# *Singapore Physics League*: Sample Problems

**Problem 1: Accelerating Car** (4 points)

A car moves forward with uniform acceleration. At some point during its journey, the car covers a distance of $s_1 = 6$ m within a time interval $t_1 = 3$ s. In the next $t_2 = 2$ s right after that, it travels a distance $s_2 = 24$ m. What is the car's acceleration?

*Leave your answer in 2 significant figures in units of m s$^{-2}$.*

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*Solution:* Let the acceleration of the car be $a$, and its initial velocity at the start of the $t_1$ time interval be $v_0$. Using standard kinematics equations, we can obtain an expression for $s_1$ in terms of $v_0$ and $a$:

$$s_1 = v_0 t_1 + \frac{1}{2} a t_1^2$$

Bearing in mind that the car's velocity at the start of the $t_2$ time interval is $v_0 + at_1$ (due to its acceleration during the $t_1$ time interval), we can determine $s_2$:

$$s_2 = (v_0 + at_1)t_2 + \frac{1}{2} a t_2^2$$

Solving both equations simultaneously for $a$ gives:

$$a = \frac{2(s_2 t_1 - s_1 t_2)}{t_1 t_2 (t_1 + t_2)} = 4.0 \text{ m s}^{-2}$$

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**Problem 2: Minecraft Physics** (4 points)

In Minecraft, one bucket of lava occupies the volume of one block. Assuming that Minecraft developers want the game to be physically accurate, what is the maximum number of ice blocks $n$ that one bucket of lava should be able to melt?

Take the density of ice to be $\rho_i = 917$ kg m$^{-3}$, the initial temperature of the ice to be 0°C, the specific latent heat of fusion of ice to be $L = 3.34 \times 10^3$ J kg$^{-1}$, the density of lava to be $\rho_l = 3100$ kg m$^{-3}$, the initial temperature of the lava to be $T_0 = 1200$ K, and the specific heat capacity of lava to be $c = 840$ J kg$^{-1}$ K$^{-1}$. You may assume that lava solidifies at $T_1 = 900$ K, and makes no thermal contact with the ice upon solidification.

*Leave your answer as an exact value.*

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*Solution:* Let the volume of one Minecraft block be $V$. One bucket of lava has a mass of $m = \rho_l V$. The thermal energy $Q_{\text{lava}}$ that can be extracted from one bucket of lava before it solidifies is given by:

$$Q_{\text{lava}} = mc(T_0 - T_1) = \rho_l V c(T_0 - T_1)$$

This energy is transferred to the ice. If the goal is to melt as many ice blocks as possible, all of the energy should be used purely for melting the ice. None of the energy should be used in raising the temperature of the water that results from the ice melting. Consequently, the heat capacity of water need not be accounted for. The heat capacity of ice is also not required, since the ice started out at its melting point 0°C.

Let $n$ be the maximum number of ice blocks melted. Then the mass of melted ice $M = \rho_i n V$. Using the specific latent heat of fusion of ice, the energy required to melt the ice $Q_{\text{ice}}$ can be determined:

$$Q_{\text{ice}} = ML = \rho_i n V L$$

Since this energy is supplied by the bucket of lava, we can write $Q_{\text{lava}} = Q_{\text{ice}}$:

$$\rho_l V c(T_0 - T_1) = \rho_i n V L$$
$$\therefore n = \frac{\rho_l c(T_0 - T_1)}{\rho_i L} \approx 255.1$$

The maximum number of blocks that can be melted has to be an integer, so we round down to obtain $n = 255$.

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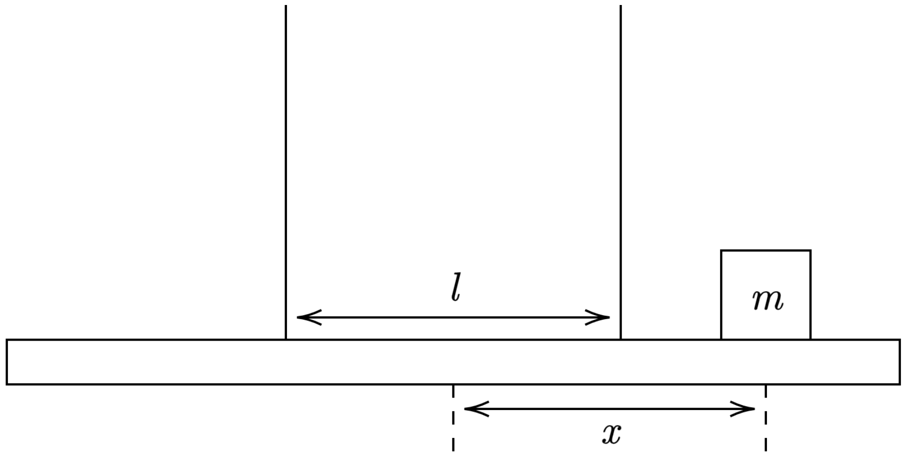
**Problem 3: Hanging Stick** (4 points)

A long uniform rod of mass $M = 1.0$ kg is held horizontal by two strings, attached to the rod at points positioned symmetrically on both sides from the rod's centre with distance $l = 0.30$ m apart. Both strings are taut and vertical, with their top ends fixed in place. A block of mass $m = 0.25$ kg initially rests on the rod's centre. It is then slowly moved towards the right. When the block reaches distance $x$ from the rod's centre, one of the strings begins to slacken. Find $x$.

*Leave your answer in 2 significant figures in units of m.*

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*Solution:* Let the tension in the left string be $T_1$, and the tension in the right string be $T_2$. Both of these string tensions pull upward on the rod. The rod also experiences its own weight $Mg$ downwards, as well as the weight of the block $mg$. Balancing forces on the rod in the vertical direction:

$$T_1 + T_2 = (M + m)g$$

Balancing torque on the rod about its centre when the block is at position $x$:

$$T_1 \frac{l}{2} + mgx = T_2 \frac{l}{2}$$

This gives us expressions for $T_1$ and $T_2$:

$$T_1 = \frac{g}{2}\left(m + M - 2m\frac{x}{l}\right)$$
$$T_2 = \frac{g}{2}\left(m + M + 2m\frac{x}{l}\right)$$

It can be seen that when $x$ is raised to a certain value, $T_1$ will become zero, indicating the point at which the left string slackens. In contrast, $T_2$ always remains positive, so the right string is taut at all times.

To find this special value of $x$ that results in slackening of the left string, we set $T_1 = 0$, which yields:

$$x = \frac{l}{2}\left(1 + \frac{M}{m}\right) = 0.75 \text{ m}$$

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**Problem 4: Inefficient Battery** (3 points)

For a voltage supply powering a load, the efficiency of the supply is defined as the ratio of power delivered to the load to the power provided by the supply.

A battery is connected to a load resistor $R = 2.0\ \Omega$. Due to substantial internal resistance $r$ present in the battery, the battery's efficiency is $\eta = 0.20$. Find the battery's internal resistance $r$.

*Leave your answer in 2 significant figures in units of* $\Omega$.

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*Solution:* Let the emf of the battery be $\varepsilon$. The total resistance of the circuit is given by $R + r$. By Ohm's Law, we can determine the current $I$ through the circuit:

$$I = \frac{\varepsilon}{R + r}$$

We can then find the total power supplied by the battery $P_{\text{battery}}$, and the power supplied to the load $P_{\text{load}}$:

$$P_{\text{battery}} = \varepsilon I \quad = \frac{\varepsilon^2}{R + r}$$
$$P_{\text{load}} = I^2 R = \frac{\varepsilon^2 R}{(R + r)^2}$$

The efficiency $\eta$ can be expressed in terms of $R$ and $r$:

$$\eta = \frac{P_{\text{load}}}{P_{\text{battery}}} = \frac{R}{R + r}$$

With this, we can determine $r = \left(\frac{1}{\eta} - 1\right) R = 8.0\ \Omega$.

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**Problem 5: Weightless Traveller** (5 points)

An intergalactic space traveller discovers a uniform spherical planet. Like the Earth, this planet rotates about its polar axis at a constant rate. A day on this planet is $T = 6.00$ hours long. When the traveller tries to measure the free-fall acceleration at the poles $g_p$ and along the equator $g_e$ on the planet's surface, he finds that $g_p = 10.0$ m s$^{-2}$ and $g_e = 8.00$ m s$^{-2}$. The traveller now stands at a point on the equator. Determine the minimum speed at which the traveller needs to run, relative to the planet's surface, so that he is essentially weightless.

*Leave your answer in 3 significant figures in units of m s$^{-1}$.*

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*Solution:* Let us take on the inertial reference frame, and denote the traveller's angular velocity as $\omega$.

When the traveller is at one of the poles, he is not in circular motion, so the measured $g_p$ is the actual gravitational field strength at the planet's surface.

On the equator, there is now a centripetal force required for his circular motion, so the measured $g = g_p - R\omega^2$. Also note that when the traveller tries to measure $g$, he travels along with the planet's surface. So his $\omega$ simply equals the angular velocity of the planet $\omega = \frac{2\pi}{T}$:

$$g_e = g_p - R\omega^2 = g_p - R\left(\frac{2\pi}{T}\right)^2$$

We can thus express $R$ in terms of the given quantities:

$$R = \frac{T^2}{4\pi^2}(g_p - g_e)$$

For the traveller to feel weightless, the traveller must move at a certain angular velocity $\omega$ (different from the planet's angular velocity) such that measured $g = 0$:

$$g_p - R\omega^2 = 0$$
$$\therefore \omega = \sqrt{\frac{g_p}{R}}$$

This $\omega$ value is viewed in the inertial reference frame, and not the planet's reference frame. As such, his required speed $v$ relative to the planet is given by:

$$\begin{aligned} v &= R\omega - R\left(\frac{2\pi}{T}\right) \\ &= \frac{T}{2\pi}(g_p - g_e)\left(\sqrt{\frac{g_p}{g_p - g_e}} - 1\right) \end{aligned}$$

noting that $R\left(\frac{2\pi}{T}\right)$ is the velocity of the planet's surface, while $R\omega$ is the traveller's velocity, both taken relative to our inertial reference frame. Substituting the relevant values, we obtain $v \approx 8500$ m s$^{-1}$.

To be precise, $v$ represents the required component of velocity directed along the planet's equator for weightlessness. As such, in the case of minimum speed, the traveller's velocity vector should purely be in the equatorial direction, which gives a minimum speed of 8500 m s$^{-1}$. If his velocity had secondary components in the radial or polar directions, he would require a slightly higher speed to achieve the same component of velocity along the equator.

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**Problem 6: Damped Pendulums** (4 points)

Two pendulums with equal lengths are set up. The bobs have identical shapes and sizes, but are made of different materials. The densities of the bobs in pendulums 1 and 2 are $\rho_1 = 3000$ kg m$^{-3}$ and $\rho_2 = 7500$ kg m$^{-3}$ respectively. They are displaced by the same small angle $\theta_0 = 5.00°$ from the vertical, and released from rest simultaneously.

Due to damping, the amplitude of oscillations decrease over time. By the time the amplitude of pendulum 1 has dropped to $\frac{1}{2}\theta_0 = 2.50°$, what is the amplitude of pendulum 2? You may assume that the magnitude of the damping force is directly proportional to the velocity of the bob.

*Leave your answer in 3 significant figures.*

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*Solution:* The pendulums are damped harmonic oscillators. As such, their amplitudes decay exponentially. We may express the amplitude of pendulum $n$, $\theta_n$, as a function of time elapsed $t$ as follows:

$$\theta_n(t) = \theta_0 e^{-\frac{b}{m_n}t}$$

where $b$ is the damping coefficient, and $m_n$ is the mass of the bob in pendulum $n$.

Let the time taken for the amplitude of pendulum 1 to be halved be $T$. We can write $\theta_1(T) = \frac{1}{2}\theta_0$, which enables us to express $b$ in terms of other quantities:

$$\theta_0 e^{-\frac{bT}{m_1}} = \frac{1}{2}\theta_0$$
$$\therefore b = \frac{m_1}{T}\ln 2$$

With the expression for $b$, we can now determine $\theta_2(T)$:

$$\theta_2(T) = \theta_0 e^{-\frac{bT}{m_2}} = 2^{-\frac{m_1}{m_2}}\theta_0$$

Since the two bobs have identical shapes and sizes, they also have the same volume, which means that the ratio $\frac{m_1}{m_2} = \frac{\rho_1}{\rho_2}$. As such, we can calculate the value of $\theta_2(T)$:

$$\theta_2(T) = 2^{-\frac{\rho_1}{\rho_2}}\theta_0 \approx 3.79°$$

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**Problem 7: $\mathbf{1+1>2}$** (3 points)

Two particles, each of rest mass $m_0$, are travelling at a velocity of $v = 0.500c$ towards each other. They collide head-on and stick together. Find the ratio of their combined rest mass before the collision to their combined rest mass after the collision.

*Leave your answer in 3 significant figures.*

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*Solution:* We opt for a conservation-of-energy approach. Since we are operating in the relativistic regime, we can write an expression for the initial total energy $U_i$:

$$U_i = 2\gamma m_0 c^2$$

Let the final combined rest mass of the two particles be $M$. Due to the symmetry of the setup, plus the fact that the two particles stick together, they must be stationary after the collision. Hence, the final energy $U_f$ is:

$$U_f = Mc^2$$

Now invoking the law of conservation of energy:

$$\begin{aligned} U_i &= U_f \\ 2\gamma m_0 &= M \end{aligned}$$

The ratio required by the question is $\frac{2m_0}{M}$, which can be calculated:

$$\frac{2m_0}{M} = \frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} \approx 0.866$$

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**Problem 8: Cubic Potential** (4 points)

For a cube with uniform charge density $\rho = 1.4$ C m$^{-3}$ and side length $x = 2.3$ cm, calculate the ratio of the electric potential at its corner to the electric potential at its centre.

*Leave your answer in 2 significant figures.*

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*Solution:* From dimensional analysis, $\frac{\rho x^2}{\varepsilon_0}$ has the same units as electric potential. Hence, we let the potential at the centre and the corner be $a\frac{\rho x^2}{\varepsilon_0}$ and $b\frac{\rho x^2}{\varepsilon_0}$ respectively, where $a$ and $b$ are dimensionless constants.

Notice that the cube can be divided into 8 smaller cubes, each of side length $\frac{x}{2}$. By principle of superposition, the potential at the original larger cube's centre is simply 8 times the potential at the corner of one of the smaller cubes. Expressing this mathematically, we have:

$$\begin{aligned} a\frac{\rho x^2}{\varepsilon_0} &= 8b\frac{\rho\left(\frac{x}{2}\right)^2}{\varepsilon_0} \\ &= 2b\frac{\rho x^2}{\varepsilon_0} \end{aligned}$$

Thus, $a = 2b$ and the required ratio $\frac{b}{a} = 0.50$.

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**Problem 9: My Abang, Usain** (4 points)

A loudspeaker is calibrated to emit a pitch at a frequency of $\nu = 442$ Hz. Usain wishes to play with the loudspeaker. First, he mounts the loudspeaker on a car, and sets the car to recede away at $u = 10.4$ m s$^{-1}$, while he stands still. He notices that the frequency of the sound he hears changes to $\nu_1$. Next, to prove that he is indeed a world-class sprinter, he stops the car, and starts to run away from it at the same speed $u$. This time, he also notes down the frequency he hears $\nu_2$. Calculate $|\nu_1 - \nu_2|$.

*Leave your answer in 3 significant figures in units of Hz.*

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*Solution:* This question investigates the Doppler effect. Consider the source of the sound as pumping out pulses at a frequency $\nu$.

In scenario 1, Usain perceives the sound frequency to be $\nu_1 = \frac{u_0}{L}$ where $u_0 = 340$ m s$^{-1}$ is the speed of sound in air and $L$ is the distance between pulses. To find $L$, consider 2 consecutive pulses. A time interval $\frac{1}{\nu}$ elapses between them, and in this time the first pulse has travelled $\frac{u_0}{\nu}$ towards Usain, while the car has travelled $\frac{u}{\nu}$ away from Usain. Hence, $L = \frac{u+u_0}{\nu}$. Therefore:

$$\nu_1 = \frac{u_0}{\left(\frac{u+u_0}{\nu}\right)} = \nu \frac{u_0}{u + u_0}$$

In scenario 2, Usain perceives the sound frequency to be $\nu_2 = \frac{u_0 - u}{L}$, where $L = \frac{u_0}{\nu}$ (since the loudspeaker is stationary). Therefore:

$$\nu_2 = \frac{u_0 - u}{\left(\frac{u_0}{\nu}\right)} = \nu \frac{u_0 - u}{u_0}$$

The required difference in frequencies is thus:

$$|\nu_1 - \nu_2| = \left|\nu \left(\frac{u_0}{u + u_0} - \frac{u_0 - u}{u_0}\right)\right| \approx 0.401 \text{ Hz}$$

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**Problem 10: Shooting a Block** (5 points)

A block of mass $M = 50$ g is placed on a plane inclined at an angle $\theta = 25°$ from the horizontal. It is prevented from sliding down the plane by a stopper. From below, a soldier takes aim at the block, aligning his rifle parallel to the plane. He fires a bullet of mass $m = 3$ g into the block, impacting it at a speed $v = 960$ m s$^{-1}$. The bullet lodges into the block in an essentially instantaneous collision. Given that the coefficient of friction (both static and kinetic) between the block and the plane is $\mu = 0.35$, find the time taken for the block to return to the stopper.

If you think that the block never returns, enter 0 as the answer.

*Leave your answer in 2 significant figures in units of s.*

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*Solution:* First, we find the speed of the block-bullet system immediately after collision, $V$, by conserving momentum:

$$mv = (M+m)V$$
$$\therefore V = \frac{m}{m+M}v$$

The block-bullet system then travels up the slope, decelerating due to friction and gravity, eventually coming to a stop. This deceleration is given by $(g\sin\theta + \mu g\cos\theta)$. Using kinematics equations, the time $t_{\text{up}}$ taken for this whole process is:

$$t_{\text{up}} = \frac{V}{g\sin\theta + \mu g\cos\theta}$$

The distance $x$ travelled along the slope in the process is:

$$x = \frac{V^2}{2(g\sin\theta + \mu g\cos\theta)}$$

Now, the block only slides back down if the force down the slope due to gravity exceeds the maximum possible static friction. This condition can be expressed mathematically:

$$g\sin\theta > \mu g\cos\theta$$
$$\therefore \mu < \tan\theta$$

Plugging in the values given in the question, we find that the condition is indeed satisfied. We may thus calculate the time taken for the block to slide back down, $t_{\text{down}}$. Note here that gravity still points down the slope but friction now points up the slope, so the acceleration of the block is given by $(g\sin\theta - \mu g\cos\theta)$. Again, using the relevant kinematics equations:

$$t_{down} = \sqrt{\frac{2x}{g\sin\theta - \mu g\cos\theta}} = \frac{V}{g}\sqrt{\frac{1}{\sin^2\theta - \mu^2\cos^2\theta}}$$

The total time taken $t$ for the whole process is thus:

$$t = t_{\mathrm{up}} + t_{\mathrm{down}} = \frac{mv}{(m+M)g}\left(\frac{1}{\sin\theta + \mu\cos\theta} + \sqrt{\frac{1}{\sin^2\theta - \mu^2\cos^2\theta}}\right)$$

Substituting the given constants yields the final answer $t \approx 27$ s.

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**Problem 11: Mysterious Process** (5 points)

Monatomic ideal gas was heated in a reversible process for which the molar heat capacity $c = 4R$, where $R$ is the molar gas constant. Over this process, the volume of the gas was doubled. By what factor did the gas pressure change?

*Leave your answer in 3 significant figures.*

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*Solution:* Consider the differential form of the first law of thermodynamics:

$$dU = dQ - p\,dV$$

The terms $dU$ and $dQ$ can be re-expressed. For a monatomic gas, since $c_v = \frac{3}{2}R$, we can write $dU = \frac{3}{2}nR\,dT$. Additionally, by definition, the molar heat capacity of a gas $c = \frac{1}{n}\frac{dQ}{dT}$, so we can express $dQ = 4nR\,dT$. Substituting these back into the first law:

$$p\,dV = \frac{5}{2}nR\,dT$$

which may be re-expressed using the ideal gas equation in order to eliminate $p$, forming an equation in $T$ and $V$ only:

$$\frac{dV}{V} = \frac{5}{2}\frac{dT}{T}$$

Let us denote the volume and temperature of the gas at the start of the process as $V_0$ and $T_0$ respectively, and $V$ and $T$ respectively at the end of the process. Integrating both sides of the equation from $V_0$ to $V$ and $T_0$ to $T$ respectively, we obtain:

$$\frac{V}{V_0} = \left(\frac{T}{T_0}\right)^{5/2}$$

To now derive a relation between gas pressure $p$ and volume $V$, we can use the ideal gas equation to write $T = \frac{pV}{nR}$, which gives $\frac{T}{T_0} = \frac{pV}{p_0V_0}$. Hence:

$$\frac{p}{p_0} = \left(\frac{V}{V_0}\right)^{-3/5}$$

Since the gas volume was doubled, $V/V_0 = 2$, so that $p = 2^{-3/5}p_0$. So the gas pressure changed by a factor of $2^{-3/5} \approx 0.660$.

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**Problem 12: Stop Spinning!** (5 points)

A uniform thin circular plate of radius $R = 10$ cm lies flat on a horizontal table. It is initially rotated at angular frequency $\omega = 480$ rad s$^{-1}$ about its centre by a motor. The motor is then switched off. Given that the coefficient of friction between the plate and the table is $\mu = 0.40$, calculate the time taken for the plate to come to a stop.

*Leave your answer in 2 significant figures in units of s.*

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*Solution:* Let $r$ denote the distance from the centre of the plate, and let $\sigma$ denote the mass per unit area of the plate. Divide the plate into concentric rings of infinitesimal thickness, and consider one such ring from $r$ to $r + dr$. The mass of this ring $dm$ can be written as $dm = \sigma(2\pi r\, dr)$.

The frictional force $df$ experienced by the ring is thus:

$$df = \mu g\, dm = 2\pi r \sigma \mu g\, dr$$

Multiplying by the distance of the ring from the centre gives the retarding torque $d\tau$ due to friction about the centre:

$$d\tau = 2\pi r^2 \sigma \mu g\, dr$$

To find the total torque $\tau$, we simply integrate from $r = 0$ to $r = R$:

$$\tau = \int_0^R 2\pi r^2 \sigma \mu g\, dr = \frac{2}{3}\pi\sigma\mu g R^3$$

The angular deceleration of the plate, $\alpha$, can be found by invoking Newton's Second Law for rotation. The moment of inertia of the plate about its central axis is $I = \frac{1}{2}mR^2$, where $m$ is the total mass of the plate. At this point, we also eliminate variable $\sigma$ by substituting $\sigma = \frac{m}{\pi R^2}$, as follows:

$$\alpha = \frac{\tau}{I} = \frac{\frac{2}{3}\pi\left(\frac{m}{\pi R^2}\right)\mu g R^3}{\frac{1}{2}mR^2} = \frac{4\mu g}{3R}$$

By definition of $\alpha$ as the rate of change of $\omega$, the time taken, $t$, for the plate to come to a stop can be determined:

$$t = \frac{\omega}{\alpha} = \frac{3R\omega}{4\mu g} \approx 9.2 \text{ s}$$

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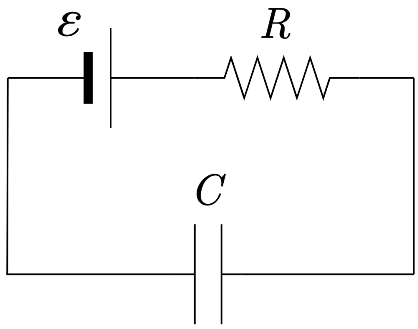
**Problem 13: Dielectric RC** (6 points)

A capacitor is connected in series with a resistor $R = 250\ \Omega$ and a battery with emf $\varepsilon = 12$ V. The space between the capacitor plates is completely filled with dielectric material with dielectric constant $\kappa = 6.0$. The circuit is initially at equilibrium.

The space between the capacitor plates is now instantaneously emptied, such that all that remains is air. At this state, its capacitance $C = 100\ \mu$F. The circuit is at a state of disequilibrium.




(a) What is the current through the resistor at the instant when the space between the capacitor plates is emptied?

*Leave your answer in 2 significant figures in units of A.* (2 points)

(b) What is the total heat generated in the resistor by the time the circuit arrives at a new state of equilibrium?

*Leave your answer in 2 significant figures in units of J.* (4 points)

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*Solution:*

(a) In the very beginning, when there was dielectric contained between the capacitor plates, its capacitance was $\kappa C$. At equilibrium, the voltage across the capacitor must equate to $\varepsilon$. Thus the initial charge $Q_0$ stored in the capacitor is:

$$Q_0 = \kappa C \varepsilon$$

At the instant when the dielectric is removed, the capacitance drops to $C$, but it still contains the same charge $Q_0$. So the voltage across the capacitor $V_0$ at this instant is given by:

$$V_0 = \frac{Q_0}{C} = \kappa\varepsilon$$

Thus, the voltage across $R$ is $\kappa\varepsilon - \varepsilon$, since the battery still provides the same potential difference $\varepsilon$. The current $I$ through the resistor can be determined using Ohm's Law:

$$I = \frac{\varepsilon(\kappa - 1)}{R} = 0.24 \text{ A}$$

(b) Let the initial (at the instant right after the dielectric is removed) and final (after the circuit returns to equilibrium) energy stored by the capacitor be $U_i$ and $U_f$ respectively. At the initial state, the charge on the capacitor is $Q_0 = \kappa C\varepsilon$. At the final state, the charge is $Q_f = C\varepsilon$, since the voltage across the capacitor must equal that of the battery at the equilibrium state. This enables us to write expressions for $U_i$ and $U_f$:

$$U_i = \frac{Q_0^2}{2C} = \frac{1}{2}C\kappa^2\varepsilon^2$$
$$U_f = \frac{Q_f^2}{2C} = \frac{1}{2}C\varepsilon^2$$

Additionally, during this process, positive charge from the capacitor flows through the battery, from its positive terminal to its negative terminal. This means that the battery does negative work on the system. The charge $\Delta q$ transported through the battery is equal to the difference between the initial and final charge on the capacitor:

$$\Delta q = C\varepsilon(\kappa - 1)$$

With this, the negative work done by the battery $W$ can be determined:

$$W = -\varepsilon\Delta q = -C\varepsilon^2(\kappa - 1)$$

Now, let the work done by the resistor on the circuit be $Q$. As the resistor dissipates heat, we expect $Q < 0$. By conservation of energy:

$$U_i + W + Q = U_f$$
$$\therefore Q = -\frac{1}{2}C\varepsilon^2(\kappa - 1)^2$$

Hence, the heat dissipated by the resistor is $\frac{1}{2}C\varepsilon^2(\kappa - 1)^2 = 0.18$ J.

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**Problem 14: A Wireless Motor?** (5 points)

An insulating circular loop of mass $m = 200$ g carries a charge $Q = 3.0$ C. The mass and charge on the loop are both uniformly distributed. The loop is free to rotate about its central axis without friction. A uniform magnetic field, parallel to the loop's axis and with strength $B = 2.0$ mT, is suddenly switched on. What is the final angular velocity of the loop?

*Leave your answer in 2 significant figures in units of rad $s^{-1}$.*

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*Solution:* Faraday's Law states that $\varepsilon = -\frac{d\phi}{dt}$, where $\varepsilon$ is the induced emf and $\phi$ is the magnetic flux through the loop. Applying it to this scenario:

$$|\varepsilon| = \pi R^2 \frac{dB}{dt}$$

where $R$ is the radius of the loop.

Since the setup is cylindrically symmetric, the electric field $E$ (induced by the activation of the magnetic field) at any point in time has the same magnitude across the loop. Moreover, the electric field would point in the circumferential direction. So, we can write a relation between $E$ and $\varepsilon$:

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = E(2\pi R)$$

Simplifying, we have:

$$E = \frac{R}{2}\frac{dB}{dt}$$

Noting again the uniformity of the electric field across the loop, the torque on the loop $\tau$ can be simply written as:

$$\tau = QER = \frac{QR^2}{2}\frac{dB}{dt}$$

As the loop has a moment of inertia $I = mR^2$, its angular acceleration $\alpha$ is given by:

$$\alpha = \frac{\tau}{I} = \frac{Q}{2m}\frac{dB}{dt}$$

To obtain the final angular velocity $\omega$ of the loop, we compute the integral of angular acceleration $\alpha$:

$$\omega = \int \alpha \, dt = \frac{QB}{2m} = 0.015 \text{ rad s}^{-1}$$

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**Problem 15: Caught in Rain** (9 points)

During a heavy thunderstorm, raindrops fall at speed $u = 12\ \text{m s}^{-1}$, with strong winds causing them to fall at an angle $\theta = 30°$ from the vertical. Galen is caught in the rain and can run on horizontal ground at velocity $v$ towards shelter. $v$ is taken to be positive when he runs against the current of the rain. The surface area of Galen's body viewed from in front is $\sigma = 15$ times larger than his body viewed from above. Assume that the rainfall is uniformly distributed.

(a) The mass of raindrops contacting Galen's body per unit time is directly proportional to $(\alpha + \beta v)$, where $\alpha$ and $\beta$ are constants to be determined, each expressed in SI units. Find the numerical value of the ratio $\alpha/\beta$.

*Leave your answer in 2 significant figures.* (3 points)

*If you were unable to solve (a), you may use $\alpha = 80$, $\beta = 20$ in the remaining parts. However, the maximum attainable score for each subsequent part is reduced by 1 point.*

There are two shelters available to Galen. Shelter X is located at distance $x = 150$ m away from Galen, but he will have to run against the current of the rain to get there. Shelter Y is at distance $y$ away, and he will run along with the current of the rain on his way there.

(b) Find the maximum value of $y$ such that Galen will get less wet by choosing to run for shelter Y rather than shelter X. Assume that Galen runs at the same speed of $v = 5\ \text{m s}^{-1}$ in every direction.

*Leave your answer in 3 significant figures in units of m.* (2 points)

(c) Suppose that Galen chooses to run towards shelter X at $v = 5\ \text{m s}^{-1}$. He ends up expending a total energy of $E_0$ throughout the journey. If he had chosen to run at $v = 10\ \text{m s}^{-1}$, he would have used a total energy of $E_1$. Find the ratio $E_1/E_0$. Neglect any resistance caused by friction, and assume that the raindrops remain stuck to Galen's body upon impact.

*Leave your answer in 3 significant figures.* (4 points)

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*Solution:*

(a) Consider separately the raindrops that hit Galen through the front of his body, and those that hit him through the top of his body. In the former case, the relative horizontal velocity between Galen's body and the raindrops is $u + v\sin\theta$, thus the rate of mass flow of raindrops through the front of his body is $\rho A\sigma(v + u\sin\theta)$. As for the latter case, the relative vertical velocity between his body and the raindrops is just $u\cos\theta$, so the rate of mass flow through the top of his body is $\rho Au\cos\theta$.

Hence, the total rate of mass flow is $\rho A\left[u(\cos\theta+\sigma\sin\theta)+\sigma v\right]$. We can thus compute $\alpha = u(\cos\theta+\sigma\sin\theta) \approx 100$ and $\beta = \sigma = 15$. Hence the ratio $\alpha/\beta \approx 6.7$.

(b) If Galen chooses to go for shelter X, he would take a time of $x/v$ to get there. The total mass of raindrops that he contacts $m_x$ along the way is given by:

$$m_x = \rho Ax\left[\frac{u}{v}(\cos\theta+\sigma\sin\theta)+\sigma\right]$$

On the other hand, if he goes for shelter Y, he would be travelling in the opposite direction. His velocity would effectively be $-v$. The journey would take a time of $y/v$, and the mass of raindrops he contacts $m_y$ can be calculated:

$$m_y = \rho Ay\left[\frac{u}{v}(\cos\theta+\sigma\sin\theta)-\sigma\right]$$

For the total mass of raindrops that contacted Galen to be lower for shelter Y than for shelter X, $m_y < m_x$ must be satisfied, which gives:

$$\frac{y}{x} < \frac{\frac{u}{v}(\cos\theta+\sigma\sin\theta)+\sigma}{\frac{u}{v}(\cos\theta+\sigma\sin\theta)-\sigma}$$

Solving for $y$ gives $y < 1040$ m. As expected, this maximum value of $y$ exceeds the value of $x$.

(c) In time $dt$, the total mass $dm$ of raindrops contacting Galen is given by:

$$dm = \rho A\left[u(\cos\theta+\sigma\sin\theta)+\sigma v\right]dt$$

Upon impact, the raindrops stick to Galen, meaning that the horizontal component of their momentum experiences change $dp = dm(v + u\sin\theta)$ (with the $+v$ direction taken to be positive). This means that by Newton's Third Law, Galen must experience a resistive force given by $F_{\text{resistive}} = -\frac{dp}{dt}$.

$$F_{\text{resistive}} = -\rho A(v+u\sin\theta)\left[u(\cos\theta+\sigma\sin\theta)+\sigma v\right]$$

By Newton's First Law, since Galen goes at a constant speed, the propulsion force produced by his body $F_{\text{propulsion}}$ must be equal and opposite to $F_{\text{resistive}}$, i.e. $F_{\text{propulsion}} = -F_{\text{resistive}}$. To relate this propulsion force to energy, the power $P$ produced by his body is given by $P = F_{\text{propulsion}}v$, so the total energy $E$ dissipated throughout his journey can be written as $E = Pt = Px/v = F_{\text{propulsion}}x$. The final expression of $E$ is as follows:

$$E = \rho Ax(v+u\sin\theta)\left[u(\cos\theta+\sigma\sin\theta)+\sigma v\right]$$

Substituting values, $E_0 \approx 1929.32\rho A$ and $E_1 \approx 4006.28\rho A$. Thus $E_1/E_0 \approx 2.08$.

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