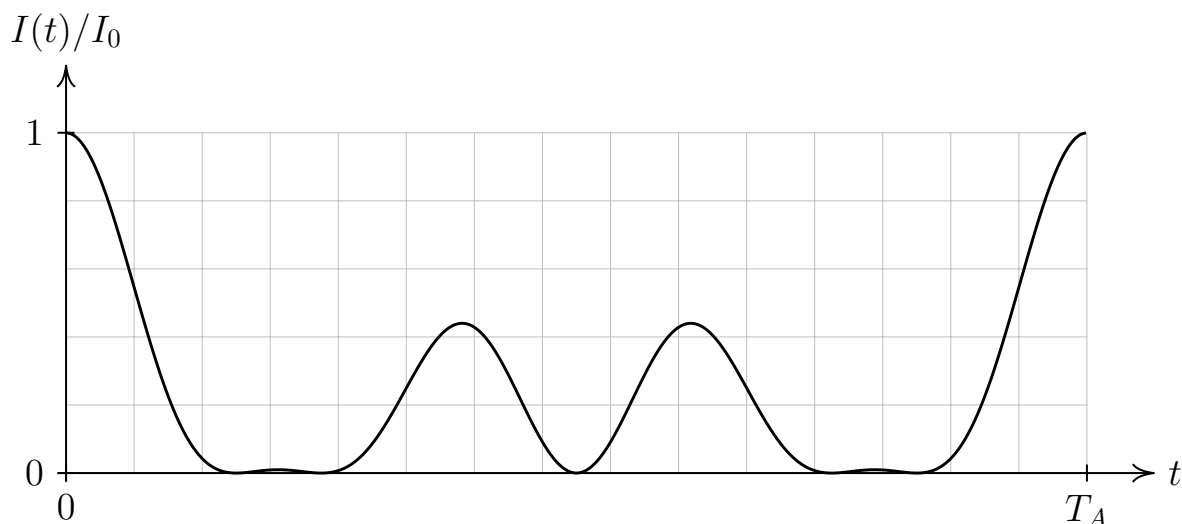
$$I(t) = I_0 \cos^2(\omega_A t) \cos^2((\omega_B - \omega_A)t)$$

One possible thing we could do is to measure the brightness (e.g. using a colour picker tool) at each time, plot a graph and then calculate the values of T_A and T_B . However, there is a much simpler approach we can do by applying some basic heuristics.

The video has 2 smaller maxima and 1 large maxima per loop, with the 2 smaller maxima being approximately equal in brightness. With these few maxima, the ratios T_A/T_B and $\omega_A/|\omega_B - \omega_A|$ are likely to be fairly simple ratios. As such, we can just try different possible ratios of $\omega_A/|\omega_B - \omega_A|$ to try and match up the intensities that we have observed.

After some trial-and-error, the best fit for the intensities observed is a 3:2 or 2:3 ratio of ω_A and $|\omega_B - \omega_A|$. The graph below shows the time-variation of the intensity (the label T_A is for the 2:3 case, which we will soon see is the correct solution).

¹The frame that immediately follows the last frame is the first frame. Since this video is at 50 frames per second, we have $t = 23.98$ s on the last frame.



The 2 smaller maxima we observed do seem to be about 44% of the brightness of the large maxima, and they occur at the correct times as well. There are also 2 very small maxima that are difficult to notice without using the provided intensity values or external tools to perform colour analysis.

If $\omega_A : |\omega_B - \omega_A| = 3 : 2$, then $T_A : T_B = 1 : 3$, while if $\omega_A : |\omega_B - \omega_A| = 2 : 3$, then $T_A : T_B = 5 : 2$. Since $T_A > T_B$, $T_A : T_B = 5 : 2$ is the correct solution.

Since the video is 24.00 s long, we must have $T_A = 24.00$ s and $T_B = 9.600$ s. Hence,

$$T_A - T_B = \boxed{14.4 \text{ s}}$$

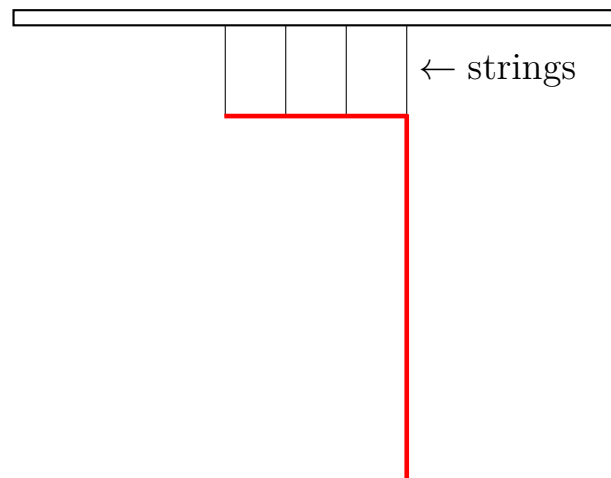
Setter: Sun Yu Chieh, yuchieh.sun@sgphysicsleague.org

Problem 39: Hanging An L

(5 points)

A rigid structure in the shape of an “L” of total mass $M = 1.70$ kg consists of two rods of identical linear mass densities, one twice the length of the other, joined perpendicularly at their ends. Four identical strings hang vertically from the ceiling and are tied to the shorter rod at equidistant positions. Determine the tension T in the rightmost string.

Hint: You may assume each string acts as an extremely stiff Hookean spring when taut and exerts no force when slack.



Leave your answer to 3 significant figures in units of N.

Solution: Both rods have the same linear mass density and the longer rod is twice the length of the shorter one, so the shorter rod has a mass of $\frac{M}{3}$.

Suppose all strings are taut. Since the strings exert linear forces, are equidistant and the rod is straight, the difference between the forces exerted by adjacent strings is the same, which we let be F . Then, the tensions of the strings from left to right are $T - 3F, T - 2F, T - F, T$ respectively. Balancing forces gives

$$T - 3F + T - 2F + T - F + T = Mg \quad \implies \quad 4T - 6F = Mg$$

Letting the length of the rod be L and balancing torques about the corner gives

$$(T - 3F)L + (T - 2F)\frac{2L}{3} + (T - F)\frac{L}{3} = \frac{Mg}{3} \cdot \frac{L}{2} \quad \implies \quad 2T - \frac{14}{3}F = \frac{1}{6}Mg$$

Solving, we get $T = \frac{11}{20}Mg$ and $F = \frac{1}{5}Mg$. However, this implies that the minimum tension, occurring at the leftmost string, is $T - 3F = -\frac{1}{20}Mg < 0$. This is a contradiction as the string is taut and so can only exert an upward force. Hence, we must have that the leftmost string is actually slack.

Now, the tensions on the strings from left to right are $0, T - 2F, T - F, T$ respectively. Balancing forces gives

$$T - 2F + T - F + T = Mg \quad \implies \quad 3T - 3F = Mg$$

and balancing torques about the corner gives

$$(T - 2F)\frac{2L}{3} + (T - F)\frac{L}{3} = \frac{Mg}{3} \cdot \frac{L}{2} \quad \implies \quad T - \frac{5}{3}F = \frac{1}{6}Mg$$

Solving again, we get $T = \frac{7}{12}Mg$ and $F = \frac{1}{4}Mg$. This time, the minimum tension is $T - 2F = \frac{1}{12}Mg > 0$, so all strings are indeed taut. Therefore, the maximum tension is

$$T = \frac{7}{12}Mg = \boxed{9.73 \text{ N}}$$

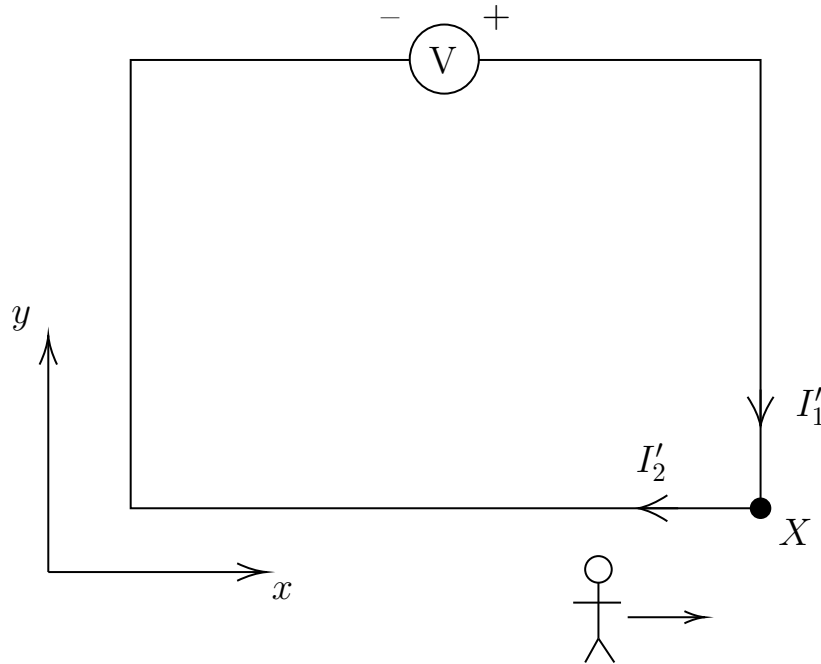
Remark. Interestingly, if we cut the two strings on the left, balancing forces and torques again gives us that the tension in both strings on the right is $\frac{1}{2}Mg$, which is lower than if both strings were uncut! This seemingly goes against intuition as cutting a string in parallel with another should increase the tension in the uncut string. This makes sense if we instead think about what happens when we reintroduce the second string from the left: the right two strings are already extended, so by introducing the third string that is initially at rest length, it necessarily has to pull on the left side of the structure upwards and rotate it slightly clockwise. But now the middle string carries a tension of $\frac{1}{3}Mg$ instead of $\frac{1}{2}Mg$ due to the extra string, and this difference is great enough (more than the tension in the reintroduced string) to increase the tension in the rightmost string. For another example of a counter-intuitive response to cutting a taut string, see [this video](#) on a physical analogy to Braess' paradox.

Setter: Oon Han, oon.han@sgphysicsleague.org

Problem 40: I of the Beholder

(4 points)

Sid has built a simple circuit, consisting of a voltage source $V = 5.00 \text{ V}$ connected to a resistive wire of resistance $R = 1.20 \Omega$. He checks that there is no current leakage and is satisfied with his design. However, Aaron, on his morning jog at $v = 0.7c$ past the circuit, notices something strange - he observes that the current I'_1 entering point X and I'_2 exiting point X are not the same! Compute this discrepancy, $I'_2 - I'_1$.



Leave your answer to 3 significant figures in units of A.

Hint: Charge density ρ and current density $\mathbf{j}$ (current per unit area) transform analogously to t and $\mathbf{x}$ respectively under a change in reference frame.

Solution: From the hint, we have

$$\begin{aligned}\rho' &= \gamma \left(\rho - \frac{v j_x}{c^2} \right) \\ j'_x &= \gamma (j_x - v \rho) \\ j'_y &= j_y \\ j'_z &= j_z\end{aligned}$$

In the unprimed frame, we have some j_x along the horizontal wire and j_y along the vertical wire, with $\rho = 0$ everywhere. In the primed frame, we can write down

$$\begin{aligned}j'_x &= \gamma j_x \\ j'_y &= j_y\end{aligned}$$

Hence the current density in the horizontal wires is increased by a factor γ , while that in the vertical wires is unchanged. However, the cross-sectional areas A_x and A_y are also transformed. Due to length contraction, we have

$$\begin{aligned} A'_x &= A_x \\ A'_y &= \frac{A_y}{\gamma} \end{aligned}$$

Therefore

$$\begin{aligned} I'_1 &= A'_y j'_y = \frac{I}{\gamma} \\ I'_2 &= A'_x j'_x = \gamma I \\ I'_2 - I'_1 &= I \left(\gamma - \frac{1}{\gamma} \right) \\ &= \frac{V}{R} \left(\gamma - \frac{1}{\gamma} \right) \\ &= \boxed{2.86 \text{ A}} \end{aligned}$$

Alternatively, one can also transform the resistances and find that

$$\begin{aligned} R'_y &= \gamma^2 R_y \\ R'_x &= \frac{R_y}{\gamma^2} \end{aligned}$$

However, this method requires some caution, as the voltages ($E' \cdot l'$) across each section are also transformed.

One may rightly ask where this discrepancy in current is coming from. In fact, it arises from a non-zero charge density in the x -wire. Specifically,

$$\rho' = \frac{\gamma v j_x}{c^2}$$

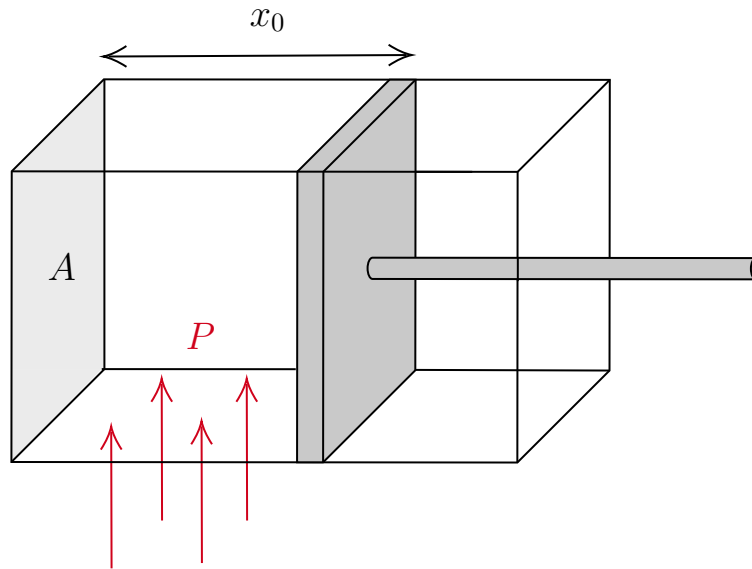
Consider a point fixed in space that intersects with point X at $t = 0$. The charge density measured at that point goes from ρ' to 0 at $t = 0$. This implies an additional “convective current” associated with the motion of the wire

$$\begin{aligned} I &= -v A_x \frac{\gamma v j_x}{c^2} \\ &= -I \gamma \frac{v^2}{c^2} \\ &= -I \left(\gamma - \frac{1}{\gamma} \right) \end{aligned}$$

which agrees with our previous result.

Problem 41: Full of Hot Air

A cuboid tank with a cross-sectional area A is placed horizontally on a hot plate and filled with an ideal monatomic gas. The tank is sealed by a frictionless piston of mass m located at an initial distance x_0 from the base of the tank. At this stage, the gas is at the same temperature and pressure as the surroundings. All faces of the tank are insulating except the face in contact with the hot plate. Let the external pressure be p_0 .



At $t = 0$, the hot plate is turned on and the gas is heated with power P . As a result, the gas expands and the piston moves to the right. It is found that the time t for the piston to move a small distance $\Delta x \ll x_0$ scales as $t \propto \Delta x^n$.

- (a) Determine the value of n in the limit $P^2 \ll \frac{(p_0 A)^3 x_0}{m}$.

Leave your answer to 2 significant figures.

(2 points)

- (b) Determine the value of n in the limit $P^2 \gg \frac{(p_0 A)^3 x_0}{m}$.

Leave your answer to 2 significant figures.

(3 points)

Solution: As the system is heated, the gas needs to occupy a larger volume to remain at atmospheric pressure. As such, the equilibrium position of the piston increases. However, the difficulty of this problem (specifically, in part 2) lies in the fact that the piston has finite mass, and hence, does not reach its equilibrium position instantly. In fact, this system is governed by two competing timescales: the timescale of the piston's oscillation τ_{osc} and the timescale of the gas's heating τ_{heat} . The two limits in the problem are the limits $\tau_{osc} \ll \tau_{heat}$ and $\tau_{osc} \gg \tau_{heat}$ respectively.

- (a) In this limit, the gas is heated very slowly and the piston remains at equilibrium

for the entire process, which is essentially isobaric. Hence

$$P = n c_p \frac{dT}{dt} = \frac{5}{2} n R \frac{dT}{dt} = \frac{5}{2} p_0 \frac{dV}{dt} = \frac{5}{2} p_0 A \dot{x}$$

i.e. the piston moves at constant speed,

$$x(t) = x_0 + \frac{2Pt}{5p_0A}$$

Hence $n = \boxed{1.0}$.

- (b) In this limit, the gas is heated extremely quickly and reaches high pressures before the piston is able to move by a significant amount. Now, the heating is approximately isochoric,

$$P = n c_v \frac{dT}{dt} = \frac{3}{2} n R \frac{dT}{dt} = \frac{3}{2} \frac{dp}{dt} A x_0.$$

$$\implies p(t) \approx p_0 + t \frac{dp}{dt} = p_0 + \frac{2Pt}{3Ax_0}$$

Hence, Newton's 2nd Law for the piston reads

$$m\ddot{x} = (p(t) - p_0)A = \frac{2Pt}{3x_0}$$

Integrating twice, we find that

$$x(t) \propto t^3$$

or in other words,

$$t \propto x^{1/3} \implies n = \frac{1}{3} \approx \boxed{0.33}$$

Note: Although the above solutions have only been justified heuristically, they produce the correct values of n in their respective limits. To obtain the same answers more rigorously, we can solve the system's equations of motion directly. They read

$$m\ddot{x} = (p - p_0)A,$$

$$\frac{dQ}{dt} + \frac{dW}{dt} = \frac{3}{2} \frac{d(pV)}{dt} \implies P - pA\dot{x} = \frac{3}{2} A(\dot{p}x + \dot{x}p)$$

with the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = 0, \quad p(0) = p_0$$

In the slow-heating limit, we substitute the ansatz

$$x(t) = x_0 + vt, \quad p(t) = p_0$$

which solves the equations of motion and fulfils all the initial conditions with vanishing error in the limit specified. However, this ansatz is inappropriate in the fast-heating limit as the error does not vanish. Instead, for the fast-heating limit, we write

$$x(t) = x_0 + \alpha t^3, \quad p(t) = p_0 + \beta t$$

which solves the system provided that

$$\alpha = P/9mx_0, \quad \beta = 2P/3Ax_0.$$

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Problem 42: Love Triangle

(4 points)

Luc, a perennial egotist, has decided to enter a *ménage à trois* with 2 clones of himself.

They start off in space, at the vertices of an equilateral triangle with circumradius $r = 100$ m. The triangle is slowly rotating at angular velocity $\omega = 1.00 \times 10^{-14} \text{ s}^{-1}$. At $t = 0$ s, they impulsively move towards their rendezvous, *adding* a velocity $v = 12.0 \text{ m s}^{-1}$ towards the centre to their current velocity. All Lucs gravitationally interact with each other.

Unfortunately, as Luc is severely underweight, weighing only $m = 50.0$ kg, his gravitational interactions are not strong enough – so everyone ends up flying into the void. After a long time, what is the angle between each Luc's initial (shortly after $t = 0$ s) and final velocity?

Leave your answer to 3 significant figures in units of degrees.

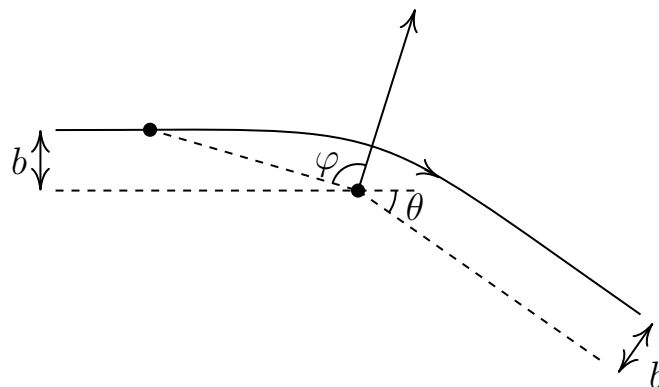
Solution: Due to the symmetry of the set-up, the masses remain in an equilateral triangle at all times, which rotates and scales. Without loss of generality, consider a Luc. The net gravitational force acting on him is simply

$$F_g = \frac{2Gm^2}{3r^2} \cos \frac{\pi}{6} = \frac{Gm^2}{\sqrt{3}r^2}$$

pointed towards the centre of the triangle. As this holds for any r , each Luc is essentially undergoing scattering due to a central force. Since he experiences almost no gravitational influence at the beginning, we can write down his impact parameter

$$b \approx \frac{\omega r^2}{v}$$

The scattering angle is then given by the well-known Rutherford formula. We briefly reproduce the derivation below.



Defining

$$\alpha = \frac{Gm^2}{\sqrt{3}}$$

and noting that the change in momentum Δp lies along the symmetry plane, we have

$$\Delta p = 2p \sin \frac{\theta}{2} = \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos \varphi \, dt$$

From conservation of angular momentum,

$$\begin{aligned} pb &= mr^2 \dot{\varphi} \\ \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos \varphi \, dt &= \int_{\frac{1}{2}(\pi-\theta)}^{\frac{1}{2}(\pi+\theta)} \frac{\alpha m}{pb} \cos \varphi \, d\varphi \\ &= \frac{2\alpha m}{pb} \cos \frac{\theta}{2} \end{aligned}$$

Thus we can find the scattering angle

$$\begin{aligned} \tan \frac{\theta}{2} &= \frac{\alpha m}{pb} \\ \theta &= 2 \tan^{-1} \left(\frac{Gm}{\sqrt{3}r^2 v \omega} \right) \\ &= \boxed{116^\circ} \end{aligned}$$

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Problem 43: Strategically Accelerating Frames

(4 points)

Aaron is currently training hard for his 2.4 km run, aiming to pass his IPPT and get a 2 month reduction to his national service. However, Sid has a better idea - by running to point P and back over the course of 2 years, he can simply make those 2 years of NS feel like only 1 year and 10 months!

Sid has a top speed of $v = 0.6c$. Assume that he runs in a straight line at constant velocity on each leg of his journey and can only change his velocity once, at point P. As Sid is worried he might be breaking confinement, he wishes to stay as close to Singapore as possible. Determine x_P , the minimum distance of point P from the origin, in light-years.

Leave your answers to 3 significant figures in units of ly.

Solution: An observer on earth sees Sid's time dilated by the Lorentz factor γ - hence, a small time dt on earth corresponds to a time dt/γ to Sid.

Intuitively, the optimal solution is for him to run one way at his top speed of v , and the other way at u , the minimum speed possible. Let us divide all units by c for convenience and use units of years for time. Denoting the time spent at v and u as t_1 and t_2 respectively, we have

$$vt_1 = ut_2 \quad (1)$$

$$\sqrt{1-v^2}t_1 + \sqrt{1-u^2}t_2 = \frac{11}{12}(t_1 + t_2) \quad (2)$$

$$(t_1 + t_2) = 2 \quad (3)$$

This is best most easily solved numerically, or with a symbolic solver like SymPy. However, we can carefully work through the algebra. Defining

$$k = \sqrt{1-v^2}$$

We can write down

$$t_1 = \frac{x_P}{v}$$

$$t_2 = 2 - \frac{x_P}{v}$$

$$u = \frac{vt_1}{t_2} = \frac{x_P v}{2v - x_P}$$

Substituting into (2):

$$k \frac{x_P}{v} + \sqrt{1-u^2} \left(2 - \frac{x_P}{v} \right) = \frac{11}{6}$$

$$\begin{aligned}
(1 - u^2) \left(2 - \frac{x_P}{v}\right)^2 &= \left(\frac{11}{6} - k \frac{x_P}{v}\right)^2 \\
\left(1 - \frac{x_P^2 v^2}{(2v - x_P)^2}\right) \frac{(2v - x_P)^2}{v^2} &= \frac{121}{36} - \frac{11k}{3} \frac{x_P}{v} + \frac{k^2 x_P^2}{v^2} \\
\frac{4v^2 - 4vx_P + x_P^2 k^2}{v^2} &= \frac{121}{36} - \frac{11k}{3} \frac{x_P}{v} + \frac{k^2 x_P^2}{v^2} \\
4v^2 - 4vx_P &= \frac{121}{36} v^2 - \frac{11k}{3} vx_P \\
\frac{23}{36} v^2 + \frac{11k - 12}{3} vx_P &= 0 \\
x_P &= \frac{23v}{12(12 - 11k)}.
\end{aligned}$$

Thus we have

$$x_P = \frac{23v}{12(12 - 11\sqrt{1 - v^2})} = \boxed{0.359 \text{ ly}}$$

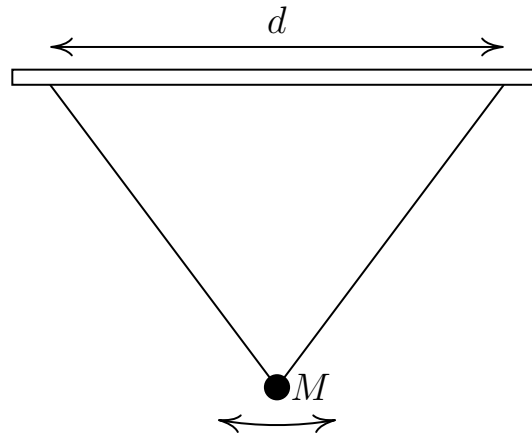
Unfortunately for Sid, this is probably outside of Singapore's borders.

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Problem 44: Rock and Roll

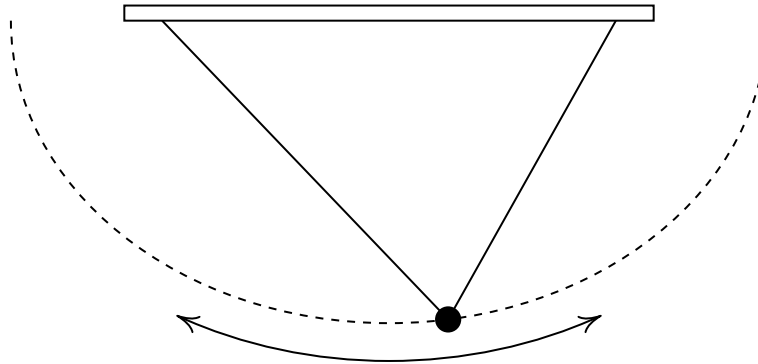
(4 points)

The ends of a massless string of length $\ell = 0.250$ m are fixed to a ceiling at a distance $d = 0.150$ m from each other. A bead of mass M is threaded through the string and positioned at equilibrium. The bead is then given a small displacement from equilibrium along the string and oscillates without friction. Determine the period T of oscillations.



Leave your answer to 3 significant figures in units of s.

Solution: Since the string is massless, it is taut on both sides, resulting in two straight segments. Since the length of the string is constant, the trajectory of the bead traces an ellipse with the fixed ends of the string as foci.



Setting the origin to be at the middle of the two fixed ends of the string with the x -axis pointing horizontally, the equation for the ellipse is

$$\frac{4x^2}{\ell^2} + \frac{4y^2}{\ell^2 - d^2} = 1$$

which can be deduced by considering the position of the bead when its on the x - and y -axis. Implicitly differentiating,

$$\frac{8x}{\ell^2} + \frac{8y}{\ell^2 - d^2} y' = 0 \quad \implies \quad y' = -\frac{x(\ell^2 - d^2)}{\ell^2 y}$$

Differentiating again,

$$y'' = -\frac{(\ell^2 - d^2)(y - xy')}{\ell^2 y^2}$$

At the equilibrium point, $x = 0$, $y = -\frac{1}{2}\sqrt{\ell^2 - d^2}$ and $y' = 0$, so

$$y''(0) = -\frac{-(\ell^2 - d^2)}{\ell^2 \left(\frac{1}{2}\sqrt{\ell^2 - d^2}\right)} = \frac{2\sqrt{\ell^2 - d^2}}{\ell^2}$$

Now, the gravitational potential energy of the bead is $U = Mgy$, so the frequency of oscillations is

$$\omega = \sqrt{\frac{U''(0)}{M}} = \sqrt{gy''(0)} = \sqrt{\frac{2g\sqrt{\ell^2 - d^2}}{\ell^2}}$$

Therefore, the period of oscillation is

$$T = 2\pi\sqrt{\frac{\ell^2}{2g\sqrt{\ell^2 - d^2}}} = \boxed{0.793 \text{ s}}$$

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Problem 45: A Light Round of Ping Pong

(5 points)

Monochromatic light of wavelength $\lambda = 600$ nm and total energy $E = 20$ J is trapped in a one-dimensional box of length $L = 1.00$ m. The two ends of the box are mirrors with mass $m = 0.10$ g and reflectance $R = 0.67$. What is the speed of each mirror after the energy of the light has halved?

Hint: The reflectance of a mirror is the ratio of the intensity of light it reflects to the intensity of light shone on it. Light not reflected is absorbed.

Leave your answer to 2 significant figures in the units of mm s^{-1} .

Solution:

We model the light as a photon gas, where each photon has energy $E_\gamma = hc/\lambda$. Therefore, the light is comprised of

$$N_0 = E/E_\gamma = 6.03 \times 10^{19} \text{ photons}$$

Before proceeding to the solution, we first make a few rough estimates. Suppose all the energy in the light is converted to the kinetic energy of one mirror. That mirror will have a speed

$$\frac{1}{2}mv^2 = E \implies v = \sqrt{\frac{2E}{m}} = 630 \text{ m/s}$$

This is a bound on the maximum speed of the mirrors. Notice that it is non-relativistic. For this reason, we will neglect the redshift of the photons when they reflect off a moving mirror.

Moreover, every photon has a probability $1 - R$ of getting absorbed each time it hits a mirror. Thus, the number of times each photon hits a mirror is a geometrically distributed random variable with expectation value $1/(1 - R)$. This corresponds to a lifetime on the order of

$$\langle t \rangle \sim \frac{L}{c} \frac{1}{1 - R} = 1.0 \times 10^{-8}$$

The distance travelled by each mirror is hence around $v\langle t \rangle \approx 6.4 \times 10^{-6}$ m. As such, we are justified in assuming that the distance between the mirrors remains as L until all the photons have been absorbed.

Now we present the solution. In a small time interval Δt , only photons which are within $c\Delta t$ of the boundaries and travelling in the correct direction will collide with

a mirror. Amongst these photons, a fraction $1 - R$ of them will be absorbed. Hence, we can write

$$\Delta N = -(1 - R)\Delta N_{\text{collide}} = -N(1 - R)\frac{c}{L}\Delta t$$

Among the $\Delta N_{\text{collide}}$ photons that contact either mirror, a proportion $(1 - R)$ of them are absorbed, transferring momentum h/λ to their mirror, and the remaining fraction R of them are reflected, transferring momentum $2h/\lambda$. Thus, the total magnitude of momentum transferred to the mirrors is

$$\Delta p = \Delta N_{\text{collide}} \left(R\frac{2h}{\lambda} + (1 - R)\frac{h}{\lambda} \right) = N\Delta t(1 + R)\frac{c}{L}\frac{h}{\lambda}$$

where we have added momentum as a scalar: $p = |\mathbf{p}_1| + |\mathbf{p}_2|$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the vector momenta of the mirrors.

Taking the limit $\Delta t \rightarrow 0$, we obtain the following differential equations:

$$\frac{dN}{dt} = -N(1 - R)\frac{c}{L}$$

$$\frac{dp}{dt} = N(1 + R)\frac{c}{L}\frac{h}{\lambda}$$

There is a slick way to solve these ODEs. Realising that $\frac{dN}{dt}$ and $\frac{dp}{dt}$ are proportional to each other, we are motivated to evaluate

$$\frac{dp}{dt} + \frac{h}{\lambda} \frac{1 + R}{1 - R} \frac{dN}{dt} = N(1 + R)\frac{c}{L}\frac{h}{\lambda} - N(1 + R)\frac{c}{L}\frac{h}{\lambda} = 0$$

Thus,

$$\frac{d}{dt} \left(p + \frac{h}{\lambda} \frac{1 + R}{1 - R} N \right) = 0$$

We make use of this conserved momentum to find the final mirror momenta,

$$\frac{h}{\lambda} \frac{1 + R}{1 - R} N_0 = p + \frac{h}{\lambda} \frac{1 + R}{1 - R} N_f$$

where $N_f = N_0/2$. This gives a final velocity

$$v = \frac{p}{2m} = \frac{1}{2m} \frac{h}{\lambda} \frac{1+R}{1-R} (N_0 - N_f) = \boxed{0.84 \text{ mm/s}}$$

Note: A common incorrect solution claims that the mirrors gain $\frac{1}{2}(E/2) = 5 \text{ J}$ of kinetic energy each, corresponding to a speed $v = \sqrt{2E/m} = 316 \text{ m/s}$. This is incorrect because some energy is converted to heat when light is absorbed by the mirror, in a process akin to an inelastic collision. Ultimately, momentum turns out to be the more useful quantity for this question, which is why modelling the light as a photon gas is useful. *Alternative Solution:* Consider each reflection of a photon (with momentum p and energy E at a mirror. The expected momentum transferred to the mirror is

$$\delta p = 2Rp + (1 - R)p$$

while the energy lost by the photons is

$$\delta E = (1 - R)E$$

Hence, the ratio of momentum transferred to energy lost at each reflection is

$$\frac{\delta p}{\delta E} = \frac{1}{c} \frac{1+R}{1-R}$$

Therefore, the momentum transferred to each mirror is proportional to the energy lost by the light:

$$\Delta p = \frac{1}{2} \frac{\Delta E}{c} \frac{1+R}{1-R}$$

Which leads us to the same answer.

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Problem 46: Lily Pond

(4 points)

Two identical planets with mass $m = 9.0 \times 10^{25}$ kg consisting entirely of water orbit each other. Each planet moves in an elliptical orbit with eccentricity $e = 0.85$ and semi-major axis $a = 2.5 \times 10^{11}$ m. The planets are covered in white lilies, making them perfectly reflective to incoming radiation. Because of this, they do not emit any blackbody radiation into space.

After a long time, tidal forces between the planets cause their orbits to become circular. Assuming that the temperature of each planet remains uniform, find the change in temperature ΔT after the planets assume circular orbits. The heat capacity of water is $c = 4184 \text{ J kg}^{-1} \text{ K}^{-1}$.

Leave your answer to 3 significant figures in units of K.

Solution:

This system can be reduced to that of one body with reduced mass $\mu = m/2$ orbiting around a fixed body with mass $M = 2m$. During the motion, tidal forces dissipate mechanical energy and convert it to thermal energy, causing the temperature of each planet to increase. However, the angular momentum of the system is conserved.

The angular momentum and energy of a planet with mass μ orbiting a fixed body with mass M in an elliptical orbit with semimajor axis a and eccentricity e are given by

$$L = \mu \sqrt{GMa(1 - e^2)},$$

$$U = -\frac{GM\mu}{2a}.$$

Substituting $a = r$ and $e = 0$, we obtain (for a circular orbit)

$$L = \mu \sqrt{GMr},$$

$$U = -\frac{GM\mu}{2r}.$$

Equating both values of L , we find that

$$r = a(1 - e^2).$$

In the reduced system, the distance between bodies is equal to the distance between bodies in the original system. Thus, we need to multiply all distances by a factor of 2 and the total change in mechanical energy of the system is

$$\Delta U = -\frac{GM\mu}{4r} + \frac{GM\mu}{4a},$$

$$\begin{aligned} &= \frac{GM\mu}{4a} \left(1 - \frac{1}{1-e^2} \right), \\ &= -\frac{Gm^2}{4a} \frac{e^2}{1-e^2}. \end{aligned}$$

This corresponds to the thermal energy gained by the planets during the motion, so

$$2mc\Delta T = \frac{Gm^2}{4a} \frac{e^2}{1-e^2}$$

Solving for ΔT ,

$$\begin{aligned} \Delta T &= \frac{Gm}{8ac} \frac{e^2}{1-e^2}, \\ &= \boxed{1.87 \text{ K}}. \end{aligned}$$

Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem 47: 67

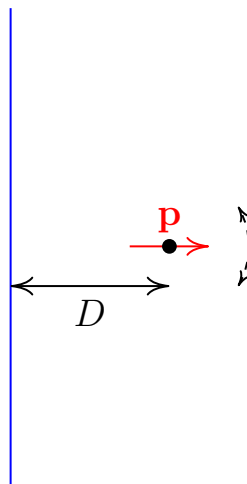
(5 points)

The 67 kid wishes to revive the 67 meme, but he has injured his arms. Determined, he instead places an electric dipole with dipole moment $p = 6.70 \times 10^{-6}$ C m and moment of inertia $I = 6.70 \times 10^{-3}$ kg m² pointing away from an infinite conducting plane at distance $D = 0.670$ m. The dipole is fixed in position but can freely rotate. The field by an electric dipole of dipole moment $\mathbf{p}$ at position vector $\mathbf{r}$ from the dipole is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{|\mathbf{r}|^3}$$

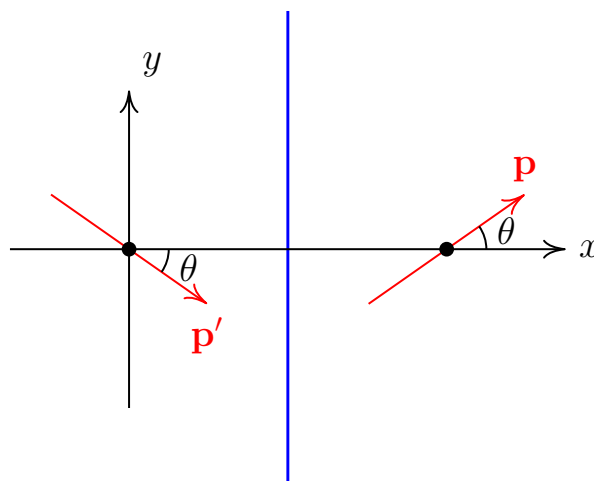
He gives the dipole a small initial angular velocity to re-enact his signature 67 gesture. Determine the period T of small oscillations of the dipole.

conducting plane



Leave your answer to 3 significant figures in units of s.

Solution: Since the conducting plane is infinite and the dipole's field goes to zero far away, it is grounded, so there is an image dipole across the conducting plane as shown below.



We adopt the coordinate system and angle convention shown above, so the position and dipole moment of the real dipole are $\mathbf{r} = 2D\hat{\mathbf{x}}$ and $\mathbf{p} = p(\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta)$ respectively. The dipole moment of the image dipole at the origin is then $\mathbf{p}' = p(\hat{\mathbf{x}} \cos \theta - \hat{\mathbf{y}} \sin \theta)$, so we can determine the field at the position of the real dipole to be

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}' \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}'}{|\mathbf{r}|^3} \\ &= \frac{1}{4\pi\epsilon_0} \frac{3p\hat{\mathbf{x}} \cos \theta - p(\hat{\mathbf{x}} \cos \theta - \hat{\mathbf{y}} \sin \theta)}{(2D)^3} \\ &= \frac{p}{32\pi\epsilon_0 D^3} (2\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta)\end{aligned}$$

Then the torque experienced by the real dipole is

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{p} \times \mathbf{E} \\ &= p(\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta) \times \frac{p}{32\pi\epsilon_0 D^3} (2\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta) \\ &= -\frac{p^2}{32\pi\epsilon_0 D^3} \hat{\mathbf{z}} \sin \theta \cos \theta \\ &= -\frac{p^2}{64\pi\epsilon_0 D^3} \hat{\mathbf{z}} \sin 2\theta\end{aligned}$$

For small oscillations, $\sin 2\theta \approx 2\theta$. Using the rotational form of Newton's second law,

$$I\ddot{\theta} = -\frac{p^2}{64\pi\epsilon_0 D^3} (2\theta) \quad \implies \quad \ddot{\theta} = -\frac{p^2}{32\pi\epsilon_0 I D^3} \theta$$

Therefore,

$$\omega = \sqrt{\frac{p^2}{32\pi\epsilon_0 I D^3}} = \frac{p}{4\sqrt{2\pi\epsilon_0 I D^3}} \quad \implies \quad T = \frac{2\pi}{\omega} = \frac{8\pi\sqrt{2\pi\epsilon_0 I D^3}}{p} = \boxed{1.26 \text{ s}}$$

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Problem 48: Drag Me Down

In clouds, raindrops grow by collecting very small droplets as they fall. These droplets are evenly distributed in the cloud with mass density $\rho_d = 1.20 \text{ g m}^{-3}$, and are small enough that they remain effectively stationary in the cloud. The raindrop experiences quadratic air drag according to the equation

$$F_D = \frac{1}{2}C_D\rho_a v^2 A,$$

where $C_D = 0.460$, $\rho_a = 1.25 \text{ kg m}^{-3}$, v is the speed of the raindrop relative to the air and A is its cross-sectional area. The raindrop has a density of $\rho_w = 1000 \text{ kg m}^{-3}$.

A raindrop with radius $r_0 = 20.0 \text{ }\mu\text{m}$ starts in the cloud with no initial velocity, then allowed to fall. Assume that the raindrop is always spherical while falling and remains in the cloud.

- (a) Immediately after its release, the magnitude of the acceleration a of the raindrop can be approximated as a quadratic function

$$a = g - \alpha t^2,$$

where a is the magnitude of the raindrop's acceleration and g is the gravitational acceleration. Find the value of α .

Leave your answer to 3 significant figures in units of m s^{-4} . (3 points)

- (b) Some time after its release, the raindrop's acceleration becomes approximately constant at β . Determine β .

Leave your answer to 3 significant figures in units of m s^{-2} . (3 points)

Solution:

The mass of the raindrop increases with time as it collects the small droplets distributed in the cloud. As such, we can write down Newton's Second Law as

$$\frac{d}{dt}(mv) = mg - \frac{1}{2}C_D\rho_a v^2 A.$$

Substituting for the mass and cross-sectional area of the raindrop in terms of its radius,

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3 \rho_w v\right) = \frac{4}{3}\pi r^3 \rho_w g - \frac{1}{2}C_D\rho_a v^2 \pi r^2.$$

Expanding the derivative and simplifying,

$$3v\dot{r} + \dot{r}v = gr - \frac{3C_D\rho_a}{8\rho_w}v^2.$$

The rate of change of the raindrop's mass is

$$\frac{d}{dt} \left(\frac{4}{3} \pi r^3 \rho_w \right) = \pi r^2 v \rho_d.$$

Simplifying this equation,

$$4\rho_w \dot{r} = \rho_d v.$$

Substituting for $\dot{r}$ in the simplified equation of motion and simplifying,

$$\dot{v} = g - \frac{3(C_d \rho_a - 2\rho_d)}{8\rho_w} \frac{v^2}{r} \equiv g - k \frac{v^2}{r},$$

where we have defined

$$k = \frac{3(C_d \rho_a + 2\rho_d)}{8\rho_w}.$$

- (a) At small times, we can expand the system perturbatively in powers of the time elapsed, using the fact that $r(0) = r_0$ and $v(0) = 0$.

To zeroth order $\dot{v} = g$. Thus, to second order in t ,

$$\dot{v} = g - k \frac{g^2 t^2}{r_0}.$$

Matching this expression with that given in the question, we find that

$$\alpha = \frac{k g^2}{r_0} = \frac{3(C_d \rho_a + 2\rho_d) g^2}{8\rho_w r_0} = \boxed{1040 \text{ m s}^{-4}}$$

- (b) We can rearrange the equation of motion to give

$$r = \frac{k v^2}{g - \dot{v}}.$$

Differentiating this equation,

$$\dot{r} = \frac{2k v \dot{v}}{g - \dot{v}} + \frac{k v^2 \ddot{v}}{(g - \dot{v})^2}.$$

Comparing this expression with that for the rate of increase of the mass of the raindrop, defining $\epsilon = \rho_d/4\rho_w$ and simplifying,

$$\frac{2k \dot{v}}{g - \dot{v}} + \frac{k v \ddot{v}}{(g - \dot{v})^2} = \epsilon.$$

Solving for $\ddot{v}$,

$$\ddot{v} = \frac{\epsilon (g - \dot{v})^2 - 2k \dot{v} (g - \dot{v})}{k v}.$$

When $\dot{v}$ is small, $\ddot{v}$ is positive, causing $\dot{v}$ to increase and the magnitude of $\ddot{v}$ to decrease. Thus, $\dot{v}$ increases with time until it reaches its limiting value at $\ddot{v} = 0$ at large times. This condition reads

$$\epsilon(g - \beta) - 2k\beta = 0.$$

where we have defined β as the limiting value of $\ddot{v}$. Solving for β ,

$$\beta = \frac{\epsilon g}{2k + \epsilon} = \frac{\rho_d g}{3C_d \rho_a + 7\rho_d} = \boxed{6.79 \times 10^{-3} \text{ m s}^{-2}}$$

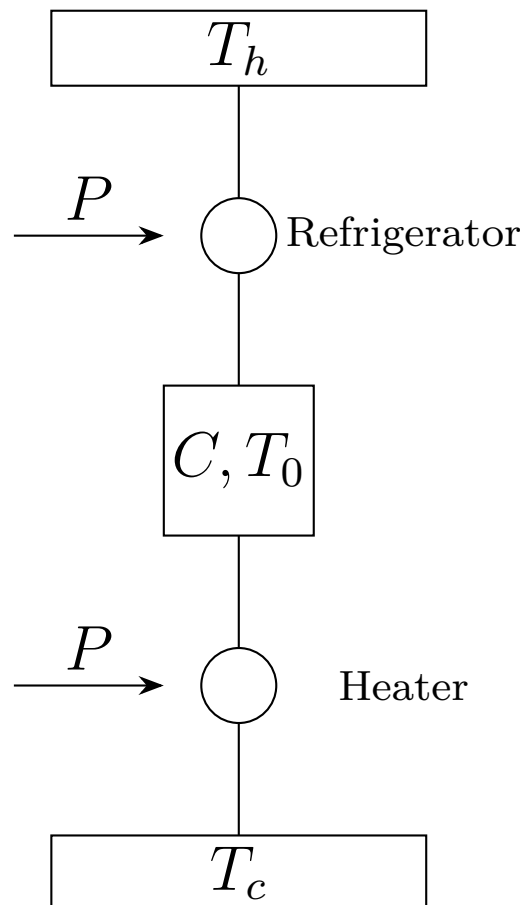
Setter: Luc Mazereeuw, luc.mazereeuw@sgphysicsleague.org

Problem 49: Would You Like it Hot or Cold?

(5 points)

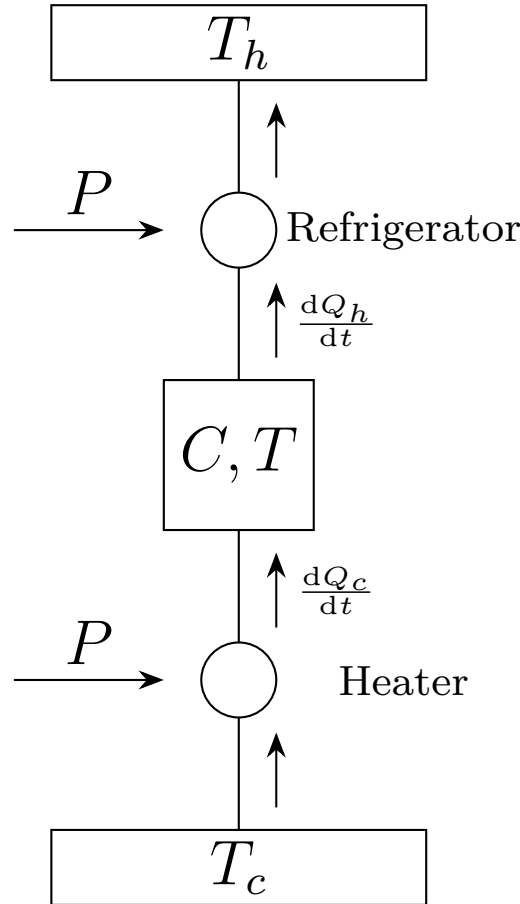
A block of heat capacity C and initial temperature $T_0 = 320$ K is connected to both a heater and a refrigerator as shown in the diagram below. The heater and refrigerator are respectively connected to a cold reservoir of temperature T_c and a hot reservoir of temperature T_h . Both have the same input power P and use reversible Carnot cycles to transfer heat from the cold reservoir to the block, or from the block to the hot reservoir.

When only the heater is turned on, it takes time t to double the block's temperature. When only the refrigerator is turned on, it takes the same time t to halve the block's temperature. If both are turned on simultaneously, what would the temperature of the block be after a long time?



Leave your answer to 3 significant figures in units of K.

Solution: First, let us study the heat flow between the reservoirs and the block:



Since ideal Carnot cycles are reversible, we require that the change in the universe's entropy be zero when either the heater or refrigerator is on. If we let the block's current temperature be T , this gives

$$-\frac{\frac{dQ_c}{dt} - P}{T_c} + \frac{\frac{dQ_c}{dt}}{T} = 0$$

$$-\frac{\frac{dQ_h}{dt}}{T} + \frac{\frac{dQ_h}{dt} + P}{T_h} = 0$$

Note that this is also the power multiplied by the coefficient of performance (COP).

Solving for $\frac{dQ_c}{dt}$ and $\frac{dQ_h}{dt}$ gives

$$\frac{dQ_c}{dt} = P \frac{T}{T - T_c}, \quad \frac{dQ_h}{dt} = P \frac{T}{T_h - T}$$

When both the heater and refrigerator are turned on, the steady-state temperature is reached when these two are equal. This means $T - T_c = T_h - T$ i.e. $T = \frac{1}{2}(T_c + T_h)$.

To find $T_H + T_c$, we will need to study the time evolution of the block's temperature when either the heater or refrigerator is turned on. When only the heater is on,

$\frac{dQ_c}{dt} = C \frac{dT}{dt}$, and substituting into the above equation gives us

$$\frac{dT}{dt} \left(1 - \frac{T_c}{T}\right) = \frac{P}{C} \implies t = \frac{C}{P} \left(T - T_0 - T_c \ln \frac{T}{T_0}\right)$$

When only the refrigerator is turned on, we have $\frac{dQ_h}{dt} = -C \frac{dT}{dt}$ since the temperature decreases, and we can integrate as follows:

$$-\frac{dT}{dt} \left(\frac{T_h}{T} - 1\right) = \frac{P}{C} \implies t = \frac{C}{P} \left(T - T_0 - T_h \ln \frac{T}{T_0}\right)$$

The heater takes time t to raise the block's temperature by T_0 , while the refrigerator takes time t to reduce it by $\frac{T_0}{2}$. Substituting into the above two equations yields

$$t = \frac{C}{P} (T_0 - T_c \ln 2) = \frac{C}{P} \left(-\frac{T_0}{2} - T_h \ln \frac{1}{2}\right)$$

$$\frac{3T_0}{2} = T_c \ln 2 - T_h \ln \frac{1}{2} = (T_c + T_h) \ln 2$$

$$\frac{1}{2} (T_c + T_h) = \frac{3}{4 \ln 2} T_0$$

This gives us a final answer of 346 K.

Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Problem 50: Ergodicity Breaking

Consider two point particles A and B somewhere strictly inside a one-dimensional frictionless chamber of length $L = 1.000$ m, enclosed by fixed walls on both ends. Particle A starts with an initial velocity of $v_0 = 1.000$ m s⁻¹, while particle B starts off stationary. Collisions between the two particles and between a particle and a wall are completely elastic. Suppose that the subsequent motion of the particles is periodic, with period T .

- (a) If $m_A = 3m_B$, find the minimum possible T .

Leave your answers to 3 significant figures in units of s. (3 points)

If you think that the motion cannot be periodic, input your answer as -1 .

- (b) If $m_A = 4m_B$, find the minimum possible T .

Leave your answers to 3 significant figures in units of s. (5 points)

If you think the motion cannot be periodic, input your answer as -1 .

Hint: A similar system is analysed in the following [video](#).

Solution: Let m_A and m_B , x_A and x_B , and v_A and v_B denote the masses, positions and velocities of the 2 particles respectively. Choose the wall closer to m_A than to m_B as the origin and the initially moving particle as the particle with mass m_A . Define y_1 , y_2 , u_A and u_B by

$$\begin{aligned} y_1 &= \sqrt{m_A} x_A & y_2 &= \sqrt{m_B} x_B \\ u_A &= \sqrt{m_A} v_A & u_B &= \sqrt{m_B} v_B \end{aligned}$$

Since all collisions are elastic, the total energy E and momentum p are conserved:

$$m_A v_A^2 + m_B v_B^2 = u_A^2 + u_B^2 = m_A v_0^2$$

$$p = m_A v_A + m_B v_B = \sqrt{m_A} u_A + \sqrt{m_B} u_B$$

Consider the motion of the particles as a point in the 2D phase space defined by the coordinates (y_1, y_2) and with velocity (u_A, u_B) .

The constraints on the positions of the particles can be expressed as

$$\begin{aligned} x_A > 0 &\implies y_1 > 0 \\ x_B < L &\implies y_2 < \sqrt{m_B} L \\ x_A < x_B &\implies y_1 < \sqrt{\frac{m_A}{m_B}} y_2 \end{aligned}$$

This describes a right-angled triangle bound by the lines $x = 0$, $y = \sqrt{m_B} L$ and $y = \sqrt{\frac{m_B}{m_A}} x$. We will now draw an analogy from this system to the motion of a single particle in a two dimensional space.

Consider a billiard ball at the coordinate (y_1, y_2) with velocity (u_A, u_B) , contained in the aforementioned triangle. Upon contact with the line $y = \sqrt{\frac{m_B}{m_A}} x$, the angle between the velocity of the billiard ball and the boundary denoted by θ does not change as

$$\begin{aligned} \cos \theta_i &= \frac{\begin{pmatrix} u_{A,i} \\ u_{B,i} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{m_A} \\ \sqrt{m_B} \end{pmatrix}}{\sqrt{u_{A,i}^2 + u_{B,i}^2} \sqrt{m_A + m_B}} \\ &= \frac{p}{\sqrt{u_{A,i}^2 + u_{B,i}^2} \sqrt{m_A + m_B}} \\ &= \frac{\begin{pmatrix} u_{A,f} \\ u_{B,f} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{m_A} \\ \sqrt{m_B} \end{pmatrix}}{\sqrt{u_{A,f}^2 + u_{B,f}^2} \sqrt{m_A + m_B}} \\ &= \cos \theta_f \end{aligned}$$

Thus, this billiard ball undergoes elastic collisions at all three boundaries, and a periodic cycle in the original system simply corresponds to the billiard ball returning to its original position with the same velocity. We can then make use of a well known trick – imagine that instead of bouncing off the walls, the trajectory of the ball instead passes through reflected copies of the original right-angled triangle. We may then solve the problem geometrically.

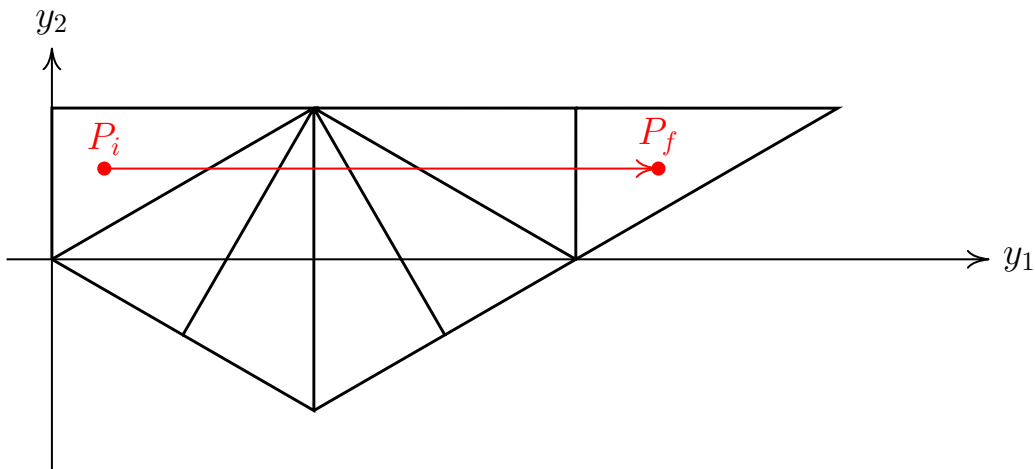


Photo 3: Trajectory of billiard ball if $(x_A, x_B) = (0.2 L, 0.6 L)$ initially and $m_A = 3m_B$

- (a) When $m_A = 3m_B$, the billiard ball returns to its original point after travelling a distance twice the length of the horizontal side of the original triangle (that contains P_i), regardless of the initial positions of A and B (see figure 1).

The period corresponding to the trajectory shown in figure 1 is

$$T = \frac{2\sqrt{m_A} L}{\sqrt{m_A} v_0} = \frac{2L}{v_0} = \boxed{2.00 \text{ s}}$$

- (b) Sketching the graph for $m_A = 4m_B$, we notice that the billiard ball returns to its original point after travelling a distance at least λ times the length of the length of the horizontal side of the original triangle, where

$$\lambda = 2 \left(1 - \cos \left(6 \tan^{-1} \sqrt{\frac{m_B}{m_A}} \right) \right) = 3.872$$

This minimum distance is achieved when $x_B = 0.296 L$, and the trajectory passes through a vertex of the triangle (see figure 2).

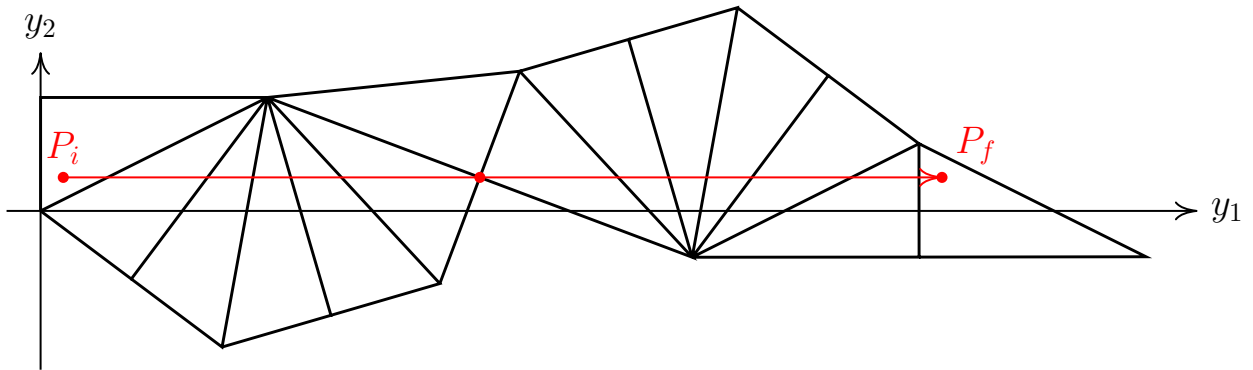


Photo 4: Trajectory of billiard ball if $(x_A, x_B) = (0.100 L, 0.296 L)$ initially and $m_A = 4m_B$

The period corresponding to the trajectory shown in figure 2 is

$$T = \frac{\lambda \sqrt{m_A} L}{\sqrt{m_A} v_0} = \frac{\lambda L}{v_0} \approx \boxed{3.87 \text{ s}}$$

Remark: According to the theorem of Kerckhoff, Masur and Smillie on ergodicity, if the initial velocity of the billiard ball were directed in a random direction, the subsequent motion of the billiard ball almost always becomes ergodic (i.e. visits every region in the state space in an equiprobable manner). In this problem, ergodicity was broken and periodicity was observed primarily because the direction of the initial velocity of the billiard ball was restricted by the constraint on the initial velocity of B .

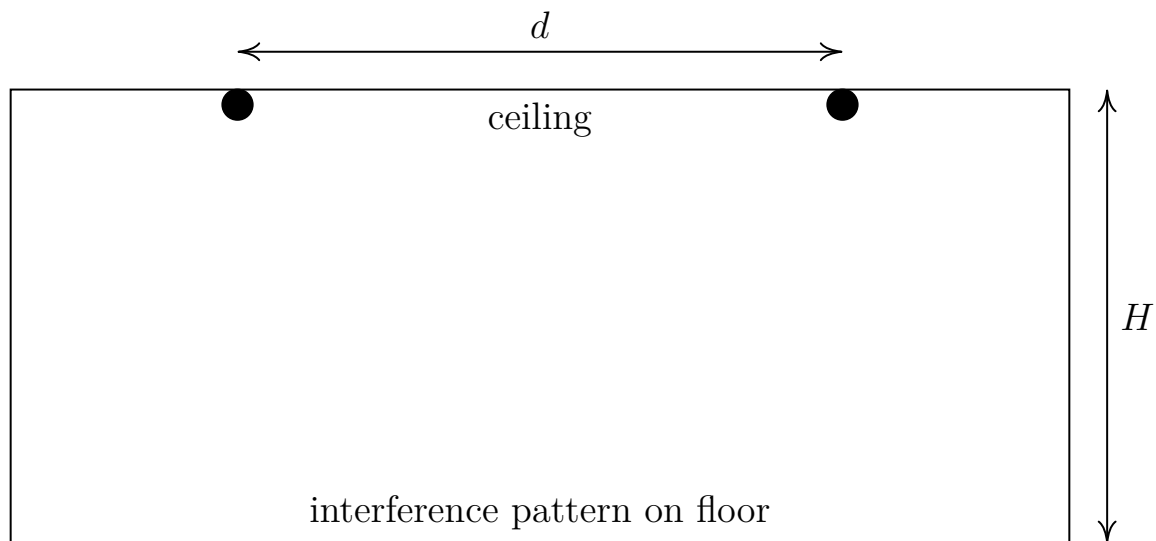
Setter: Raphael Teng Zhi Xiang, raphael.teng@sgphysicsleague.org

Problem 51: Micromacromicrowave

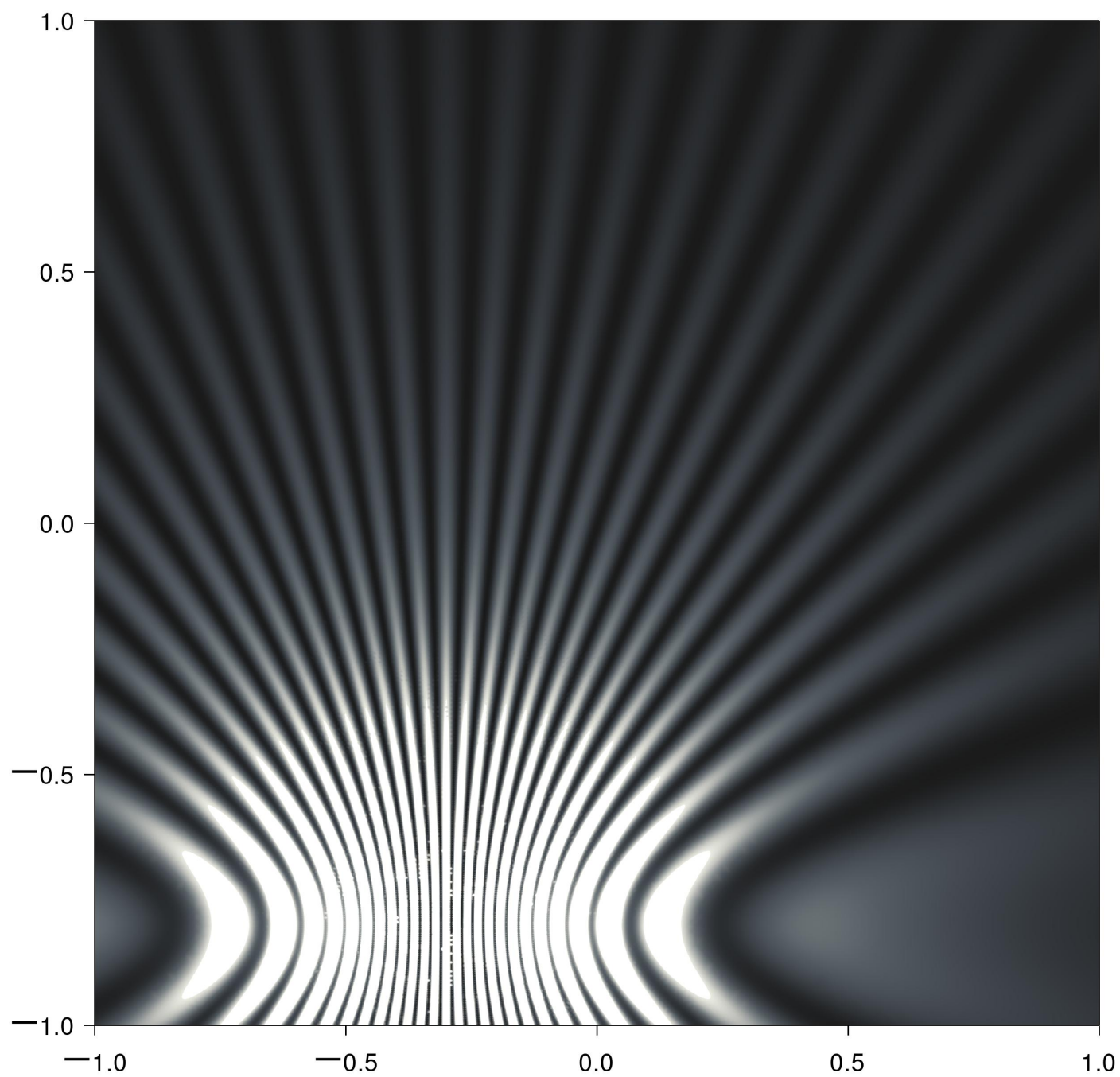
(5 points)

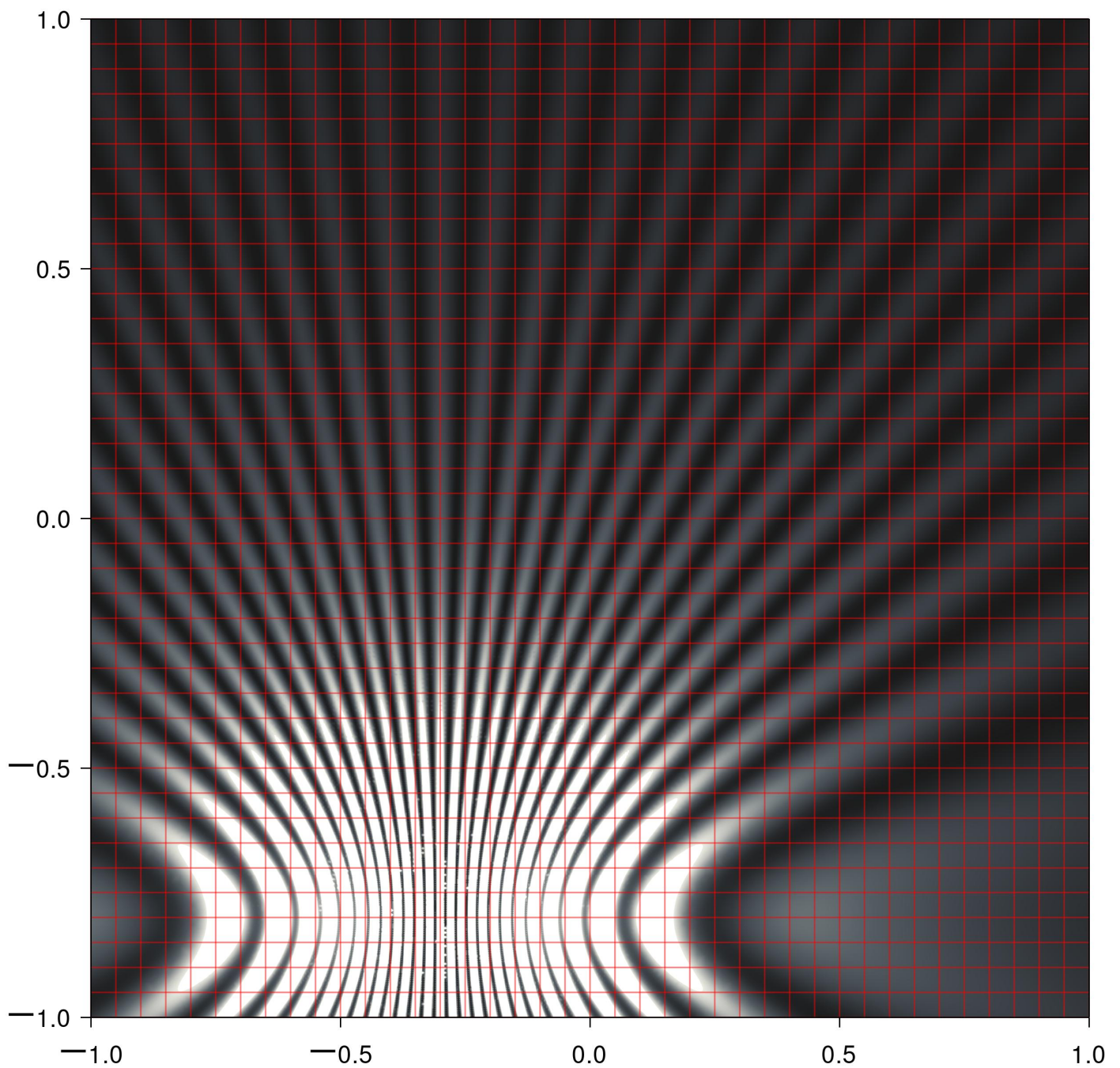
Aaron's microwave broke down suddenly, so he decides to build a new one from scratch. Unfortunately, he does not have access to the magnetrons used in conventional microwaves. Instead, Aaron fixes two small bulbs to the ceiling of the microwave, a distance d apart. Each bulb isotropically emits microwaves of wavelength λ . With his 20/20 vision, Aaron is able to observe the interference pattern the microwaves form on the floor, which can be found on the next two pages (both pictures are the same, but one of them has grid lines for annotations). Note that coordinates are given in metres.

If the ceiling is $H = 19$ cm above the floor, determine $5\lambda + \frac{d}{2}$.

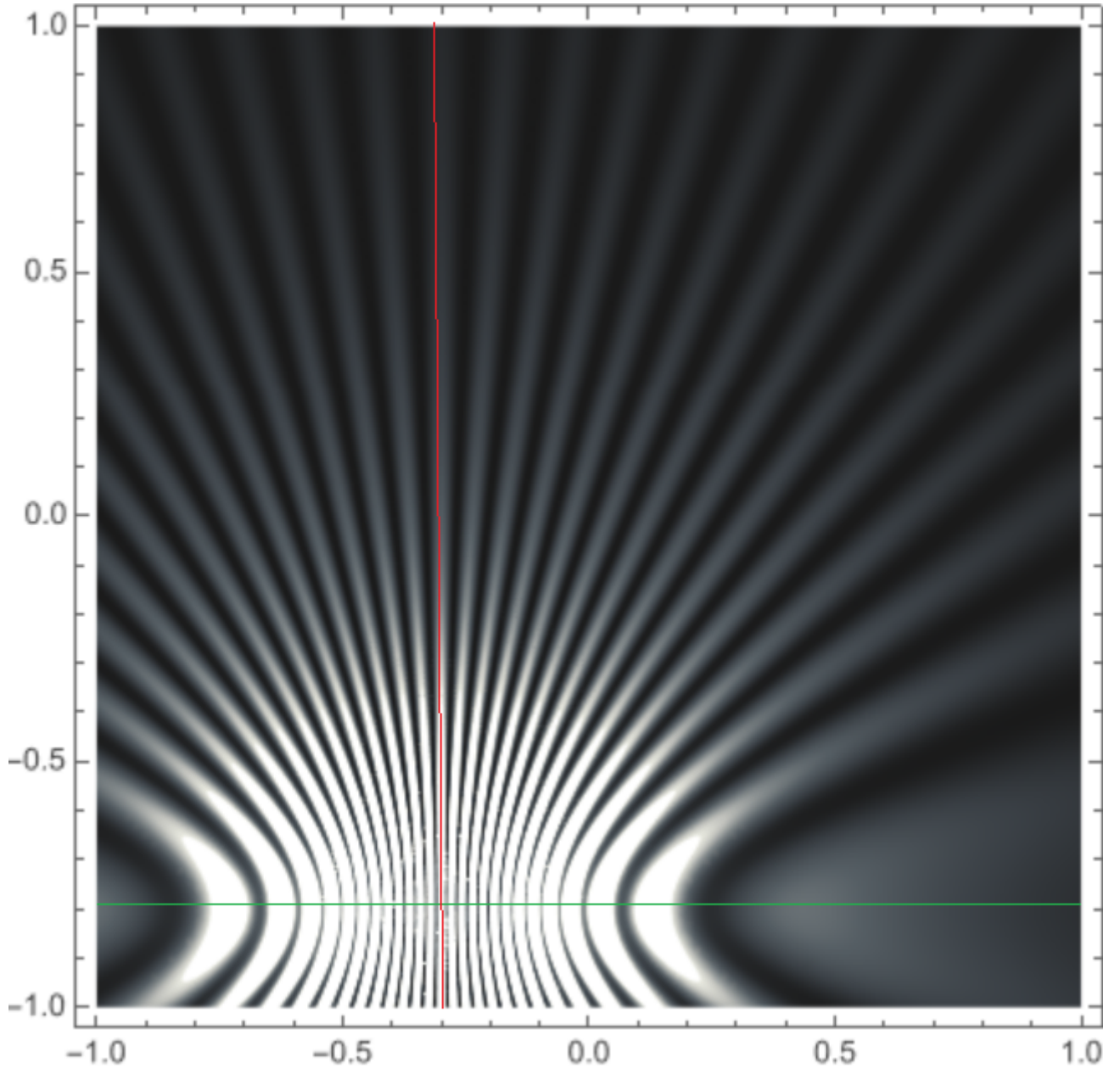


Leave your answer to 2 significant figures in units of m.





Solution: Firstly, note that by symmetry, the midpoint of the two bulbs is right above the intersection of the axes of symmetry of the interference pattern, so we can pinpoint it as shown below.



We can also conclude that the line connecting the two bulbs is right above the green line as all the bright spots are along it.

To get λ and d , we need two equations. Hence, let us analyze the interference pattern near the centre of the pattern and far away from it. We take the midpoint of the bulbs as our new origin, the green axis as the new x -axis and the red axis as the new y -axis. Then the bulbs are at $(-\frac{d}{2}, 0, 0)$ and $(\frac{d}{2}, 0, 0)$. Near the center of the pattern, such that $x, y \ll H, d$, the equation of the maxima is

$$\Delta\phi = 2n\pi = \frac{2\pi}{\lambda} \left(\sqrt{H^2 + \left(x + \frac{d}{2}\right)^2 + y^2} - \sqrt{H^2 + \left(x - \frac{d}{2}\right)^2 + y^2} \right)$$

$$2n\pi \approx \frac{2\pi}{\lambda} \left(\frac{d}{\sqrt{H^2 + \frac{d^2}{4}}} \right) x$$

where n is a nonnegative integer.

These are just parallel vertical lines, with the distance between adjacent fringes as

$$\Delta x \approx \frac{\lambda \sqrt{H^2 + \frac{d^2}{4}}}{d}$$

Measuring along the green axis, once can measure a distance of about 8 cm between the 5 central fringes, giving $\Delta x \approx 2$ cm.

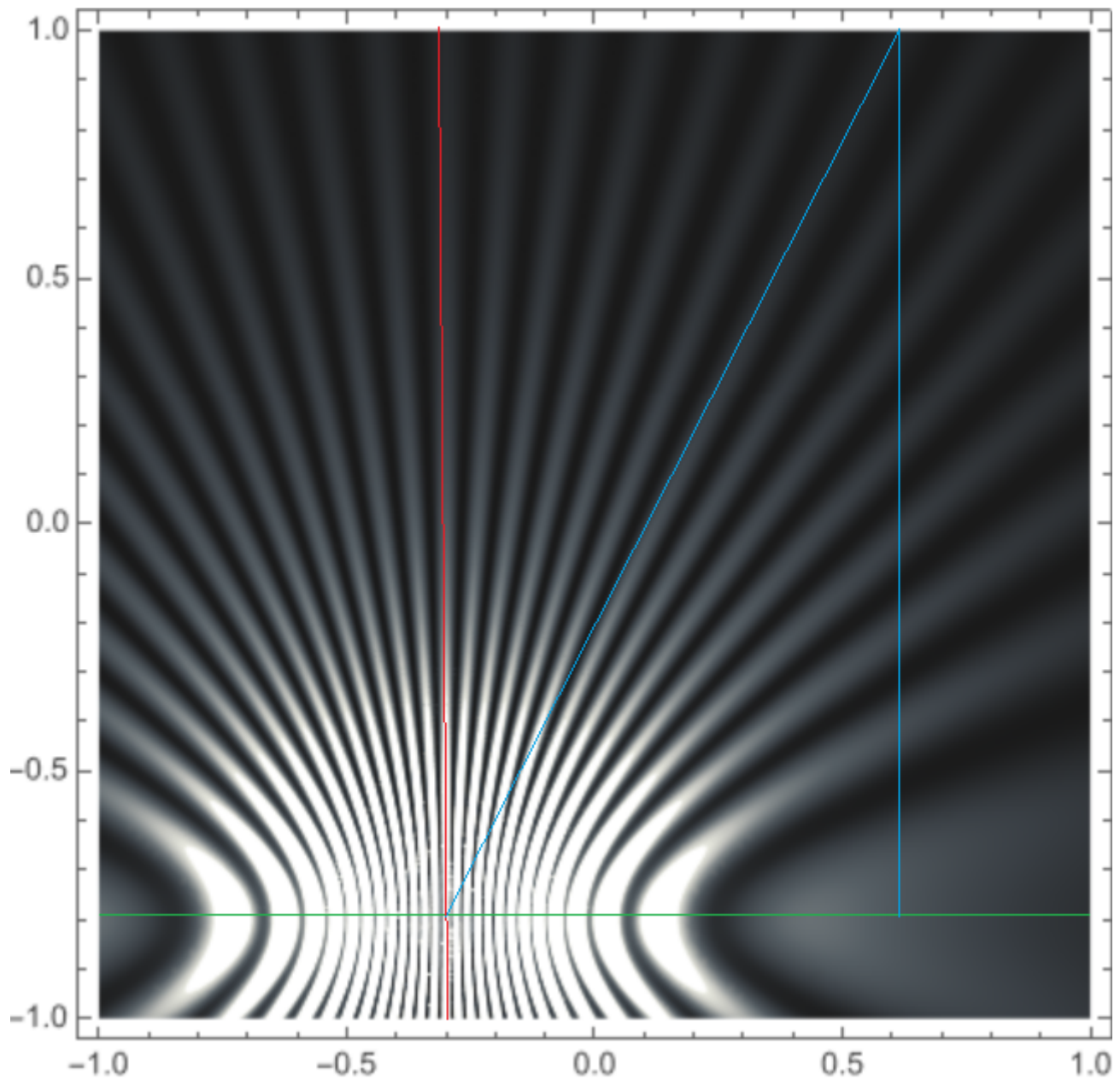
Now notice that the interference maxima further away appear to be conic sections. In fact, if we take $x, y \gg H, d$, we have $2n\pi \approx \frac{2\pi}{\lambda} \left(\sqrt{\left(x + \frac{d}{2}\right)^2 + y^2} - \sqrt{\left(x - \frac{d}{2}\right)^2 + y^2} \right)$. This is just the equation of a hyperbola centered at $(0, 0, -H)$, with foci at $(-\frac{d}{2}, 0, -H)$ and $(\frac{d}{2}, 0, -H)$! The semi-major axis of this hyperbola is just $\frac{n\lambda}{2}$ from the definition of a hyperbola.

At this point, one might measure the semi-major axis of the hyperbolae directly. But remember that this hyperbola equation is only valid far away from the centre (and the vertices are not sufficiently far away!). For an accurate result, we will need to use the asymptotes of the hyperbolae instead.

From the definition of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the focal length is $f = \sqrt{a^2 + b^2}$ and the asymptotes have gradients $\pm \frac{b}{a}$. We already know the focal length is $\frac{d}{2}$ as mentioned above. Therefore, if we take the asymptote with a positive gradient g , we will get

$$b = \frac{ng\lambda}{2}, d = 2\sqrt{\left(\frac{n\lambda}{2}\right)^2 + \left(\frac{ng\lambda}{2}\right)^2} = n\lambda\sqrt{1 + g^2}$$

We need to choose a hyperbola that is sufficiently far away yet not too blurred. Taking the hyperbola in blue with $n = 6$, we obtain $g \approx 2.10$. Note that we need to take the gradient with respect to our own green/red axes and not the ones in the diagram.



Now we have $d \approx 6\sqrt{1 + 2.10^2}\lambda$ and we can substitute this into the above equation for Δx . Finally (with some margin for error), we obtain

$$d \approx 40.9 \text{ cm}, \lambda \approx 2.93 \text{ cm}$$

$$5\lambda + \frac{d}{2} \approx \boxed{0.35 \text{ m}}$$

The actual values used to generate the interference pattern were $d = 40 \text{ cm}$ and $\lambda = 3 \text{ cm}$.

Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Problem 52: Let Him Cook

(5 points)

James is tired of the food from the army cookhouse, and sneaked some ready-to-eat meals into camp. To heat the food, he has to use the ancient oven in the pantry. The heating element in the oven is maintained at a constant temperature $T_0 = 373$ K, and the surrounding air outside the oven at $T_2 = 298$ K. The food has heat capacity $C = 750$ J K⁻¹, and an initial temperature $T_1 = T_2$. Given that the ready-to-eat meal has to reach a minimum temperature of $T_f = 348$ K for it to be cooked, what is the minimum heat Q_0 that has to be supplied by the oven?

James has a lot of ideal heat engines with him.

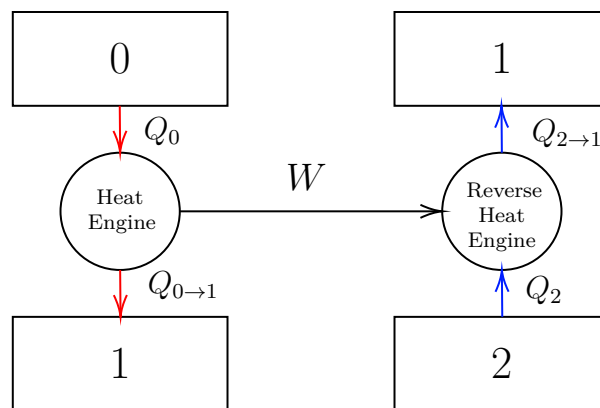
Leave your answer to 3 significant figures in units of kJ.

Solution: Let's denote the heating element, food, and surrounding with ①, ② and ③ respectively.

Let Q_i denote the heat supplied by the i^{th} component, and $Q_{i \rightarrow j}$ denotes the heat supplied by the i^{th} component that reaches the j^{th} component. For direct heating, all the heat supplied by ① reaches ②. We can therefore write

$$Q_{0 \rightarrow 1} = Q_0 = C(T_f - T_i).$$

This is, however, not ideal. Using a heat engine, we can create work with the temperature difference between ① and ②. Subsequently, we use this work to extract heat from ③ (the surrounding) to ② using a reverse heat engine.



The net effect of this heat engine set-up is that total heat Q_0 and Q_2 are extracted from ① and ③ respectively, and $dQ_{0 \rightarrow 1} + dQ_{2 \rightarrow 1} > dQ_{0 \rightarrow 1}$ ($dQ_{2 \rightarrow 1} > 0$) is infinitesimally pumped into ②. From here, it is immediately obvious that this new heating method is more efficient.

By [Clausius Inequality](#), which is valid for any cyclic thermodynamic process, we have

$$\oint \frac{dQ}{T} \leq 0 \quad \text{and} \quad \sum_k \frac{Q_k}{T_k} \leq 0,$$

where the former is for continuous heat transfers, and the latter is for discrete heat transfers. Both forms are required in our case since the temperature of the food is continuously increasing while being heated, while the temperature of the reservoirs remain constant. We can therefore write

$$\frac{Q_0}{T_0} - \int_{T_i}^{T_f} \left(\frac{dQ_{0 \rightarrow 1}}{T_1} + \frac{dQ_{2 \rightarrow 1}}{T_1} \right) + \frac{Q_2}{T_2} \leq 0.$$

As $dQ_{0 \rightarrow 1} + dQ_{2 \rightarrow 1} = C dT_1$, the integral evaluates to $C \ln(T_f/T_i)$. By conservation of energy, we further have

$$\int_{T_i}^{T_f} (dQ_{0 \rightarrow 1} + dQ_{2 \rightarrow 1}) = C(T_f - T_i) = Q_0 + Q_2.$$

Joining the two equations above and solving for Q_0 , we obtain

$$Q_0 \geq C \frac{T_0}{T_0 - T_2} \left[T_f - T_i - T_2 \ln \left(\frac{T_f}{T_i} \right) \right] = \boxed{14.1 \text{ kJ}}.$$

Note: If we were to instead directly heat the food, we have $Q_0 = C(T_f - T_i) = 45 \text{ kJ}$ which is more than twice the optimised value. James is indeed cooking.

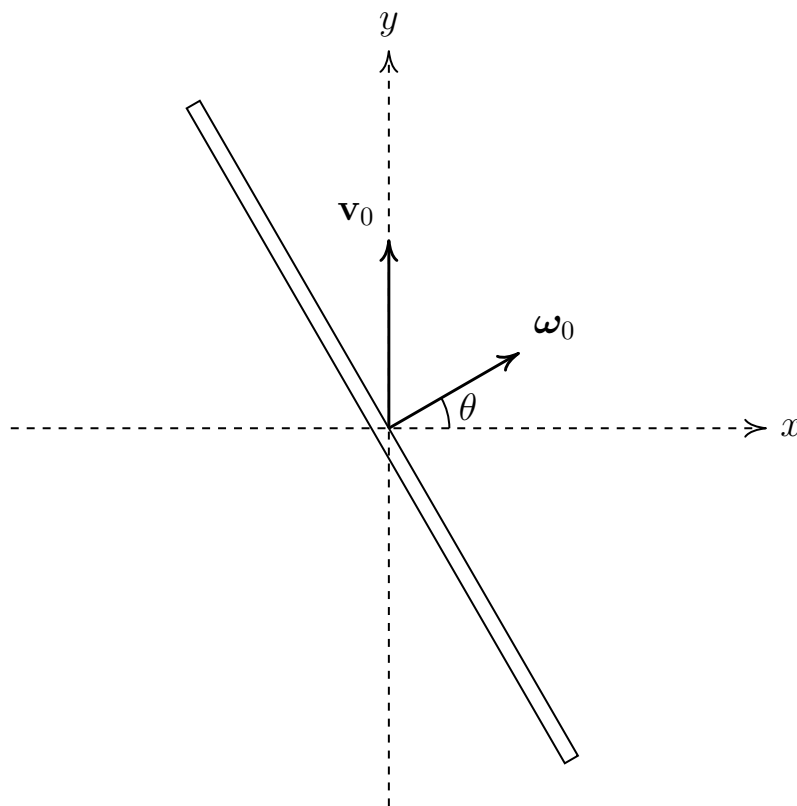
Setter: Li Xinrui, xinrui.li@sgphysicsleague.org

Problem 53: Kansei Dorifto?!

Ryab watched too much Initial D and wanted to try out some sick drift moves himself. Unfortunately, he can't afford a car with his NS pay, so he begrudgingly tries to drift a thin roll of tape like in [this video](#).

Ryab places a roll of tape (modelled as a thin ring) on a horizontal table and flicks it, giving it a translational velocity $\mathbf{v}_0 = v_0 \hat{\mathbf{y}}$ and a backspin angular velocity $\omega_0 = v_0/R$, where R is the radius of the roll of tape. The angular velocity vector $\boldsymbol{\omega}_0$ is at an angle θ anticlockwise from the x -axis, as shown in the **top-view** diagram below.

After some time, the roll transitions to pure rolling without slipping, with final velocity $\mathbf{v}_f$. The coefficient of kinetic friction between the tape roll and the table is $\mu = 0.86$.



- (a) Ryab wants the angle between $\mathbf{v}_0$ and $\mathbf{v}_f$ to be 86° . What is the required angle θ to achieve this?

Leave your answer to 3 significant figures in degrees.

(3 points)

- (b) If instead Ryab wants to maximise the displacement of the roll of tape from its starting position up to the instant it first attains pure rolling without slipping, what should the angle θ be?

Leave your answer to 3 significant figures in degrees.

(4 points)

Solution:

- (a) Firstly, we note that the frictional force $\mathbf{F}$ is antiparallel to the velocity of the contact point of the roll of tape with the ground, which is $\mathbf{v} + \boldsymbol{\omega} \times (-R\hat{\mathbf{z}}) = \mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}$. If we let it have mass m , the translational and rotational equations of motion are

$$\mathbf{F} = -\mu mg \frac{\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}}{|\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}|} = m \frac{d\mathbf{v}}{dt}$$

$$mR^2 \frac{d\boldsymbol{\omega}}{dt} = (-R\hat{\mathbf{z}}) \times \mathbf{F} = \mu mg \frac{R\hat{\mathbf{z}} \times \mathbf{v} - R^2\boldsymbol{\omega}}{|\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}|}$$

Of note is the fact that

$$\frac{d}{dt}(R\hat{\mathbf{z}} \times \boldsymbol{\omega}) = R\hat{\mathbf{z}} \times \frac{d\boldsymbol{\omega}}{dt} = -\left(\frac{\mu g}{R^2}\right) \left(\frac{R^2\mathbf{v} + R^3\hat{\mathbf{z}} \times \boldsymbol{\omega}}{|\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}|}\right) = \frac{d\mathbf{v}}{dt}$$

Hence one realises that

$$\frac{d}{dt}(\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}) = -2\mu g \frac{\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}}{|\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}|}$$

$$\frac{d}{dt}(|\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega}|) = -2\mu g$$

Let $\mathbf{v} + R\hat{\mathbf{z}} \times \boldsymbol{\omega} = \mathbf{u}$. Because $\omega_0 = \frac{v_0}{R}$, at time $t = 0$ we have $u_0 = v_0\sqrt{2(1 + \cos\theta)}$ and the direction of $\mathbf{u}$ always fixed at an angle $\frac{\theta}{2}$ anticlockwise from the $+y$ -axis. The roll of tape stops slipping and attains its final velocity at time $t = \frac{v_0}{2\mu g}\sqrt{2(1 + \cos\theta)}$. From the translational equation of motion above we see that

$$\mathbf{v}_f - \mathbf{v}_0 = -\mu g \left(\frac{v_0}{2\mu g} \sqrt{2(1 + \cos\theta)} \right) \hat{\mathbf{u}} = -v_0 \sqrt{\frac{1 + \cos\theta}{2}} \hat{\mathbf{u}} = -v_0 \cos \frac{\theta}{2} \hat{\mathbf{u}}$$

From this, one realises that the vectors $\mathbf{v}_0$, $\mathbf{v}_f$ and $(\mathbf{v}_f - \mathbf{v}_0)$ form a right triangle, with $\mathbf{v}_f \perp (\mathbf{v}_f - \mathbf{v}_0)$ and the angle between $\mathbf{v}_0$ and $(\mathbf{v}_f - \mathbf{v}_0)$ being $\frac{\theta}{2}$. Thus, the angle between $\mathbf{v}_0$ and $\mathbf{v}_f$ is $90^\circ - \frac{\theta}{2} = 86^\circ$, which gives $\theta = \boxed{8.00^\circ}$.

- (b) Since the acceleration of the tape is constant, its trajectory must be a parabola. We can hence define a y' -axis parallel to $\mathbf{u}$. Using the right-angled triangle mentioned above, we can get

$$x' = v_0 t \sin \frac{\theta}{2}, \quad y' = v_0 t \cos \frac{\theta}{2} - \frac{1}{2} \mu g t^2$$

We can then find the tape's displacement:

$$r^2 = x'^2 + y'^2 = v_0^2 t^2 - \mu g v_0 t^3 \cos \frac{\theta}{2} + \frac{1}{4} \mu^2 g^2 t^4$$

Here $t = \frac{v_0}{2\mu g} \sqrt{2(1 + \cos \theta)} = \frac{v_0}{\mu g} \cos \frac{\theta}{2}$ is the time taken to stop slipping as seen in part (a). We can hence write

$$r^2 = v_0^2 t^2 - \frac{3}{4} \mu^2 g^2 t^4$$

Differentiating with respect to t and noting that $t \neq 0$, we have

$$\frac{d}{dt} (r^2) = 0$$

$$2v_0^2 = 3\mu^2 g^2 t^2$$

$$t = \sqrt{\frac{2}{3}} \frac{v_0}{\mu g} = \frac{v_0}{\mu g} \cos \frac{\theta}{2}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{2}{3}}$$

Thus we get $\theta = \boxed{70.5^\circ}$.

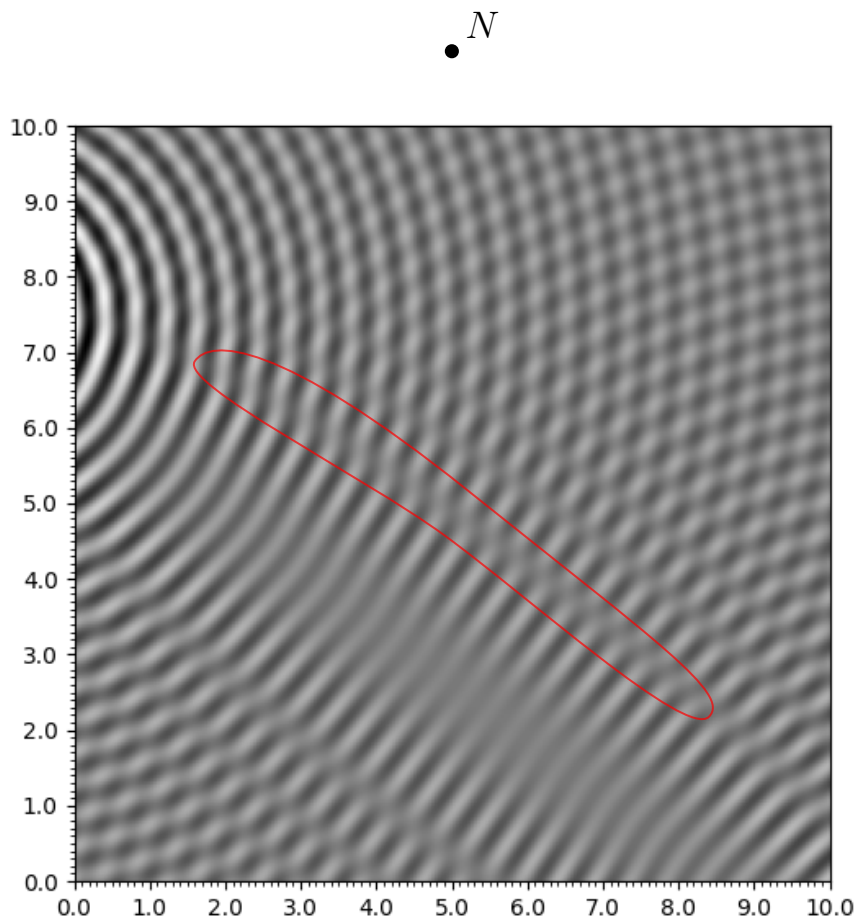
Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Problem 54: Mesmerizer

(6 points)

Miku and Teto are having a singing competition to see who can hold a note for the longest time. Both of them can be approximated as point sources in the 2-dimensional plane, each emitting a pure frequency f . The [video](#) shows the high and low pressure regions inside a $L \times L$ square, where $L = 10.0$ m, where black indicates regions with lower pressures and white indicates regions with higher pressures.

Note that both Miku and Teto are situated at points M and T outside of the square. Furthermore, **the video has been slowed down** by a factor of $\gamma = 1000$ (i.e. 1 s of the video corresponds to 1 ms of the actual physical setup).



The figure shows one example frame of the video. Observe the grey “wave” highlighted in the figure. Determine the speed of the “wave” v_w when it reaches point $N(5.0, 11.0)$ (shown in the figure above).

Leave your answer to 2 significant figures in units of m s^{-1} .

Hint: Clearly, it is not possible to find v_w by direct measurement at point N . Instead, try working out the positions of M and T .

Solution: Consider an arbitrary point P that is at distances of r_M and r_T from M and T respectively. Then we can express the combined waves from Miku and Teto as:

$$\psi(r_M, r_T, t) = \frac{A_M}{r_M} \cos(kr_M - \omega t + \phi_M) + \frac{A_T}{r_T} \cos(kr_T - \omega t + \phi_T)$$

where A_M and A_T are the amplitudes of the waves, ϕ_M and ϕ_T are the initial phases of the waves, k is the wavenumber and ω is the angular frequency.

Since grey represents regions with low deviations from atmospheric pressure, these are places where the waves destructively interfere. The grey “waves” hence represent regions where the phase of the waves differ by π rad. Hence, we obtain:

$$\pm(kr_M - \omega t + \phi_M) = (2n + 1)\pi + (kr_T - \omega t + \phi_T)$$

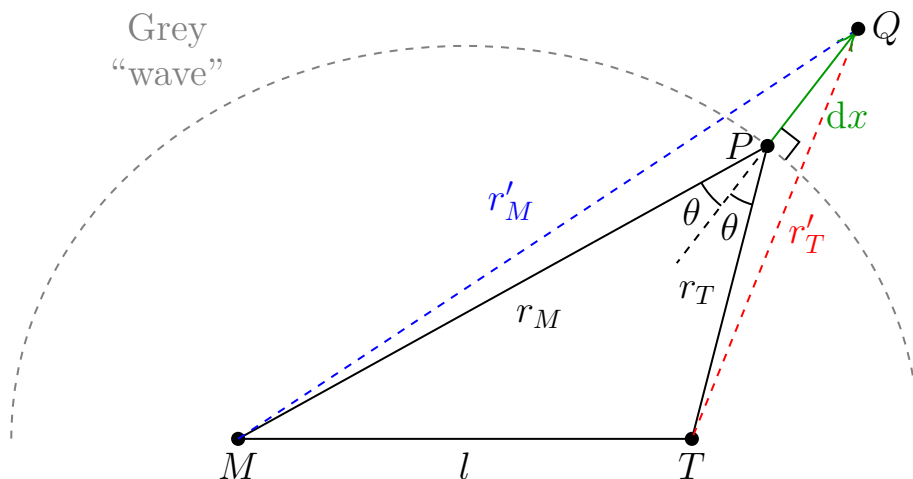
for some integer n . Choosing the positive sign leads to the equation for a hyperbola, while the negative sign is the equation for an ellipse. Since the grey “waves” are elliptical in shape, we choose the negative sign.

$$\begin{aligned} -(kr_M - \omega t + \phi_M) &= (2n + 1)\pi + kr_T - \omega t + \phi_T \\ 2\omega t - (\phi_M + \phi_T + (2n + 1)\pi) &= kr_M + kr_T \\ r_M + r_T &= \frac{1}{k}(2\omega t - (\phi_M + \phi_T + (2n + 1)\pi)) \\ r_M + r_T &= 2v_s t - \delta_n \end{aligned}$$

where δ_n is some constant that is different for each grey “wave”. Here, we have made use of the relation $\omega/k = v_s$, the speed of the sound wave.

This is the equation of an ellipse, with semimajor axis $a = \frac{1}{2}(2v_s t - \delta_n)$ and two foci at M and T .

To work out the speed of the grey wave, we consider a gray “wave” at the instant it passes an arbitrary point P , as well as a short time dt later.



Let l be the distance between M and T . After a short time dt , the grey “wave” at point P will have moved a distance dx to point Q . The speed of the grey wave is hence given by $v_w = \frac{dx}{dt}$.

From the reflection property of ellipses, we know that the angle bisector of $\angle MPT$ is perpendicular to the ellipse, and thus, Q lies on the bisector as well. Let $\angle MPT = 2\theta$, then, using the cosine rule on $\triangle PQT$:

$$\begin{aligned} r'_T &= \sqrt{r_T^2 + (dx)^2 - 2r_T dx \cos(\angle TPQ)} \\ &= r_T \sqrt{1 + \frac{2 \cos \theta}{r_T} dx + \left(\frac{dx}{r_T}\right)^2} \\ &\approx r_T \left(1 + \frac{\cos \theta}{r_T} dx\right) \\ &= r_T + dx \cos \theta \end{aligned}$$

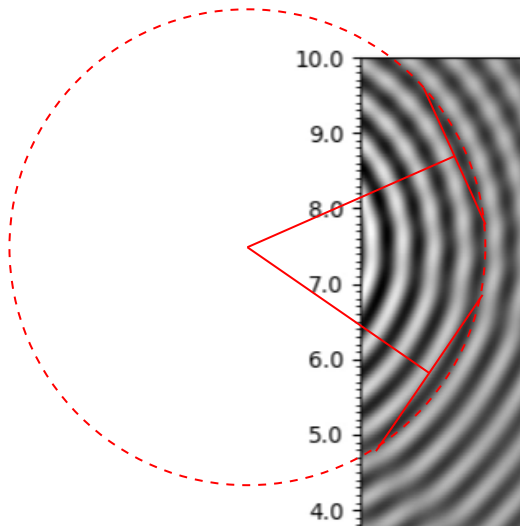
Similarly,

$$r'_M = r_M + dx \cos \theta$$

Since the equation of the ellipse is $r_M + r_T = 2v_s t - \delta_n$,

$$\begin{aligned} r'_M + r'_T &= 2v_s(t + dt) - \delta_n \\ r_M + r_T + 2 dx \cos \theta &= 2v_s t - \delta_n + 2v_s dt \\ 2 dx \cos \theta &= 2v_s dt \\ v_w = \frac{dx}{dt} &= \frac{v_s}{\cos \theta} \end{aligned}$$

v_s is the speed of sound in air, which is 340 m s^{-1} from the SPhL Data Sheet (alternatively, the speed can also be directly measured from the video). However, to evaluate θ , we need to know where M and T are.



The easiest way to find M and T is to find the centres of the concentric circles that emanate from M and T . Given any circle/circular arc, the perpendicular bisector of any chord passes through the center of the circle. Hence, simply by drawing 2 chords, we can find the centre of the circle as the intersection of the 2 perpendicular bisectors. The above diagram shows an example on how to find the centres of one of the circles.

With some basic measurements, the coordinates of M and T are $M(-1.5, 7.5)$ and $T(12.5, -3.0)$ (or vice versa). Since point N is at $(5.0, 11.0)$, we can obtain:

$$\overrightarrow{MN} = \begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{TN} = \begin{pmatrix} -6.0 \\ 14.0 \end{pmatrix}$$

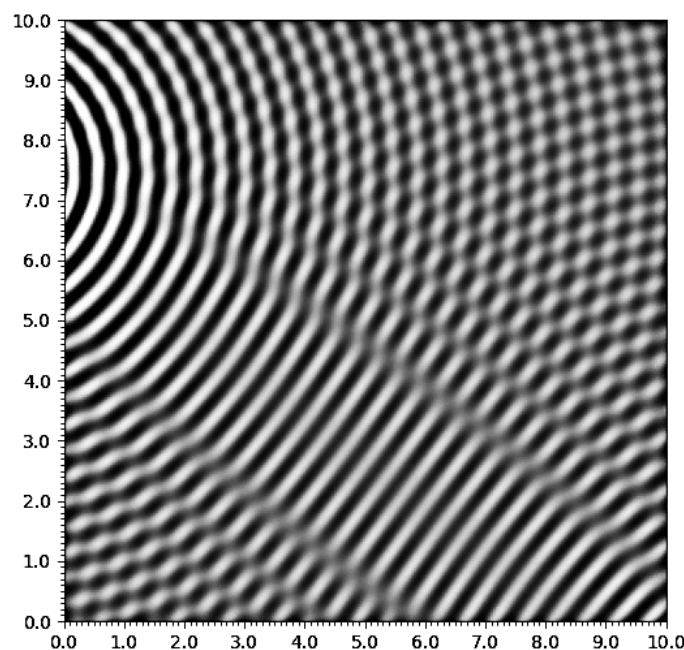
$$\theta = \frac{1}{2} \cos^{-1} \left(\frac{\overrightarrow{MN} \cdot \overrightarrow{TN}}{|\overrightarrow{MN}| |\overrightarrow{TN}|} \right)$$

$$\approx 0.78433 \text{ rad}$$

$$v_w = \frac{340 \text{ m s}^{-1}}{\cos(0.77336 \text{ rad})}$$

$$\approx \boxed{480 \text{ m s}^{-1}}$$

Alternative Solution: For more accurate determination of M and T , we can take one frame of the video and use image editing software to raise the contrast, sharpness and highlights:



This allows for us to choose points along the circles with much higher accuracy. To achieve greater precision with our numbers, we should aim to analyse several circles for their centres, then averaging the coordinates. Finally, it is also possible to sample more points along each circle, then performing regression analysis to find the circle centres instead for greater precision.

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Problem 55: Simp 4

(6 points)

While chasing Amy around Jake, the dog managed to break free of its leash. As such, Amy, Jake, and his dog decide to play one final game of catch, with the same rules as before:

- A “catch” occurs when two characters occupy the same position on the plane.
- If Amy catches Jake before his dog catches her, she wins.
- If Jake’s dog catches Amy before she catches Jake, Jake and his dog win.

Since Jake does not have a replacement leash for the dog, the following rules are used:

- Jake must remain stationary.
- Both Jake’s dog and Amy can move freely in the plane.
- Amy plans her path to catch Jake before the game starts and follows it exactly.
- Jake’s dog knows Amy’s planned path before the game starts and can anticipate her movements.

Amy, Jake, and his dog begin the game in a line, with the dog between Amy and Jake. The starting distance between Amy and Jake’s dog is r_A and that between Jake and his dog is r_J . Amy and the dog’s top speeds are v_A and v_D respectively, and they can change their velocities instantaneously.



They start the game with $r_A/r_J = \pi/2e$. Assuming all characters play optimally, Amy wins if $v_A/v_D > \chi$. Determine χ .

Leave your answer to 3 significant figures.

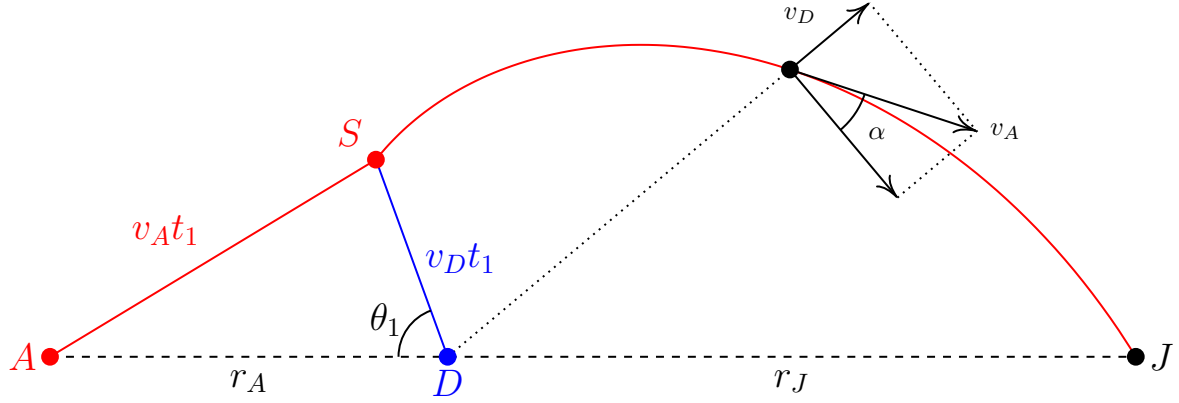
Solution:

The region that the dog could be in is a circle that expands with time, with its radius increasing at a rate of v_D . Thus, in order for Amy to avoid the dog, her plan must stay outside of this region; she must also catch Jake before the region includes him.

Firstly, we note that if Amy catches Jake before the dog can reach him, then she can choose a slightly different path and a lower top speed, which would still allow her to catch Jake while avoiding the dog. As such, at her minimum top speed, she catches Jake just as the dog’s region encompasses him.

Therefore, the total time that Amy takes to catch Jake is fixed at $T = r_J/v_D$, and the problem boils down to minimising her distance travelled while staying outside the dog's region. For convenience, we choose the origin to be the starting point of the dog.

Amy's optimal path consists of a straight line segment outside of the dog's region, followed by a spiral slightly outside the edge of the dog's region until she catches Jake. The geometry of this path is shown below:



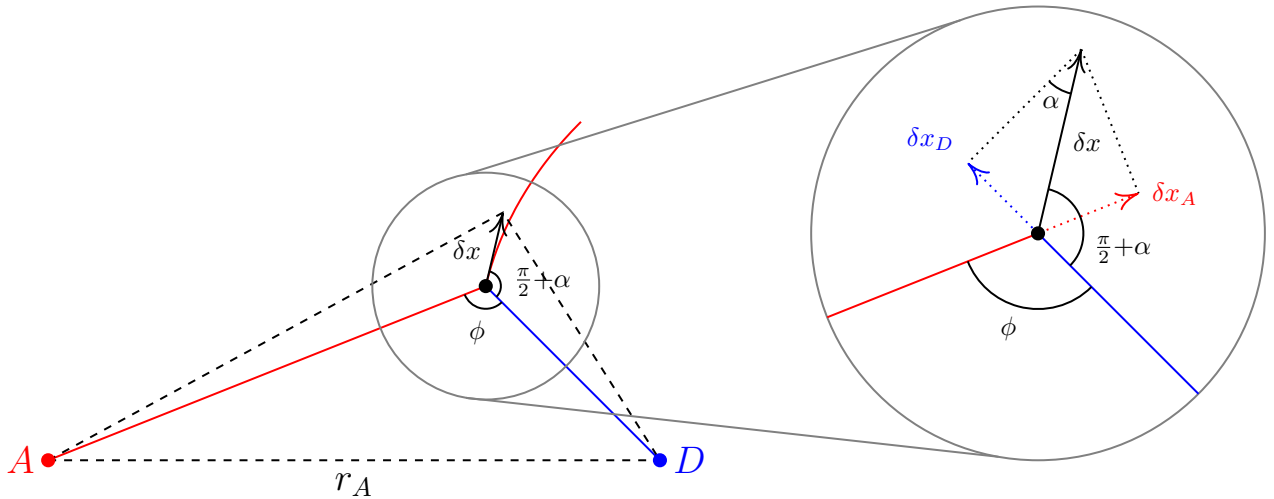
where t_1 is the time at which for Amy switches from the straight line segment to the spiral at point S .

Along the spiral, Amy uses part of her velocity to travel away from the origin at speed v_D to stay outside of the dog's region, and uses the remaining $\sqrt{v_A^2 - v_D^2}$ to travel in the tangential direction.

The pitch angle of the spiral is

$$\alpha = \sin^{-1} \frac{v_D}{v_A},$$

Suppose Amy reaches the spiral and performs a sharp transition onto the spiral. We will show that if Amy had aimed for a point that is further along the spiral by a small amount δx , she would have been able to use less speed.



$$\begin{aligned}
\text{Extra distance for Amy} &= \delta x_A \approx \delta x \cos \left(\phi + \alpha - \frac{\pi}{2} \right) \\
&= \delta x \sin (\phi + \alpha) \\
\text{Extra time for Amy} &\approx \frac{\delta x}{v_A} \sin (\phi + \alpha)
\end{aligned}$$

$$\begin{aligned}
\text{Extra distance for dog} &= \delta x_D \approx \delta x \sin(\alpha) \\
\text{Extra time for dog} &\approx \frac{\delta x}{v_D} \sin(\alpha) = \frac{\delta x}{v_A}
\end{aligned}$$

Since the transition is not smooth, $\phi + \alpha < \frac{\pi}{2}$, so that $\sin(\phi + \alpha) < 1$ and thus, Amy will take less time to reach the new point compared to the dog. Therefore, Amy can use a lower speed to reach the new point. Hence, in order for Amy to use the lowest possible speed, the transition needs to be smooth.

We thus must have $\phi = \frac{\pi}{2} - \alpha$, so the cosine rule, applied to the angles θ_1 and $\frac{\pi}{2} - \alpha$ give

$$v_A^2 t_1^2 = v_D^2 t_1^2 + r_A^2 - 2r_A v_D t_1 \cos \theta_1$$

and

$$\begin{aligned}
r_A^2 &= v_A^2 t_1^2 + v_D^2 t_1^2 - 2v_A v_D t_1^2 \cos \left(\frac{\pi}{2} - \alpha \right), \\
&= v_A^2 t_1^2 + v_D^2 t_1^2 - 2v_A v_D t_1^2 \frac{v_D}{v_A}, \\
&= v_A^2 t_1^2 - v_D^2 t_1^2.
\end{aligned}$$

respectively.

Solving the second equation for t_1 ,

$$t_1 = \frac{r_A}{\sqrt{v_A^2 - v_D^2}}.$$

Substituting this into the first equation, we get that $\cos \theta_1 = 0$, so $\theta_1 = \frac{\pi}{2}$.

On the spiral, Amy has a radius of $v_D t$ at any time t , and her tangential velocity around the origin is constant at $\sqrt{v_A^2 - v_D^2}$ (as can be derived from the second diagram above). Thus, her polar angle around the origin increases (clockwise) at a rate of

$$\frac{d\phi}{dt} = \frac{\sqrt{v_A^2 - v_D^2}}{r} = \frac{\sqrt{v_A^2 - v_D^2}}{v_D t}.$$

Rearranging and integrating,

$$\int_{\theta_1}^{\pi} d\phi = \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \int_{t_1}^T \frac{dt}{t}$$

$$\pi - \theta_1 = \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \ln \frac{T}{t_1}.$$

This is the expression for a logarithmic spiral. Substituting our expressions for T , t_1 , and θ_1 ,

$$\frac{\pi}{2} = \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \ln \left(\frac{r_J}{r_A} \frac{\sqrt{v_A^2 - v_D^2}}{v_D} \right) \equiv u \ln \left(\frac{r_J}{r_A} u \right),$$

where

$$u = \frac{\sqrt{v_A^2 - v_D^2}}{v_D}.$$

Substituting our (very specific) value for r_J/r_A and solving, we find that $u = \pi/2$. Using our definition for u ,

$$\frac{\pi}{2} = \frac{\sqrt{v_A^2 - v_D^2}}{v_D} = \sqrt{\left(\frac{v_A}{v_D}\right)^2 - 1},$$

so the minimum ratio of Amy and the dog's top speeds is

$$\frac{v_A}{v_D} = \sqrt{1 + \frac{\pi^2}{4}} = \boxed{1.86}.$$

Amy finds the strength within herself to run faster than she ever has, evading the dog and catching Jake. They end up happily ever after..?

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Half Hour Rush M1: Faro Shuffle

(3 points)

A faro shuffle, pictured below, consists of interweaving two halves of a deck together, such that the cards of each half alternate. Jindong performs a faro shuffle on his deck of $n = 52$ cards, sliding the cards slightly more than halfway into the weave, and places the deck on a frictionless table. Each card has mass $m = 1.70$ g, and static friction coefficient $\mu_s = 0.44$. Determine the minimum force F he must exert to separate the 2 halves by pulling on them.



Leave your answer to 3 significant figures, in units of N.

Solution: The i^{th} card from the top of the deck supports the weight of all the cards above it; hence the frictional force between it and the card below is

$$f_i = i\mu_s mg$$

We can then compute the force required to overcome friction as

$$\begin{aligned} F &= \sum_{i=1}^n i\mu_s mg \\ &= \frac{1}{2}n(n-1)\mu_s mg \\ &= \boxed{9.73 \text{ N}} \end{aligned}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Half Hour Rush M2: Ribbon Spread

(3 points)

Jindong shows off his latest trick - ribbon spreading, without hands! He puts his patented frictionless deck of cards on a rough table and quickly pushes it sideways, giving it an initial velocity $v_0 = 10 \text{ cm s}^{-1}$ along the x -axis. Each card has a coefficient of kinetic friction with the table of $\mu_k = 0.562$ and negligible friction with other cards. You may treat the cards as thin rigid plates with width $w = 6.35 \text{ cm}$. Given the deck of cards consists of $N = 52$ cards, vertically stacked, determine the furthest distance $x_{\max}$ along the x -axis a playing card travels during the subsequent motion.

Leave your answer to 2 significant figures in units of cm.

Solution: As the card stack slides across the table, the bottom card decelerates rapidly, while the cards above it maintain velocity v_0 and pivot off the edge after traveling a distance $w/2$ relative to the first card. Let us compute the stopping distance x_s for a single card.

$$\begin{aligned} f &= \mu_k mg \\ x_s &= \frac{v_0^2}{2f} \\ &\approx 0.091 \text{ cm} \end{aligned}$$

This is much smaller than the half-width of a single card, and the stopping distance for a card with n cards above it will be even smaller (by a factor of $1/(n+1)$). Hence, we can treat the stopping of cards as essentially instantaneous, with each successive card traveling an additional distance $w/2$. The distance that the furthest card travels is then simply

$$\begin{aligned} x_{\max} &= \frac{w}{2}(N-1) \\ &= \boxed{160 \text{ cm}} \end{aligned}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Half Hour Rush M3: House of Cards

(5 points)

After mastering the faro shuffle and ribbon spread, Jindong decides to build a tower of cards to celebrate. The tower is composed of equilateral triangles, with the bottom layer on a rough surface. For added stability, Jindong glues the cards that make up each horizontal layer together, so that they can be treated as a single rigid body that evenly transfers the weight of the cards above to the cards below.

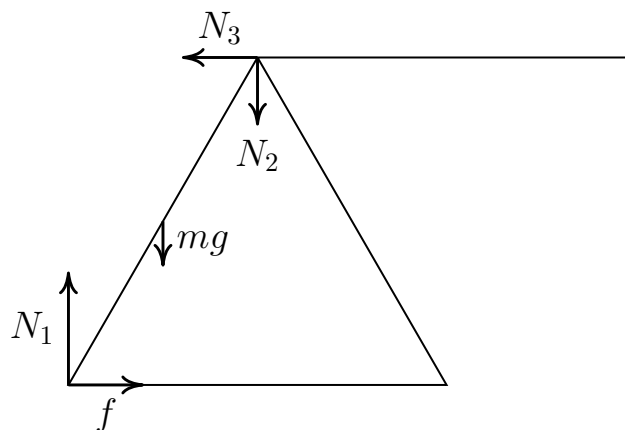


The static friction coefficient between the backs and edges of the cards is $\mu_s = 0.562$, while there is no friction between the faces and edges of the cards. You may assume the cards are rigid and have rounded edges. Determine the maximum number of levels, n , Jindong can build the tower to before it collapses.

Leave your answer as an integer.

Answer -1 if the card tower can be built arbitrarily high.

Solution:



Clearly, the first cards to slip must be on the side of the tower. From the free body

diagram, we write down the force and torque balances:

$$\begin{aligned} N_1 &= N_2 + mg \\ N_3 &= f \\ \frac{\sqrt{3}}{2}(N_3 + f) &= \frac{1}{2}(N_1 + N_2) \end{aligned}$$

In order for the card to not slip, we have

$$\begin{aligned} f &\leq \mu_s N_1 \\ \frac{\sqrt{3}}{6}(N_1 + N_2) &\leq \mu_s N_1 \\ N_2 &\leq \frac{mg}{2} \frac{6\mu_s - \sqrt{3}}{\sqrt{3} - 3\mu_s} \end{aligned}$$

Labelling layers from the top by i , each layer i supports the $\frac{3}{2}i(i-1)$ cards above it. This weight is evenly distributed across the $2i$ cards of that layer.

$$\begin{aligned} N_2 &= \frac{\frac{3}{2}i(i-1)mg}{2i} \\ &= \frac{3}{4}(i-1)mg \end{aligned}$$

The cards most likely to fail are at the second lowest level of the tower. Thus, we arrive at the condition

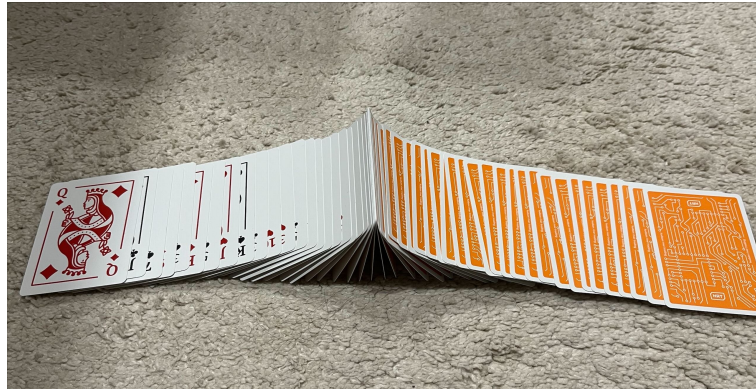
$$\begin{aligned} \frac{3}{4}(n-2)mg &\leq \frac{mg}{2} \frac{6\mu_s - \sqrt{3}}{\sqrt{3} - 3\mu_s} \\ n &= \left\lfloor \frac{2(2\sqrt{3} - 3\mu_s)}{3(\sqrt{3} - 3\mu_s)} \right\rfloor \\ &= \boxed{25} \end{aligned}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Half Hour Rush M4: Soliton Solitaire

(5 points)

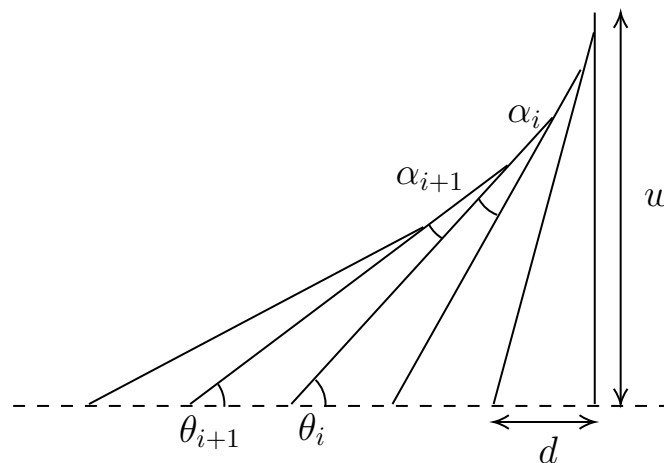
Jindong prepares his final flourish - by performing a ribbon spread, then turning the bottom card over, he can generate a “wave of cards” that travels down the spread.



Each card is very thin, has width $w = 6.35$ cm, and the gap between each card in the spread is $d = 0.10$ cm. You may assume that the table is sufficiently rough that cards do not slip, and that the total number of cards in the spread is very large. Jindong initiates the wave by turning over the first card at an angular speed of $\omega = 0.50 \text{ rad s}^{-1}$. Determine the speed of the wave when its peak passes the $n = 520^{\text{th}}$ card.

Leave your answer to 2 significant figures in units of m s^{-1} .

Solution: Notice that when the first card is stationary, the wave remains stably stationary, since the cards are symmetric around the peak. Hence, this is a statics problem - the speed of the wave is determined entirely by geometry.



Let the peak of the wave be at card 1. Indexing the cards away from the peak with i , we can write down the following recurrence relation between the angles of the cards:

$$\frac{\sin(\pi - \theta_i)}{w} = \frac{\sin \alpha_{i+1}}{d}$$

$$\theta_{i+1} = \theta_i - \alpha_{i+1}$$

$$= \theta_i - \sin^{-1} \left(\frac{d}{w} \sin \theta_i \right)$$

Since i becomes relatively large and d/w is small, we can approximate this relation with a more tractable differential equation:

$$\begin{aligned} \frac{d\theta}{di} &= -\sin^{-1} \left(\frac{d}{w} \sin \theta \right) \\ &\approx -\frac{d}{w} \sin \theta \end{aligned}$$

Now, to get the distance from the first card to the peak as a function of the angle of the first card, we simply integrate:

$$\begin{aligned} n - 1 &= \int_{\pi/2}^{\theta_0} -\frac{1}{\frac{d}{w} \sin \theta} d\theta \\ &= -\frac{w}{d} \left[\ln \left(\tan \frac{\theta}{2} \right) \right]_{\pi/2}^{\theta_0} \\ &= -\frac{w}{d} \ln \left(\tan \frac{\theta_0}{2} \right) \\ n - 1 &= -\frac{w}{d} \ln \left(\tan \frac{\theta_0}{2} \right) \\ \theta_0 &= 2 \tan^{-1} \left(e^{-\frac{d}{w}(n-1)} \right) \end{aligned}$$

Since

$$\theta_0 = \frac{\pi}{2} - \omega t$$

We can find the speed of the wave as

$$\begin{aligned} v &= d \frac{dn}{dt} \\ &= -\omega d \left(\frac{d\theta_0}{dn} \right)^{-1} \\ &= \frac{\omega w}{2} e^{\frac{d}{w}(n-1)} \left(e^{-\frac{2d}{w}(n-1)} + 1 \right) \\ &= \omega w \cosh \left(\frac{d}{w}(n-1) \right) \\ &= \boxed{56 \text{ m s}^{-1}} \end{aligned}$$

Setter: Ng Le Xi, lexi.ng@sgphysicsleague.org

Half Hour Rush E1: Nyan Cat[®]

(3 points)

Nyan Cat[®] is a cartoon cat with a Pop-Tart torso. Nyan Cat[®] loves to fly through space, leaving a magnificent rainbow trail of width $w = 7.62$ cm. Nyan Cat[®] has a mass of $m = 48.0$ g and may be treated as a point particle.

Nyan Cat[®] decides to charge itself with an electric charge of $q = 1.00 \times 10^5$ C and fly through a region of space where there exists a uniform magnetic field $B = 5.50 \times 10^{-10}$ T. Nyan Cat[®] initially starts flying at a speed of $v = 5.00$ m s⁻¹ perpendicular to the magnetic field.

Determine the area A of Nyan Cat[®]'s rainbow trail once it loops back on itself, if it does. The centre of Nyan Cat[®]'s trail is formed by its trajectory and the trail lies in the same plane as the trajectory. Neglect all forces other than the magnetic force.

Leave your answer to 3 significant figures in units of m².

If you think the trail does not loop back on itself, enter -1.

Solution: The magnetic force does no work on Nyan Cat[®], so its speed remains constant. Furthermore, the magnetic force is always perpendicular to Nyan Cat[®]'s velocity, so it must be travelling in a circle. Equating the centripetal force to the magnetic force, the radius of the trajectory is

$$\frac{mv^2}{R} = qvB \quad \implies \quad R = \frac{mv}{qB} = 4363.6 \text{ m}$$

The area of the trail is then

$$A = 2\pi R w = \boxed{2090 \text{ m}^2}.$$

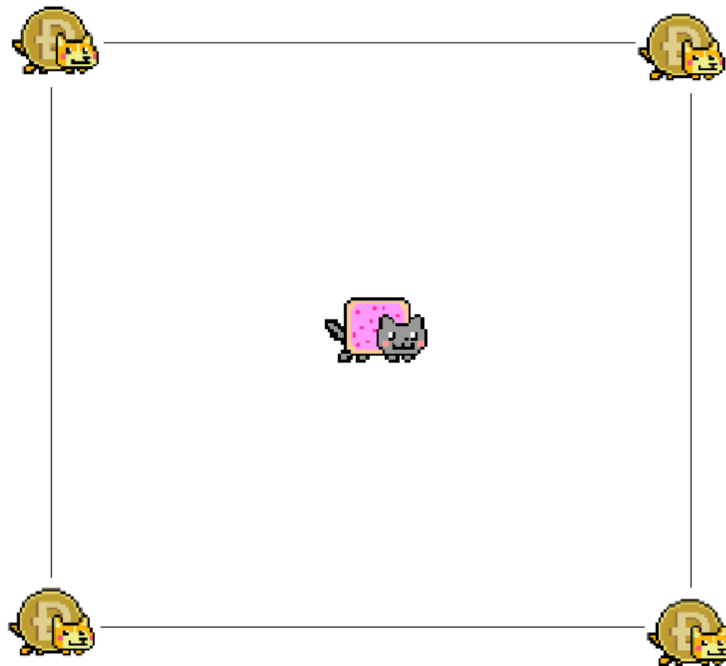
Setter: Oon Han, oon.han@sgphysicsleague.org

Half Hour Rush E2: It Goes in the Square Hole

(4 points)

Nyan Cat[®] is trying to fly through a massless square frame of length $l = 3000$ m as shown in the diagram below (Nyan Cat[®]'s velocity is into the page). **The square frame is free to move in space.** However, as Nyan Cat[®] is not square enough, Square Hole Girl wants to stop it from flying through.

While Nyan Cat[®] is still far away, she attaches a Nyan Doge[®] with the same mass and charge as Nyan Cat[®] to each corner of the frame. If Nyan Cat[®] travels along the axis of symmetry perpendicular to the square, find the minimum velocity u it must have (when far from the frame) in order to pass through it. Nyan Cat[®]'s mass is 48.0 g and its charge is now $q = 1.00$ C.



Leave your answer to 3 significant figures in units of m s^{-1} .

Solution: Since there are no external forces acting on the Nyan Cat[®] and Nyan Doges[®] in the direction perpendicular to the page, the total momentum along that direction has to be conserved. In addition, due to symmetry the square can only move into and out of the page.

The minimum velocity corresponds to the case where Nyan Cat[®] barely reaches the centre of the square frame. In this case both Nyan Cat[®] and the square move at the same velocity v , so conservation of momentum gives

$$mu = mv + 4mv$$

$$v = \frac{u}{5}$$

Energy is also conserved, so we have

$$\frac{1}{2}mu^2 = \frac{1}{2}m\left(\frac{u}{5}\right)^2 + \frac{1}{2}(4m)\left(\frac{u}{5}\right)^2 + 4\left(\frac{\sqrt{2}q^2}{4\pi\epsilon_0 l}\right)$$

$$\frac{2}{5}mu^2 = \frac{\sqrt{2}q^2}{\pi\epsilon_0 l}$$

$$u = \sqrt{\frac{5\sqrt{2}q^2}{2\pi m\epsilon_0 l}} = \boxed{29700 \text{ m s}^{-1}}$$

Setter: Siddhaarth Dharani, siddhaarth.dharani@sgphysicsleague.org

Half Hour Rush E3: Megalovania

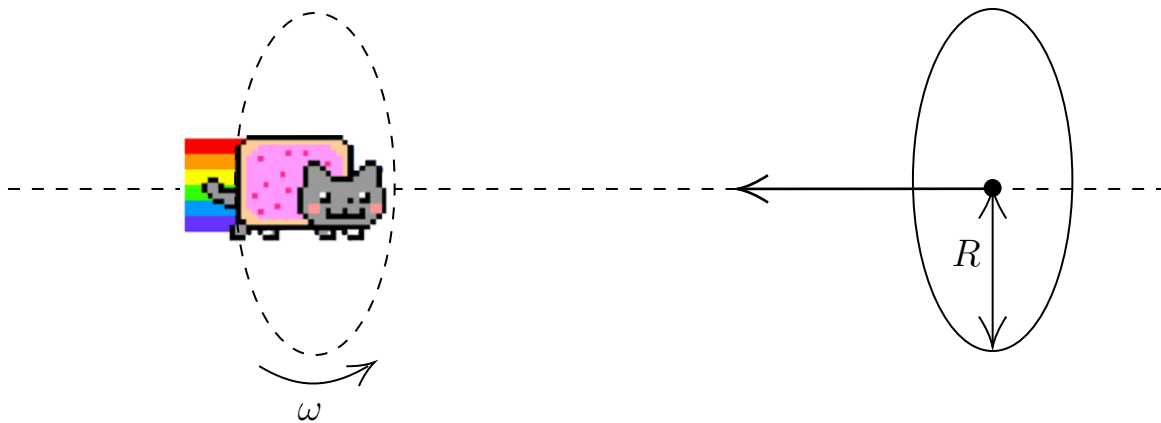
(4 points)

Angered by Nyan Cat[®]'s shenanigans, Sans[®] (from the indie game *Undertale*[®]) challenges Nyan Cat[®] to a boss fight. Unfortunately, Sans[®] ran out of bones, so he instead uses a circular, insulating and uniformly charged ring of radius $R = 0.240$ m, mass $M = 0.0960$ kg and electric charge $Q = 1.75 \times 10^3$ C as an obstacle. In retaliation, Nyan Cat[®] strips itself of its electric charge and, using its power of being fictional, charges itself with a *magnetic charge* $q_m = 1.25 \times 10^3$ A m, the hypothetical analogue of electric charge. As a result, Nyan Cat[®] produces a magnetic field around itself given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 q_m}{4\pi r^2} \hat{\mathbf{r}}$$

where Nyan Cat[®] is centred at the origin.

Sans[®] attacks Nyan Cat[®] by propelling the initially non-rotating ring from very far away at a constant velocity $v = 12.0$ m s⁻¹ towards it. The ring's axis passes through Nyan Cat[®], who remains fixed in position. To Sans[®]' surprise, the ring begins to spin! Determine the angular velocity ω of the ring when its centre reaches Nyan Cat[®]. You may ignore the self-inductance of the ring.



Leave your answer to 3 significant figures in units of rad s⁻¹.

Solution: The ring spins because as it approaches the magnetic charge, the magnetic flux through the ring increases, inducing a (non-conservative) electric field in it. Since the ring is charged, the electric field exerts a torque about its centre, causing it to rotate.

Applying Ampere's law on the loop of the ring,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \implies E = \frac{1}{2\pi R} \frac{d\Phi_B}{dt}$$

so the torque acting on the ring is

$$\tau = QER = \frac{Q}{2\pi} \frac{d\Phi_B}{dt}$$

The rotational form of Newton's second law gives the angular acceleration as

$$\tau = I\alpha = MR^2\alpha \quad \implies \quad \alpha = \frac{Q}{2\pi MR^2} \frac{d\Phi_B}{dt}$$

Finally, integrating over time, we get the final angular velocity as

$$\begin{aligned} \omega &= \int \alpha \, dt \\ &= \int \frac{Q}{2\pi MR^2} \, d\Phi_B \\ &= \frac{Q\Delta\Phi_B}{2\pi MR^2} \end{aligned}$$

which conveniently only depends on the initial and final values of the magnetic flux through the ring!

Now, from the expression for the magnetic field, we can infer that the analogue for the magnetic flux around the magnetic charge is $\Phi_{B,q} = \mu_0 q_m$. When the ring is far away, the flux through the ring is clearly $\Phi_{B,i} = 0$. Just before the centre of the ring reaches the charge, the flux through the ring is *half* of that around the charge, so $\Phi_{B,f} = \frac{1}{2}\Phi_{B,q} = \frac{1}{2}\mu_0 q_m$. Therefore, the final angular velocity is

$$\omega = \frac{\mu_0 Q q_m}{4\pi MR^2} = \boxed{39.6 \text{ rad s}^{-1}}$$

Note: One can also solve this problem using the concept of **canonical angular momentum** – where the flux through the ring is treated as an additional angular momentum $L_{\text{mag}} = Q\Phi_B/2\pi$, and total angular momentum is conserved.

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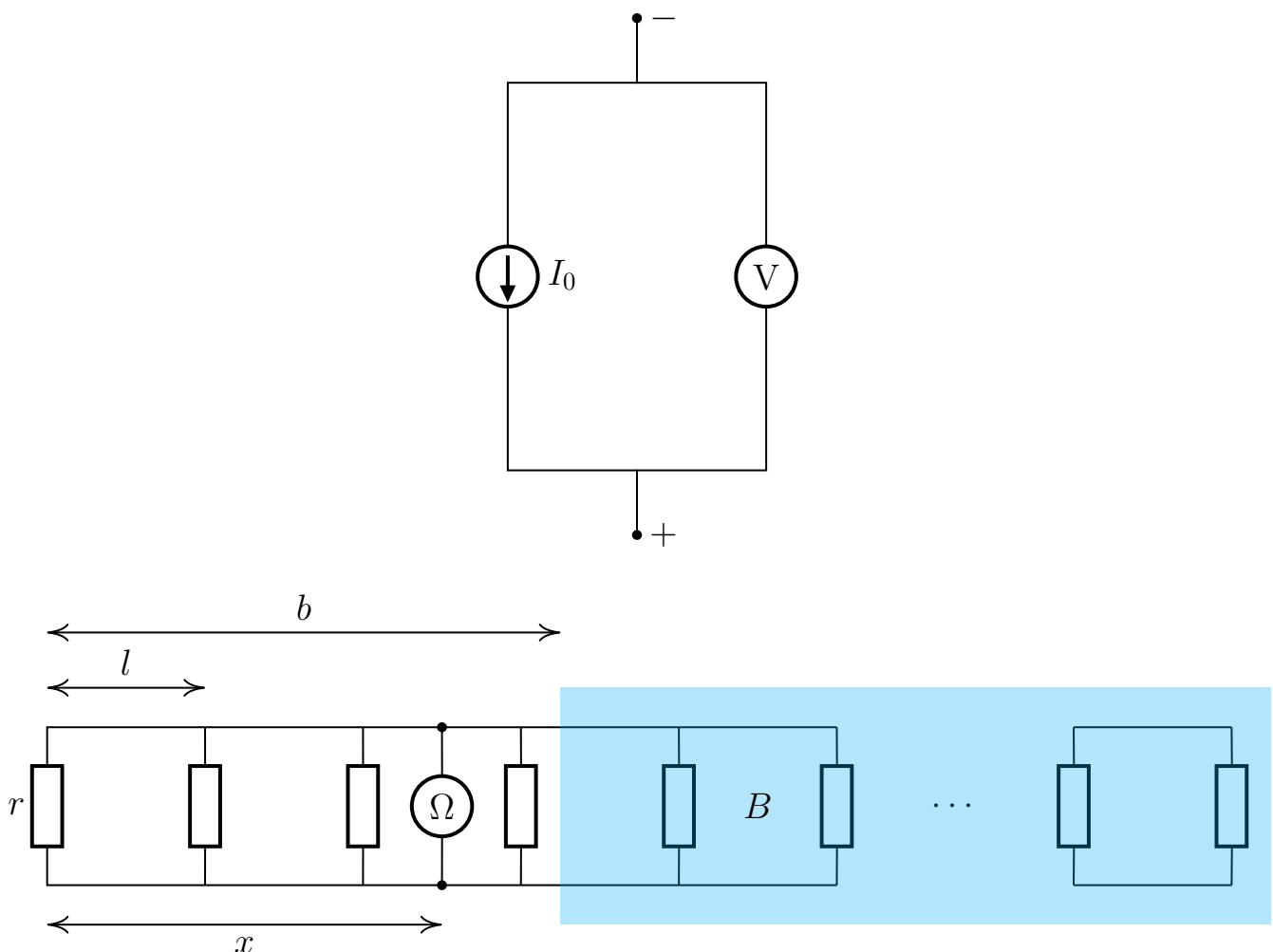
Half Hour Rush E4: I Am A Parallelogram

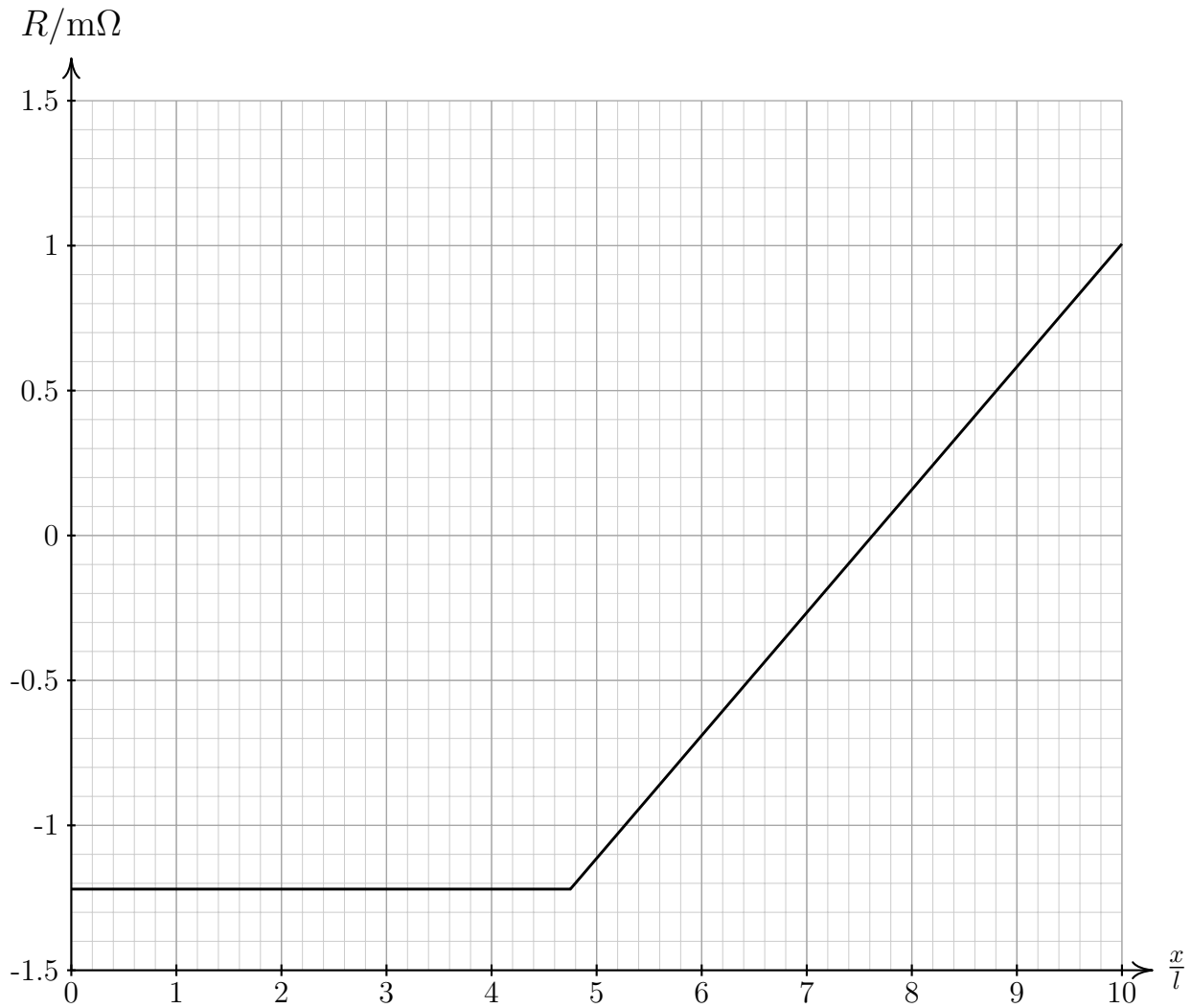
(5 points)

Being obsessed with parallelograms, Nyan Cat[®] decides to build a parallelogram circuit as shown below. Being traumatised by squares earlier, it builds N identical resistors r in parallel, with adjacent resistors spaced a distance l from each other. Let the origin be at the bottom left corner. Nyan Cat[®] uses an ohmmeter to measure the effective resistance of the circuit and gets a reading of $2.50 \text{ m}\Omega$. As shown below, the ohmmeter consists of an ideal current source I_0 in parallel with (and very close to) an ideal voltmeter. The negative end is connected to the top side of the circuit and the positive end is connected directly below to the bottom side. The ohmmeter reading is the voltmeter reading divided by I_0 .

Nyan Cat[®] then decides to turn on a uniform magnetic field in the blue region $x > b$ which increases linearly with time, $B = \alpha t$. Curiously, now the resistance reading on the ohmmeter R not only changes, but also depends on the position of the two terminals x as shown in the (incomplete) graph of R against $\frac{x}{l}$ on the next page. Find r .

Leave your answer to 3 significant figures in units of $\text{m}\Omega$.





Solution: When the magnetic field is turned on, the resistance reading drops below 0 mΩ because the current changes direction, changing the sign of the voltmeter reading. This happens because the changing magnetic field induces an emf in the circuit, creating a nonzero potential difference between the resistors outside the magnetic field and those inside it. This is also why the reading changes with the position of the ohmmeter.

From the graph, we can see that there is a kink at $x = \frac{19}{4}l$ (between the 5th and 6th resistors), and hence $x > \frac{19}{4}l$ must be the region where the magnetic field acts. Let us label the currents through resistor i as I_i , numbering the resistors from left to right. Then from Faraday's Law we have

$$I_1 = I_2 = I_3 = I_4 = I_5$$

$$I_6 r - I_5 r = \frac{1}{4} \alpha A$$

$$I_{n+1} r - I_n r = \alpha A, 6 \leq n \leq N - 1$$

where A is the area of one of the squares in the resistor grid. From current conservation, we also have

$$\sum_{i=1}^N I_i = I_0$$

We can combine all the equations to get

$$5I_1 + (N-5)I_6 + \frac{(N-6)(N-5)}{2r}\alpha A = I_0$$

$$NI_1 + \frac{\alpha A}{r} \left(\frac{N-5}{4} + \frac{(N-6)(N-5)}{2} \right) = I_0$$

$$I_1 r = \frac{1}{N} \left(I_0 r - \alpha A \left(\frac{N-5}{4} + \frac{(N-6)(N-5)}{2} \right) \right)$$

For $x < \frac{19}{4}l$, this is the reading that the voltmeter will measure. However, inside the magnetic field we need to use Faraday's Law again:

$$V - I_1 r = \left(\frac{x}{l} - \frac{19}{4} \right) \alpha A$$

Hence the ohmmeter reading, $\frac{V}{I_0}$, is

$$\frac{V}{I_0} = \begin{cases} \frac{1}{N} \left(r - \frac{\alpha A}{I_0} \left(\frac{N-5}{4} + \frac{(N-6)(N-5)}{2} \right) \right), & 0 \leq \frac{x}{l} < \frac{19}{4} \\ \frac{1}{N} \left(r - \frac{\alpha A}{I_0} \left(\frac{N-5}{4} + \frac{(N-6)(N-5)}{2} \right) \right) + \left(\frac{x}{l} - \frac{19}{4} \right) \frac{\alpha A}{I_0}, & \frac{x}{l} \geq \frac{19}{4} \end{cases}$$

From the graph, we see that the gradient $\frac{\alpha A}{I_0}$ is about $\frac{(0.90 - (-1.22)) \text{ m}\Omega}{9.750 - 4.750} = 0.424 \text{ m}\Omega$ and that before the kink the resistance measurement is about $-1.22 \text{ m}\Omega$. We also know that before the magnetic field is turned on, the ohmmeter obviously measures the effective resistance as $\frac{r}{N} = 2.50 \text{ m}\Omega$. With this, we can find N :

$$2.50 \text{ m}\Omega - (0.424 \text{ m}\Omega) \left(\frac{N-5}{4N} + \frac{(N-6)(N-5)}{2N} \right) = -1.22 \text{ m}\Omega$$

$$\frac{N-5}{4N} + \frac{(N-6)(N-5)}{2N} = \frac{2.50 + 1.22}{0.424} \approx 8.7$$

With this, we can find N either via solving the quadratic or guess-and-check, giving $N = 27$. Note that due to the answer's sensitivity to the gradient it is possible to get ± 1 to this answer, for which both cases are accepted too. With this, we get

$$r = \boxed{67.5 \text{ m}\Omega}$$

The answers of $70.0 \text{ m}\Omega$ for $N = 28$ and $65.0 \text{ m}\Omega$ for $N = 26$ are also accepted.

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Half Hour Rush X1: Spacing Out

(3 points)

After defeating Thor in [Avengers: Infinity War \(2018\)](#), Thanos uses the Space Stone to transport himself to distant locations in search of the remaining Infinity Stones. He teleports himself and all the air around him to a location extremely far away, leaving behind a perfect vacuum which rapidly collapses.

Thanos teleports a spherical region of air, of radius 2.00 m. The surrounding environment is at a pressure $P = 1.00 \text{ atm}$. Determine the total energy E released (as heat and sound) during the collapse of the vacuum.

Leave your answer to 2 significant figures in units of kJ.

Solution: In order to collapse the vacuum, the atmosphere does work on its boundary equal to PV , where V is the volume of the ball. This work is then dissipated as heat and sound once the vacuum fully collapses. Using the formula for the volume of a sphere,

$$E = \frac{4}{3}\pi R^3 P = \boxed{3400 \text{ kJ}}$$

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Half Hour Rush X2: Hammer Time

(3 points)

Thor is determined to get revenge on Thanos. He decides to try reforging his broken hammer, Jonathan.

Thor brings a piece of Jonathan close to the Sun, which can be modelled as an infinite black plane at constant temperature $T_0 = 5700^\circ\text{C}$. The fragment is then left to reach an equilibrium temperature of T_1 .

Assuming that the piece is an ideal blackbody in the shape of a cube with side length $L = 1\text{ m}$, determine T_1 . Note that there is no air in space.

Leave your answer to 3 significant figures in units of $^\circ\text{C}$.

Solution: Consider 2 infinite planes of temperature T_0 (measured in Kelvin). If the piece of Jonathan is placed in between those 2 planes, it would eventually acquire a temperature of T_0 too. If we let the power received by the piece from each plane be P , it is receiving a total power of $2P$, so

$$2P = \epsilon\sigma AT_0^4$$

When one plane is removed, the amount of thermal radiation received by the piece is halved, so it equilibrates at a new temperature T_1 given by

$$P = \epsilon\sigma AT_1^4$$

Eliminating P from both equations gives

$$\epsilon\sigma AT_0^4 = \frac{1}{2}\epsilon\sigma AT_1^4$$

Hence $T_1 = \frac{T_0}{2^{\frac{1}{4}}} = \boxed{4750^\circ\text{C}}$

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Half Hour Rush X3: Particularly Strange

(4 points)

Sadly, Thor was unable to reforge his hammer, as the sun was not hot enough. Instead, Thor plans to travel to an alternate timeline to steal a different version of Jonathan. However, while handling a vial of Pym Particles, he accidentally breaks it, causing the particles to fall to the ground with near zero initial velocity. The Pym particles can be modelled as point masses with $m = 10^{-31}$ kg, released at a height $h = 10$ m above ground.

Thor initially expects all the particles to fall straight down. However, upon closer inspection, he realises that the landing positions of the Pym Particles form a distribution due to quantum uncertainty! Find Δr , the minimum standard deviation of the horizontal displacement from the expected landing point.

Assume that gravity causes each particle to accelerate downwards, air resistance is negligible, and ignore any higher-order quantum effects in the vertical axis.

Leave your answer to 2 significant figures in units of m.

Hint: The Heisenberg Uncertainty Principle states that:

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

Where h is Planck's constant and x and p refer to position and momentum in one axis.

Solution: By the uncertainty principle, there will be uncertainties in the initial horizontal momentum and position of each particle. In the 2D plane of the ground, we need to consider the uncertainties in both the x and y axis, which is equal by symmetry.

Using Heisenberg's uncertainty Principle in the x axis, we have

$$\Delta x_0 \Delta p_0 \geq \frac{h}{4\pi}$$

After each particle is released, the uncertainty of its eventual horizontal displacement grows due to its velocity.

Taking $\Delta p_0 = m\Delta v_0$ we have:

$$\begin{aligned} (\Delta x)^2 &= (\Delta x_0)^2 + (v\Delta t_0)^2 \\ &= (\Delta x_0)^2 + \frac{2H}{g} \Delta v_0^2 \\ &\geq 2\Delta x_0 \Delta v_0 \sqrt{\frac{2H}{g}} \quad \text{by AM-GM inequality} \end{aligned}$$

$$= \frac{2h}{4m\pi} \sqrt{\frac{2H}{g}}$$

Combining both the uncertainties in the x and y axis gives

$$\begin{aligned}(\Delta r)^2 &= (\Delta x_0)^2 + (\Delta y)^2 \\&= \frac{2h}{4m\pi} \sqrt{\frac{2H}{g}} + \frac{2h}{4m\pi} \sqrt{\frac{2H}{g}} \\ \Delta r &= \sqrt{\frac{h}{m\pi} \sqrt{\frac{2H}{g}}} \\&= \boxed{0.055 \text{ m}}\end{aligned}$$

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Half Hour Rush X4: Forged from Flames

(5 points)

After failing several times, Thor finally decides to seek the help of Eitri, the dwarf, to reforge Jonathan. He travels to Nidavellir, where a giant neutron star is encased by a Dyson Sphere to harness its energy. The Dyson Sphere can be approximated as a thin spherical shell with negligible thickness.

However, Thor finds out that the outer surface of the Dyson sphere has become completely covered with a thick layer of ice, stopping it from working. Thor measures that the net radiative power transferred from the neutron star to the Dyson sphere is 10^{25} W.

The Dyson sphere and the neutron star are concentric spheres, both of which have emissivity $\epsilon = 0.5$. The neutron star surface is at constant temperature $T_1 = 10^6$ K, while the Dyson sphere is kept at a constant temperature $T_2 = 273.15$ K by the surrounding ice, which absorbs excess heat from the sphere via conduction. The region between the neutron star and Dyson sphere is a perfect vacuum.

Given that the inner radius of the Dyson sphere is $r_2 = 10^5$ m, find r_1 , the radius of the neutron star.

Leave your answer to 2 significant figures in km.

Hint: Kirchhoff's Law of Thermal Radiation applies here, so each object's absorptivity is equal to its emissivity.

Solution:

The neutron star radiates heat energy towards the Dyson Sphere, but only a fraction ϵ is absorbed by it. The rest of the energy is reflected back towards the neutron star, which also only absorbs a portion of it. Hence, one needs to account for the radiation being reflected between the 2 shells.

Let M_1, M_2 represents the total heat flux leaving the neutron star and the Dyson sphere respectively, which includes both emitted radiation and reflected radiation. Similarly, N_1, N_2 will represent the total heat flux incident on the neutron star and the Dyson sphere. Since each object has emissivity ϵ , it will reflect $1 - \epsilon$ of the total incident radiation. We have

$$M_i = \sigma \epsilon T_i^4 + (1 - \epsilon) N_i$$

The total heat $\dot{Q}$ leaving neutron star is given by $4\pi r_1^2(M_1 - N_1)$ and vice versa for the Dyson Sphere(except that heat is absorbed instead), so rearranging we have

$$\dot{Q} = 4\pi r_1^2(\sigma \epsilon T_1^4 - \epsilon N_1)$$

$$-\dot{Q} = 4\pi r_2^2(\sigma\epsilon T_2^4 - \epsilon N_2)$$

Representing $\sigma\epsilon T_i^4 = R_i$ and $4\pi r_i^2 = A_i$

$$\dot{Q}\left(\frac{1}{A_1} + \frac{1}{A_2}\right) = R_1 - R_2 - \epsilon(N_1 - N_2) \quad (1)$$

$$M_1 - M_2 = R_1 - R_2 + (1 - \epsilon)(N_1 - N_2) \quad (2)$$

Define the view factor $F_{A \rightarrow B}$ to be the proportion of radiation leaving surface A and striking surface B . Since the neutron star, is entirely enclosed within the Dyson Sphere, we have

$$F_{\text{Star} \rightarrow \text{Dyson}} = 1$$

Due to the reciprocity relation,

$$A_1 F_{\text{Star} \rightarrow \text{Dyson}} = A_2 F_{\text{Dyson} \rightarrow \text{Star}}$$

Hence, we have $M_1 = N_2$. Since the total heat $\dot{Q}$ can also be represented in terms of M_1 and M_2

$$\begin{aligned} \dot{Q} &= A_1 F_{\text{Star} \rightarrow \text{Dyson}} M_1 - A_2 F_{\text{Dyson} \rightarrow \text{Star}} M_2 \\ &= A_1(M_1 - M_2) \end{aligned}$$

You can substitute this back into equation (2) to arrive at an expression for $N_1 - N_2$.

$$N_1 - N_2 = \frac{\frac{\dot{Q}}{A_1} - (R_1 - R_2)}{1 - \epsilon}$$

Eliminating $N_1 - N_2$ from equation (1) and solving for $\dot{Q}$, we arrive at

$$\dot{Q} = \frac{R_1 - R_2}{\frac{1-\epsilon}{A_1} + \frac{1-\epsilon}{A_2} + \frac{\epsilon}{A_1}}$$

Substituting $\sigma\epsilon T_i^4 = R_i$ and $4\pi r_i^2 = A_i$ back into this, we may solve for inner radius as follows:

$$\dot{Q} = \frac{4\pi r_1^2 r_2^2 \sigma (T_1^4 - T_2^4) \epsilon}{r_1^2 (1 - \epsilon) - r_2^2}$$

$$\begin{aligned} r_1 &= \sqrt{\frac{\dot{Q}}{\dot{Q}\epsilon + 4\pi r_2^2 \sigma (T_1^4 - T_2^4) \epsilon}} r_2 \\ &= \boxed{5.3 \text{ km}} \end{aligned}$$

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